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Cauchy Integral on Time Scales: Constructive Sense

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ABSTRACT: We study the Cauchy integral on time scales in a constructive sense. We provide a definition of the Cauchy Δ-integral, Cauchy ∇-integral and the Cauchy $\diamondsuit_\alpha$-integral. We establish the Cauchy criteria of integrability using these definitions and also establish a few results.

Key Words: Cauchy Δ-integral, Cauchy ∇-integral, Cauchy $\diamondsuit_\alpha$-integral, time scales.

Contents

1. Introduction

Stefan Hilger under the supervision of Bernd Aulbach, in his Ph.D. Thesis- "Ein Maßkettenkalkül mit Anwendungen auf Zentrumsmannigfaltigkeiten", 1988, introduced the theory of time scale Calculus, which was later published as [10].

Hilger's main motivation was the analogy between discrete and continuous analysis and the aim to unify them. Hilger, in [10], states a system of axioms for the development of the theory of time scale, and introduces the delta derivative, and also a descriptive sense of the integral (named the Cauchy integral).

More than a decade after the so-called delta derivative was formulated, another derivative called the nabla derivative was introduced by Atici and Guseinov [3], which was previously hinted in the works of Calvin and Bohner [2], who introduced a so-called alpha derivative which consisted both the delta and nabla derivative as special cases.

For an excellent introduction to this subject with theoretical developmental summary and rich history, the reader is referred to the following [4,6,10,11,12].

In 2005, Sheng et. al. [15] formulated a derivative on time scales, called the diamond-alpha ($\diamondsuit_\alpha$) derivative, which was a linear combination of the previously defined Δ and ∇ derivatives (one may also refer [16]). In 2007, this derivative was re-defined by Roger and Sheng [14] independently of the standard Δ and ∇ derivatives.

Various integration notions in their constructive sense is discussed in literature, however the Cauchy integral, in constructive sense, has not garnered much attention. In this article we attempt to give a constructive definition of the Cauchy Δ-integral, Cauchy ∇-integral and the Cauchy $\diamondsuit_\alpha$-integral on time scales. We establish the Cauchy criteria of integrability, used in time scale literature (see [8] or [9]), and also discuss a few results using these definitions.

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2. Preliminary

In this section, we recall a few definitions and results on the theory of time scales (one may refer [4,5,6,10] for more insight).

A time scale $\mathbf{T}$ is any non-empty closed subset of $\mathbb{R}$.

Definition 2.1. *[10] Forward Jump Operator: The forward jump operator denoted by σ is a mapping, $\sigma : \mathbf{T} \to \mathbf{T}$ defined by $\sigma(t) = \inf\{r \in \mathbf{T} : r > t\}$.*

Definition 2.2. *[10] Backward Jump Operator: The backward jump operator denoted by ρ is a mapping, $\rho : \mathbf{T} \to \mathbf{T}$ defined by $\rho(t) = \sup\{r \in \mathbf{T} : r < t\}$.*

Definition 2.3. *[1, pp. 5] Forward Shift: For $f : \mathbf{T} \to \mathbb{R}$, the forward shift is defined as $f^\sigma : \mathbf{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for any $t \in \mathbf{T}$ i.e. $f^\sigma = f \circ \sigma$.*

Definition 2.4. *[4] Backward Shift: For $f : \mathbf{T} \to \mathbb{R}$, the forward shift is defined as $f^\rho : \mathbf{T} \to \mathbb{R}$ by $f^\rho(t) = f(\rho(t))$ for any $t \in \mathbf{T}$ i.e. $f^\rho = f \circ \rho$.*

Definition 2.5. *[10, pp. 27] $\mathbf{T}^k$ is defined as,*

$$\mathbf{T}^k = \begin{cases} \mathbf{T} \setminus \big(\rho(\sup \mathbf{T}), \sup \mathbf{T}\big] & if \quad \sup \mathbf{T} < \infty \\ \mathbf{T} & otherwise. \end{cases}$$

Definition 2.6. *[3, pp. 77] $\mathbf{T}_k$ is defined as,*

$$\mathbf{T}_k = \begin{cases} \mathbf{T} \setminus \big[\inf \mathbf{T}, \sigma(\inf \mathbf{T})\big) & if \quad \inf \mathbf{T} > -\infty \\ \mathbf{T} & otherwise. \end{cases}$$

Note 1. $\mathbf{T}_k^k = \mathbf{T}_k \cap \mathbf{T}^k$.

Assuming $p \leq q$, intervals in $\mathbf{T}$ are defined as [4]-
$[p, q] = [p, q]_\mathbf{T} = \{t \in \mathbf{T} : p \leq t \leq q\}$; $(p, q) = (p, q)_\mathbf{T} = \{t \in \mathbf{T} : p < t < q\}$
$[p, q) = [p, q)_\mathbf{T} = \{t \in \mathbf{T} : p \leq t < q\}$; $(p, q] = (p, q]_\mathbf{T} = \{t \in \mathbf{T} : p < t \leq q\}$.
Throughtout the article $[p, q]$, $[p, q)$, $(p, q]$ and (p, q) will denote intervals on $\mathbf{T}$.

Definition 2.7. *[10, pp. 27] Delta Derivative: Let function f be a mapping, $f : \mathbf{T} \to \mathbb{R}$ and $t \in \mathbf{T}^k$. The number $f^\Delta(t)$, provided it exist is called the delta derivative of f at t, if for any $\epsilon > 0$, there exist a neighbourhood $W = (t - \delta, t + \delta) \cap \mathbf{T}$ of t for $\delta > 0$ such that,*

$$\big|f(\sigma(t)) - f(r) - f^\Delta(t)(\sigma(t) - r)\big| \leq \epsilon\big|\sigma(t) - r\big| \quad for \ \ all \ \ r \in W.$$

Theorem 2.8. *[4] Δ-Mean Value Theorem: Suppose that f is continuous on $[p, q]$ and has a delta derivative at each point of $[p, q)$. Then there exist $\xi_1, \xi_2 \in [p, q)$ such that,*

$$f^\Delta(\xi_1)(q - p) \leq f(q) - f(p) \leq f^\Delta(\xi_2)(q - p).$$

Definition 2.9. *[3, pp. 77] Nabla Derivative: Let function f be a mapping, $f : \mathbf{T} \to \mathbb{R}$ and $t \in \mathbf{T}_k$. The number $f^\nabla(t)$, provided it exist is called the nabla derivative of f at t, if for any $\epsilon > 0$, there exist a neighbourhood $W = (t - \delta, t + \delta) \cap \mathbf{T}$ of t for $\delta > 0$ such that,*

$$\big|f(\rho(t)) - f(r) - f^\nabla(t)(\rho(t) - r)\big| \leq \epsilon\big|\rho(t) - r\big| \quad for \ \ all \ \ r \in W.$$

Theorem 2.10. *[4] ∇-Mean Value Theorem: Suppose that f is continuous on $[p, q]$ and has a nabla derivative at each point of $(p, q]$. Then there exist $\xi_1, \xi_2 \in (p, q]$ such that,*

$$f^\nabla(\xi_1)(q - p) \leq f(q) - f(p) \leq f^\nabla(\xi_2)(q - p).$$

Definition 2.11. *[14, pp. 231] Diamond-Alpha ($\diamondsuit_\alpha$) Derivative: Let function f be a mapping, $f:\mathbf{T}\to\mathbb{R}$ and $t\in\mathbf{T}_k^k$. The number $f^{\diamondsuit_\alpha}(t)$, provided it exist is called the diamond-alpha derivative of f at t, if for any $\epsilon>0$, there exist a neighbourhood $W=(t-\delta,t+\delta)\cap\mathbf{T}$ of t for $\delta>0$ such that for any $r\in W$,*

$$\Big|\alpha\big[f^\sigma(t)-f(r)\big]\big(\rho(t)-r\big)+(1-\alpha)\big[f^\rho(t)-f(r)\big]\big(\sigma(t)-r\big)-f^{\diamondsuit_\alpha}(t)\big(\sigma(t)-r\big)\\ \big(\rho(t)-r\big)\Big|<\epsilon\Big|\big(\sigma(t)-r\big)\big(\rho(t)-r\big)\Big|.$$

Here $\alpha\in[0,1]$.

The diamond-alpha derivative as a linear combination of the Δ and ∇ derivatives [15, pp. 397], [16, pp. 65] is given as- for any $t\in\mathbf{T}_k^k$,

$$f^{\diamondsuit_\alpha}(t)=\alpha f^\Delta(t)+(1-\alpha)f^\nabla(t),\qquad \alpha\in[0,1].$$

If the number $f^{\diamondsuit_\alpha}(t)$ exist, then we call this the diamond-alpha derivative of f at t.

We now proceed to give the definition of the Cauchy integral on time scales in the following sections.

3. Cauchy Δ-integral

In this section, we define the Cauchy Δ-integral on time scales. We establish the Cauchy criterion of Δ-integrability and formulate a few results.

Let $[p,q]_\mathbf{T}=[p,q]$ be a closed interval on $\mathbf{T}$ such that $p<q$. Let $\mathfrak{P}$ be the collection of all possible partitions of $[p,q]$.

Before proceeding we first establish a few preliminary information required for the definition.

Let $\mathcal{Y}\in\mathfrak{P}$, $\mathcal{Y}=\big\{p=t_0<t_1<\ldots<t_n=q\big\}$, with t_0, t_1, …, t_n being the points of division. We consider subintervals of the form $[t_{h-1},\rho(t_h)]$, for $1\le h\le n$, and from each subinterval we choose t_{h-1} and call it the tag point of the respective subinterval. For $\mathcal{Y}\in\mathfrak{P}$, we define a point-interval collection as $\breve{\mathcal{Y}}=\big\{\big(t_{h-1},[t_{h-1},\rho(t_h)]\big)\big\}_{h=1}^n$, and call it the tagged partition.
We define the mesh of $\mathcal{Y}$ as, mesh-$(\mathcal{Y})=\max_{1\le h\le n}(t_h-t_{h-1})>0$.
For some $\delta>0$, $\mathcal{Y}_\delta$ will represent a partition of $[p,q]$ with mesh δ satisfying the property: For each $h=1,2,\ldots,n$ we have either- $(t_h-t_{h-1})\le\delta$ or $(t_h-t_{h-1})>\delta\wedge\rho(t_h)=t_{h-1}$. Henceforth, $\breve{\mathcal{Y}}_\delta$ will mean a tagged partition with mesh δ satisfying the above property.

We now form the Cauchy Δ-sum, $\overline{C}$, of the function f evaluated at the tags as-

$$\overline{C}=\sum_{h=1}^n f(t_{h-1})\big(t_h-t_{h-1}\big).$$

The limit of the above sum, provided it exist, as length of the subinterval tends to 0 is said to be the Cauchy Δ-integral of f over $[p,q]$, written as $\overline{C}\int_p^q f(t)\Delta t$.

Definition 3.1. *Cauchy Δ-integral: A bounded function $f:[p,q]_\mathbf{T}\to\mathbb{R}$ is Cauchy Δ-integrable if there exists a number $\overline{I}\in\mathbb{R}$ such that for any $\epsilon>0$ there exists a $\delta>0$ such that for any tagged partition $\breve{\mathcal{Y}}_\delta$, we have $\big|\overline{C}-\overline{I}\big|<\epsilon$.*
Here $\overline{I}=\overline{C}\int_p^q f(t)\Delta t$ and $\overline{C}=\sum_{h=1}^n f(t_{h-1})\big(t_h-t_{h-1}\big)$.

The set of all Cauchy Δ-integrable functions on $[p,q]$ will be denoted by $\mathfrak{C}_\Delta[p,q]$.

Example 3.2. *Suppose f is a constant function on $[p,q]$, given by $f(t)=r$, $r\in\mathbb{R}$. Then for any partition on $[p,q]$,*

$$\overline{C}=\sum_{h=1}^n f(t_{h-1})\big(t_h-t_{h-1}\big)=r\big(q-p\big),$$

and,

$$\overline{C}\int_p^q f(t)\Delta t=r\big(q-p\big).$$

Considering cases when $\mathbf{T}=\mathbb{R}$ and when $\mathbf{T}=\mathbb{Z}$-

1. When $\mathbf{T}=\mathbb{R}$, the Cauchy Δ-integral coincides with the usual Cauchy integral in $\mathbb{R}$.
2. When $\mathbf{T}=\mathbb{Z}$, $\overline{C}\int_p^q f(t)\Delta t=\sum_{h=1}^n f(t_{h-1})$.

Theorem 3.3. *If $f\in\mathfrak{C}_\Delta[p,q]$, then the value of integral, $\overline{I}$, is unique.*

Proof. Let us assume that f on $[p,q]$ has two integral values, say $\overline{I}'$ and $\overline{I}''$, both satisfying the definition, and let $\epsilon>0$.
Then, $\exists\ \delta'_{\frac{\epsilon}{2}}>0$ such that for any tagged partition $\breve{\mathcal{Y}}_{\delta'_{\frac{\epsilon}{2}}}$, the respective Cauchy Δ-sum, $\overline{C}'$ satisfies, $\left|\overline{C}'-\overline{I}'\right|<\frac{\epsilon}{2}$.
Also, $\exists\ \delta''_{\frac{\epsilon}{2}}>0$ such that for any tagged partition $\breve{\mathcal{Y}}_{\delta''_{\frac{\epsilon}{2}}}$, the respective Cauchy Δ-sum, $\overline{C}''$ satisfies, $\left|\overline{C}''-\overline{I}''\right|<\frac{\epsilon}{2}$.
Now, let $\delta_\epsilon=\min\left\{\delta'_{\frac{\epsilon}{2}},\delta''_{\frac{\epsilon}{2}}\right\}>0$ and let $\breve{\mathcal{Y}}_{\delta_\epsilon}$ be the tagged partition. Since length of the partition of $\breve{\mathcal{Y}}_{\delta_\epsilon}$ is lesser or equal to the length of the partitions of $\breve{\mathcal{Y}}_{\delta'_{\frac{\epsilon}{2}}}$ and $\breve{\mathcal{Y}}_{\delta''_{\frac{\epsilon}{2}}}$, thus taking $\overline{C}$ to be the respective Cauchy Δ-sum we have,

$$\left|\overline{C}-\overline{I}'\right|<\frac{\epsilon}{2}\quad \textit{and}\quad \left|\overline{C}-\overline{I}''\right|<\frac{\epsilon}{2},$$

whence it follows from triangle inequality that,

$$\begin{aligned}\left|\overline{I}'-\overline{I}''\right|&=\left|\overline{I}'-\overline{C}+\overline{C}-\overline{I}''\right|\\&\le\left|\overline{I}'-\overline{C}\right|+\left|\overline{C}-\overline{I}''\right|<\epsilon.\end{aligned}$$

Since $\epsilon>0$ is arbitrary, thus we conclude that $\overline{I}'=\overline{I}''$. □

Theorem 3.4. *If f is continuous on the interval $[p,q]$, then f is Cauchy Δ-integrable on $[p,q]$.*

Proof. Suppose that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are any two partitions of the interval $[p,q]$, whose subintervals have length less than δ.
Together, $\mathcal{Y}_1\cup\mathcal{Y}_2$ also form a subinterval whose length is less than δ, and $\mathcal{Y}_1,\mathcal{Y}_2\subset\mathcal{Y}_1\cup\mathcal{Y}_2$.
Firstly, considering partition $\mathcal{Y}_1$ and $\mathcal{Y}_1\cup\mathcal{Y}_2$-

$$\begin{aligned}\left|\overline{C}_{\mathcal{Y}_1}-\overline{C}_{\mathcal{Y}_1\cup\mathcal{Y}_2}\right|&=\left|\sum_{h=1}^n f(t_{h-1})\big(t_h-t_{h-1}\big)-\sum_{h=1}^n\sum_{j=1}^{i_h} f(\tilde{t}_{h_{j-1}})\big(\tilde{t}_{h_j}-\tilde{t}_{h_{j-1}}\big)\right|\\&=\left|\sum_{h=1}^n\sum_{j=1}^{i_h}\Big[f(t_{h-1})-f(\tilde{t}_{h_{j-1}})\Big]\big(\tilde{t}_{h_j}-\tilde{t}_{h_{j-1}}\big)\right|.\end{aligned}$$

Since given f is continuous on $[p,q]\Rightarrow f$ is uniformly continuous on $[p,q]$, and given $\epsilon^*>0$ we have $\delta>0$ so that $\big|f(r)-f(s)\big|<\epsilon^*$ for $r,s\in[p,q]$ within δ of each other.
Hence,

$$\left|\overline{C}_{\mathcal{Y}_1}-\overline{C}_{\mathcal{Y}_1\cup\mathcal{Y}_2}\right|<\epsilon^*\big(q-p\big).$$

Similarly,

$$\left|\overline{C}_{\mathcal{Y}_2}-\overline{C}_{\mathcal{Y}_1\cup\mathcal{Y}_2}\right|<\epsilon^*\big(q-p\big).$$

Thus the associated sums, $\overline{C}_{\mathcal{Y}_1}$ and $\overline{C}_{\mathcal{Y}_2}$ are within $\epsilon^*\big(q-p\big)$ of $\overline{C}_{\mathcal{Y}_1\cup\mathcal{Y}_2}$, and hence within $2\epsilon^*\big(q-p\big)$ of each other.
Let us choose $\epsilon = \frac{\epsilon^*}{2\left(q-p\right)}$, then we get,

$$\begin{aligned}\left|\overline{C}_{\mathcal{Y}_1} - \overline{C}_{\mathcal{Y}_2}\right| &< 2\epsilon^*\big(q-p\big)\\ &= \epsilon,\end{aligned}$$

which is the desired result. □

Theorem 3.5. *Cauchy Criterion for Cauchy Δ-integrability: A continuous function f on the interval $[p,q]$ is Cauchy Δ-integrable if and only if for each $\epsilon > 0$ there exists a positive constant δ so that, if $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are any Cauchy partitions of $[p,q]$ whose subintervals have length less than δ, the associated Cauchy Δ-sums are within in ϵ of each other:*

$$\left|\overline{C}_{\mathcal{Y}_1} - \overline{C}_{\mathcal{Y}_2}\right| < \epsilon.$$

Proof. Proof is similar to Theorem 3.4. □

Theorem 3.6. *Convergence for Cauchy Δ-integrable functions: If $\big\{f_h\big\}$ is a sequence of continuous functions converging uniformly to the function f on $[p,q]$, then f is Cauchy Δ-integrable on $[p,q]$ and $\overline{C}\int_p^q f(t)\Delta t = \lim \overline{C}\int_p^q f_h(t)\Delta t$.*

Proof. Given $\big\{f_h\big\}$ is a sequence of continuous function converging uniformly to function f, thus from Weierstrass's Theorem f is continuous,

$$\big|f(t) - f(\tilde{t})\big| \leq \big|f(t) - f_h(t)\big| + \big|f_h(t) - f_h(\tilde{t})\big| + \big|f_h(\tilde{t}) - f(\tilde{t})\big|.$$

Thus, from uniform convergence and by continuity of f_h we conclude that f is Cauchy Δ-integrable on $[p,q]$.
As for the second conclusion,

$$\left|\overline{C}\int_p^q f_h(t)\Delta t - \overline{C}\int_p^q f(t)\Delta t\right| \leq \overline{C}\int_p^q \big|f_h(t) - f(t)\big|\Delta t,$$

here $\overline{C}\int_p^q \big|f_h(t) - f(t)\big|\Delta t$ can be made arbitrarily small by uniform convergence: Given $\epsilon > 0$, we have a number K so that $\big|f_h - f\big| < \epsilon$ whenever $h \geq K$, throughout the interval $[p,q]$. □

4. Cauchy ∇-integral

In this section, we define the Cauchy ∇-integral on time scales. We establish the Cauchy Criterion of ∇-integrability and formulate a few results.

Let $[p,q]_{\mathbf{T}} = [p,q]$ be a closed interval on $\mathbf{T}$ such that $p < q$. Let $\mathfrak{P}$ be the collection of all possible partitions of $[p,q]$.

Before proceeding we first establish a few preliminary information required for the definition.

Let $\mathcal{Y} \in \mathfrak{P}$, $\mathcal{Y} = \big\{p = t_0 < t_1 < \ldots < t_n = q\big\}$, with t_0, t_1, $\ldots$, t_n being the points of division. We consider subintervals of the form $[\sigma(t_{h-1}), t_h]$, for $1 \leq h \leq n$, and from each subinterval we choose $\sigma(t_{h-1})$ and call it the tag point of the respective subinterval. For $\mathcal{Y} \in \mathfrak{P}$, we define a point-interval collection as $\breve{\mathcal{Y}} = \big\{\big(\sigma(t_{h-1}), [\sigma(t_{h-1}), t_h]\big)\big\}_{h=1}^{n}$, and call it the tagged partition.
We define the mesh of $\mathcal{Y}$ as, mesh-$(\mathcal{Y}) = \max_{1\leq h\leq n}(t_h - t_{h-1}) > 0$. For some $\delta > 0$, $\mathcal{Y}_\delta$ will represent a partition of $[p,q]$ with mesh δ satisfying the property: For each $h = 1, 2, \ldots, n$ we have either- $(t_h - t_{h-1}) \leq \delta$ or $(t_h - t_{h-1}) > \delta \wedge t_h = \sigma(t_{h-1})$. Henceforth, $\breve{\mathcal{Y}}_\delta$ will mean a tagged partition with mesh δ satisfying

the above property.

We now form the Cauchy ∇-sum, $\underline{C}$, of the function f evaluated at the tags as-

$$\underline{C} = \sum_{h=1}^{n} f\big(\sigma(t_{h-1})\big)\big(t_h - t_{h-1}\big) = \sum_{h=1}^{n} f^{\sigma}(t_{h-1})\big(t_h - t_{h-1}\big).$$

The limit of the above sum, provided it exist, as length of the subinterval tends to 0 is said to be the Cauchy ∇-integral of f over $[p,q]$, written as $\underline{C}\int_p^q f(t)\nabla t$.

Definition 4.1. *Cauchy ∇-integral: A bounded function $f:[p,q]_{\mathbf{T}} \to \mathbb{R}$ is Cauchy ∇-integrable if there exists a number $\underline{I} \in \mathbb{R}$ such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any tagged partition $\breve{\mathcal{Y}}_\delta$, we have $\big|\underline{C} - \underline{I}\big| < \epsilon$.*
Here $\underline{I} = \underline{C}\int_p^q f(t)\nabla t$ and $\underline{C} = \sum_{h=1}^{n} f\big(\sigma(t_{h-1})\big)\big(t_h - t_{h-1}\big) = \sum_{h=1}^{n} f^{\sigma}(t_{h-1})\big(t_h - t_{h-1}\big)$.

The set of all Cauchy ∇-integrable functions on $[p,q]$ will be denoted by $\mathfrak{C}_{\nabla}[p,q]$.

Example 4.2. *Suppose f is a constant function on $[p,q]$, given by $f(t) = r$, $r \in \mathbb{R}$. Then for any partition on $[p,q]$,*

$$\underline{C} = \sum_{h=1}^{n} f^{\sigma}(t_{h-1})\big(t_h - t_{h-1}\big) = r\big(q-p\big),$$

and,

$$\underline{C}\int_p^q f(t)\nabla t = r\big(q-p\big).$$

Considering cases when $\mathbf{T} = \mathbb{R}$ and when $\mathbf{T} = \mathbb{Z}$-

1. When $\mathbf{T} = \mathbb{R}$, the Cauchy ∇-integral coincides with the usual Cauchy integral in $\mathbb{R}$.
2. When $\mathbf{T} = \mathbb{Z}$,

$$\underline{C}\int_p^q f(t)\nabla t = \sum_{h=1}^{n} f(t_h).$$

Theorem 4.3. *If $f \in \mathfrak{C}_{\nabla}[p,q]$, then the value of integral, $\underline{I}$, is unique.*

Proof. Proof is similar to Theorem 3.3. □

Theorem 4.4. *If f is continuous on the interval $[p,q]$ then f is Cauchy ∇-integrable on $[p,q]$.*

Proof. Suppose that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are any two partitions of the interval $[p,q]$ whose subintervals have length less than δ.
Together, $\mathcal{Y}_1 \cup \mathcal{Y}_2$ also form a subinterval whose length is less than δ, and $\mathcal{Y}_1, \mathcal{Y}_2 \subset \mathcal{Y}_1 \cup \mathcal{Y}_2$.
Firstly, considering partition $\mathcal{Y}_1$ and $\mathcal{Y}_1 \cup \mathcal{Y}_2$-

$$\begin{aligned}
\Big|\underline{C}_{\mathcal{Y}_1} - \underline{C}_{\mathcal{Y}_1\cup\mathcal{Y}_2}\Big| &= \left|\sum_{h=1}^{n} f\big(\sigma(t_{h-1})\big)\big(t_h - t_{h-1}\big) - \sum_{h=1}^{n}\sum_{j=1}^{i_h} f\big(\sigma(\tilde{t}_{h_{j-1}})\big)\big(\tilde{t}_{h_j} - \tilde{t}_{h_{j-1}}\big)\right| \\
&= \left|\sum_{h=1}^{n}\sum_{j=1}^{i_h}\Big[f\big(\sigma(t_{h-1})\big) - f\big(\sigma(\tilde{t}_{h_{j-1}})\big)\Big]\big(\tilde{t}_{h_j} - \tilde{t}_{h_{j-1}}\big)\right| \\
&= \left|\sum_{h=1}^{n}\sum_{j=1}^{i_h}\Big[f^{\sigma}(t_{h-1}) - f^{\sigma}(\tilde{t}_{h_{j-1}})\Big]\big(\tilde{t}_{h_j} - \tilde{t}_{h_{j-1}}\big)\right|.
\end{aligned}$$

Since given f is continuous on $[p,q] \Rightarrow f$ is uniformly continuous on $[p,q]$, and given $\epsilon^* > 0$ we have $\delta > 0$ so that $\big|f(r) - f(s)\big| < \epsilon^*$ for $r, s \in [p,q]$ within δ of each other.
Hence,

$$\Big|\underline{C}_{\mathcal{Y}_1} - \underline{C}_{\mathcal{Y}_1 \cup \mathcal{Y}_2}\Big| < \epsilon^*\big(q-p\big).$$

Similarly,

$$\Big|\underline{C}_{\mathcal{Y}_2} - \underline{C}_{\mathcal{Y}_1 \cup \mathcal{Y}_2}\Big| < \epsilon^*\big(q-p\big).$$

Thus the associated sums, $\underline{C}_{\mathcal{Y}_1}$ and $\underline{C}_{\mathcal{Y}_2}$ are within $\epsilon^*\big(q-p\big)$ of $\underline{C}_{\mathcal{Y}_1 \cup \mathcal{Y}_2}$, and hence within $2\epsilon^*\big(q-p\big)$ of each other.
Let us choose $\epsilon = \frac{\epsilon^*}{2\left(q-p\right)}$, then we get,

$$\begin{aligned}\Big|\underline{C}_{\mathcal{Y}_1} - \underline{C}_{\mathcal{Y}_2}\Big| &< 2\epsilon^*\big(q-p\big)\\ &= \epsilon,\end{aligned}$$

which is the desired result. □

Theorem 4.5. *Cauchy Criterion for Cauchy ∇-integrability: A continuous function f on the interval $[p,q]$ is Cauchy ∇-integrable if and only if for each $\epsilon > 0$ there exists a positive constant δ so that, if $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are any Cauchy partitions of $[p,q]$ whose subintervals have length less than δ, the associated Cauchy ∇-sums are within in ϵ of each other:*

$$\Big|\underline{C}_{\mathcal{Y}_1} - \underline{C}_{\mathcal{Y}_2}\Big| < \epsilon.$$

Proof. Proof is similar to Theorem 4.4. □

Theorem 4.6. *Convergence for Cauchy ∇-integrable functions: If $\big\{f_h\big\}$ is a sequence of continuous functions converging uniformly to the function f on $[p,q]$, then f is Cauchy ∇-integrable on $[p,q]$ and $\underline{C}\int_p^q f(t)\nabla t = \lim \underline{C}\int_p^q f_h(t)\nabla t$.*

Proof. Proof is similar to Theorem 3.6. □

5. Fundamental Theorem of Calculus

Theorem 5.1. *If F is a Δ-differentiable function on the interval $[p,q]$, and F^{Δ} is continuous on $[p,q]$ then, F^{Δ} is Cauchy Δ-integrable on $[p,q]$ and,*

$$\overline{C}\int_p^q F^{\Delta}(t)\Delta t = F(q) - F(p).$$

Proof. Given F^{Δ} is continuous on $[p,q]$, we conclude that F^{Δ} is Cauchy Δ-integrable on $[p,q]$ from Theorem 3.4.
Thus, given $\epsilon > 0$ there exist $\delta_1 > 0$ such that, for partition $\mathcal{Y}_{\delta_1} \in \mathfrak{P}$ we get,

$$\big|\overline{C} - \overline{I}\big| < \epsilon \quad \Rightarrow \left|\sum_{h=1}^{n} F^{\Delta}(t_{h-1})\big(t_h - t_{h-1}\big) - \overline{C}\int_p^q F^{\Delta}(t)\Delta t\right| < \epsilon.$$

Given subintervals of $\mathcal{Y}_{\delta_1}$ are of the form $[t_{h-1}, t_h)$, applying the Δ-Mean Value Theorem (Theorem 2.8) on each subinterval implies that there exists $r_h \in (t_{h-1}, t_h)$ such that

$$F(t_h) - F(t_{h-1}) = F^{\Delta}(r_h).(t_h - t_{h-1}).$$

Because F^{Δ} is continuous by assumption, thus it is uniformly continuous on $[p,q]$, hence for $\delta_2 > 0$, $\left|F^{\Delta}(u) - F^{\Delta}(v)\right| < \epsilon$ whenever $u, v \in [p,q]$ satisfying $\left|u - v\right| < \delta_2$.
Letting δ be the smaller of the two numbers δ_1 and δ_2, we may conclude that,

$$\begin{aligned}\left|F(q) - F(p) - \overline{C}\int_p^q F^{\Delta}(t)\Delta t\right| &= \left|\sum_{h=1}^{n}[F(t_h) - F(t_{h-1})] - \sum_{h=1}^{n} F^{\Delta}(t_{h-1})(t_h - t_{h-1})\right.\\ &\qquad \left. + \sum_{h=1}^{n} F^{\Delta}(t_{h-1})(t_h - t_{h-1}) - \overline{C}\int_p^q F^{\Delta}(t)\Delta t\right|\\ &\le \sum_{h=1}^{n}\left|F^{\Delta}(r_h) - F^{\Delta}(t_{h-1})\right|(t_h - t_{h-1}) + \left|\sum_{h=1}^{n} F^{\Delta}(t_{h-1})\big(t_h - t_{h-1}\big) - \overline{C}\int_p^q F^{\Delta}(t)\Delta t\right|\\ &< \epsilon(q-p) + \epsilon,\end{aligned}$$

which is the desired result. □

Theorem 5.2. *If F is a ∇-differentiable function on the interval $[p,q]$, and F^{∇} is continuous on $[p,q]$ then, F^{∇} is Cauchy ∇-integrable on $[p,q]$ and,*

$$\underline{C}\int_p^q F^{\nabla}(t)\nabla t = F(q) - F(p).$$

Proof. Given F^{∇} is continuous on $[p,q]$, we conclude that F^{∇} is Cauchy ∇-integrable on $[p,q]$ from Theorem 4.4.
Thus, given $\epsilon > 0$ there exist $\delta_1 > 0$ such that, for partition $\mathcal{Y}_{\delta_1} \in \mathfrak{P}$ we get,

$$\left|\underline{C} - \underline{I}\right| < \epsilon \quad \Rightarrow \left|\sum_{h=1}^{n} F^{\nabla}(\sigma(t_{h-1}))\big(t_h - t_{h-1}\big) - \underline{C}\int_p^q F^{\nabla}(t)\nabla t\right| < \epsilon.$$

Given subintervals of $\mathcal{Y}_{\delta_1}$ are of the form $(t_{h-1}, t_h]$, applying the ∇-Mean Value Theorem (Theorem 2.10) on each subinterval implies that there exists $r_h \in (t_{h-1}, t_h)$ such that

$$F(t_h) - F(t_{h-1}) = F^{\nabla}(r_h).(t_h - t_{h-1}).$$

Because F^{∇} is continuous by assumption, thus it is uniformly continuous on $[p,q]$, hence for $\delta_2 > 0$, $\left|F^{\nabla}(u) - F^{\nabla}(v)\right| < \epsilon$ whenever $u, v \in [p,q]$ satisfying $\left|u - v\right| < \delta_2$.
Letting δ be the smaller of the two numbers δ_1 and δ_2, we may conclude that,

$$\begin{aligned}\left|F(q) - F(p) - \underline{C}\int_p^q F^{\nabla}(t)\nabla t\right| &= \left|\sum_{h=1}^{n}[F(t_h) - F(t_{h-1})] - \sum_{h=1}^{n} F^{\nabla}(\sigma(t_{h-1}))(t_h - t_{h-1})\right.\\ &\qquad \left. + \sum_{h=1}^{n} F^{\nabla}(\sigma(t_{h-1}))(t_h - t_{h-1}) - \underline{C}\int_p^q F^{\nabla}(t)\nabla t\right|\\ &\le \sum_{h=1}^{n}\left|F^{\nabla}(r_h) - F^{\nabla}(\sigma(t_{h-1}))\right|(t_h - t_{h-1}) + \left|\sum_{h=1}^{n} F^{\nabla}(\sigma(t_{h-1}))\big(t_h - t_{h-1}\big) - \underline{C}\int_p^q F^{\nabla}(t)\nabla t\right|\\ &< \epsilon(q-p) + \epsilon,\end{aligned}$$

which is the desired result. □

Theorem 5.3. *Let f is a function which is Cauchy Δ-integrable on $[p,q]$. For $r \in [p,q]$, let*

$$F(r) = \overline{C}\int_p^r f(t)\Delta t.$$

Then F is continuous on $[p, q]$. If f is continuous at $t_0 \in [p, q)$, then F is Δ-differentiable at t_0 and

$$F^{\Delta}(t_0) = f(t_0).$$

Proof. Select $D > 0$ so that $|f(r)| \leq D$ for all $r \in [p, q]$. If $r, u \in [p, q]$ and $|r - u| < \frac{\epsilon}{D}$, where $r < u$, say then

$$|F(u) - F(r)| = \left| \int_r^u f(t)\Delta t \right| \leq \int_r^u |f(t)|\Delta t \leq \int_r^u D\Delta t = D(u - r) < \epsilon.$$

This shows that F is uniformly continuous on $[p, q]$.
Suppose that f is continuous at $t_0 \in [p, q)$. If t_0 is right-scattered, then we have

$$\begin{aligned}
F^{\Delta}(t_0) = \frac{F(\sigma(t_0)) - F(t_0)}{\sigma(t_0) - t_0} &= \frac{1}{\sigma(t_0) - t_0} \left[\int_p^{\sigma(t_0)} f(t)\Delta t - \int_p^{t_0} f(t)\Delta t \right] \\
&= \frac{1}{\sigma(t_0) - t_0} \int_{t_0}^{\sigma(t_0)} f(t)\Delta t = f(t_0),
\end{aligned}$$

which is the desired result.
Further, consider the case when t_0 is right-dense. In this case,

$$F^{\Delta}(t_0) = \lim_{r \to t_0} \frac{F(r) - F(t_0)}{r - t_0}.$$

On the other hand,

$$\frac{F(r) - F(t_0)}{r - t_0} = \frac{1}{r - t_0} \left[\int_p^r f(t)\Delta t - \int_p^{t_0} f(t)\Delta t \right] = \frac{1}{r - t_0} \int_{t_0}^r f(t)\Delta t.$$

Therefore, it suffices to prove that

$$\lim_{r \to t_0} \frac{1}{r - t_0} \int_{t_0}^r f(t)\Delta t = f(t_0). \tag{5.1}$$

Let $\epsilon > 0$. Since f is continuous at t_0, $\exists\, \delta > 0$ such that $t \in [p, q]$ and $|t - t_0| < \delta$ imply $|f(t) - f(t_0)| < \epsilon$. Then,

$$\begin{aligned}
\left| \frac{1}{r - t_0} \int_{t_0}^r f(t)\Delta t - f(t_0) \right| &= \left| \frac{1}{r - t_0} \int_{t_0}^r [f(t) - f(t_0)]\Delta t \right| \\
&\leq \frac{1}{|r - t_0|} \left| \int_{t_0}^r [f(t) - f(t_0)]\Delta t \right| \leq \frac{\epsilon}{|r - t_0|} \left| \int_{t_0}^r \Delta t \right| \\
&= \epsilon
\end{aligned}$$

for all $r \in [p, q]$ such that $|r - t_0| < \delta$, and $r \neq t_0$. Hence Eq. 5.1 follows. □

Theorem 5.4. *Let f is a function which is Cauchy ∇-integrable on $[p, q]$. For $r \in [p, q]$, let*

$$F(r) = \int_p^r f(t)\nabla t.$$

Then F is continuous on $[p, q]$. If f is continuous at $t_0 \in (p, q]$, then F is ∇-differentiable at t_0 and

$$F^{\nabla}(t_0) = f(t_0).$$

Proof. Proof is similar to Theorem 5.3. □

6. Cauchy $\diamondsuit_\alpha$-integral

In this section, we define the Cauchy $\diamondsuit_\alpha$-integral on time scales. We and establish the Cauchy Criterion of $\diamondsuit_\alpha$-integrability and formulate a few results.

Let $[p,q]_{\mathbf{T}} = [p,q]$ be a closed interval on $\mathbf{T}$ such that $p < q$. Let $\mathfrak{P}$ be the collection of all possible partitions of $[p,q]$.

Before proceeding we first establish a few preliminary information required for the definition.

Let $\mathcal{Y} \in \mathfrak{P}$, $\mathcal{Y} = \{p = t_0 < t_1 < \ldots < t_n = q\}$, with t_0, t_1, …, t_n being the points of division.
We consider subintervals of the form $[t_{h-1}, \rho(t_h)]$, for $1 \le h \le n$ with corresponding tag t_{h-1}; and subintervals of the form $[\sigma(t_{h-1}), t_h]$ for $1 \le h \le n$ with corresponding tag $\sigma(t_{h-1})$. We define the mesh of $\mathcal{Y}$, as mesh-$(\mathcal{Y}) = \max_{1\le h\le n}(t_h - t_{h-1}) > 0$. For some $\delta > 0$, $\mathcal{Y}_\delta$ will represent a partition of $[p,q]$ with mesh δ satisfying the property: For each $h = 1, 2, \ldots, n$ we have either- $(t_h - t_{h-1}) \le \delta$ or $(t_h - t_{h-1}) > \delta \wedge \{\sigma(t_{h-1}) = t_h \textit{ or } \rho(t_h) = t_{h-1}\}$. Henceforth, $\mathcal{Y}_\delta$ will mean a partition $\mathcal{Y}$ with mesh δ satisfying the above property.

We now form the Cauchy $\diamondsuit_\alpha$-sum, $C_{\diamondsuit_\alpha}$, of the function f evaluated at the tags as-

$$\begin{aligned} C_{\diamondsuit_\alpha} &= \sum_{h=1}^{n} \Big(\alpha f(t_{h-1}) + \big(1-\alpha\big) f(\sigma(t_{h-1}))\Big)\big(t_h - t_{h-1}\big) \\ &= \alpha \overline{C} + (1-\alpha)\underline{C}, \end{aligned}$$

here $\alpha \in [0,1] \subset \mathbb{R}$.
The limit of the above sum, provided it exist, as length of the subinterval tends to 0 is said to be the Cauchy $\diamondsuit_\alpha$-integral of f over $[p,q]$, written as $C_{\diamondsuit_\alpha} \int_p^q f(t)\diamondsuit_\alpha t$.

Definition 6.1. *Cauchy $\diamondsuit_\alpha$-integral: A bounded function $f : [p,q]_{\mathbf{T}} \to \mathbb{R}$ is Cauchy $\diamondsuit_\alpha$-integrable if there exists a number $I_{\diamondsuit_\alpha} \in \mathbb{R}$ such that, for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any partition $\mathcal{Y}_\delta$, we have $\left|C_{\diamondsuit_\alpha} - I_{\diamondsuit_\alpha}\right| < \epsilon$.*
Here $I_{\diamondsuit_\alpha} = C_{\diamondsuit_\alpha} \int_p^q f(t)\diamondsuit_\alpha t$ and $C_{\diamondsuit_\alpha} = \sum_{h=1}^{n} \Big(\alpha f(t_{h-1}) + \big(1-\alpha\big) f(\sigma(t_{h-1}))\Big)\big(t_h - t_{h-1}\big)$.

The set of all Cauchy $\diamondsuit_\alpha$-integrable functions on $[p,q]$ will be denoted by $\mathfrak{C}_{\diamondsuit_\alpha}[p,q]$.

Example 6.2. *Suppose f is a constant function on $[p,q]$, given by $f(t) = r$, $r \in \mathbb{R}$. Then for any partition on $[p,q]$,*

$$C_{\diamondsuit_\alpha} = \sum_{h=1}^{n} \Big(\alpha r + \big(1-\alpha\big) r\Big)\big(t_h - t_{h-1}\big) = r\big(q-p\big),$$

and,

$$C_{\diamondsuit_\alpha} \int_p^q f(t)\diamondsuit_\alpha t = r\big(q-p\big).$$

Corollary 6.3. *Let f be Cauchy $\diamondsuit_\alpha$-integrable on $[p,q]$-*

1. *If $\alpha = 1$, then f is Cauchy Δ-integrable on $[p,q]$.*
2. *If $\alpha = 0$, then f is Cauchy ∇-integrable on $[p,q]$.*
3. *If $\alpha = \frac{1}{2}$, then for $\mathbf{T} = \mathbb{R}$, the Cauchy $\diamondsuit_\alpha$-integral coincides with the usual Cauchy integral in $\mathbb{R}$ and for $\mathbf{T} = \mathbb{Z}$, the Cauchy $\diamondsuit_\alpha$-integral becomes,*

$$C_{\diamondsuit_\alpha} \int_p^q f(t)\diamondsuit_\alpha t = \sum_{h=1}^{n} \left(\frac{f(t_{h-1}) + f(t_h)}{2}\right).$$

Theorem 6.4. *If $f \in \mathfrak{C}_{\diamondsuit_\alpha}[p,q]$, then the value of integral, $I_{\diamondsuit_\alpha}$, is unique.*

Proof. Let us assume that f on $[p,q]$ has two integral values, say $I'_{\diamondsuit_\alpha}$ and $I''_{\diamondsuit_\alpha}$, both satisfying the definition and let $\epsilon > 0$.
Then, $\exists\ \delta'_{\frac{\epsilon}{2}} > 0$ such that for any partition $\mathcal{Y}_{\delta'_{\frac{\epsilon}{2}}}$, the respective Cauchy $\diamondsuit_\alpha$-sum, $C'_{\diamondsuit_\alpha}$ satisfies, $\left|C'_{\diamondsuit_\alpha} - I'_{\diamondsuit_\alpha}\right| < \frac{\epsilon}{2}$.
Also, $\exists\ \delta''_{\frac{\epsilon}{2}} > 0$ such that for any partition $\mathcal{Y}_{\delta''_{\frac{\epsilon}{2}}}$, the respective Cauchy $\diamondsuit_\alpha$-sum, $C''_{\diamondsuit_\alpha}$ satisfies, $\left|C''_{\diamondsuit_\alpha} - I''_{\diamondsuit_\alpha}\right| < \frac{\epsilon}{2}$.
Now, let $\delta_\epsilon = \min\left\{\delta'_{\frac{\epsilon}{2}}, \delta''_{\frac{\epsilon}{2}}\right\} > 0$ and let $\mathcal{Y}_{\delta_\epsilon}$ be the partition. Since length of the partition of $\mathcal{Y}_{\delta_\epsilon}$ is lesser or equal to the length of the partitions of $\mathcal{Y}_{\delta'_{\frac{\epsilon}{2}}}$ and $\mathcal{Y}_{\delta''_{\frac{\epsilon}{2}}}$, thus taking $C_{\diamondsuit_\alpha}$ to be the respective Cauchy $\diamondsuit_\alpha$-sum we have,

$$\left|C_{\diamondsuit_\alpha} - I'_{\diamondsuit_\alpha}\right| < \frac{\epsilon}{2} \quad and \quad \left|C_{\diamondsuit_\alpha} - I''_{\diamondsuit_\alpha}\right| < \frac{\epsilon}{2},$$

whence it follows from triangle inequality that,

$$\begin{aligned}\left|I'_{\diamondsuit_\alpha} - I''_{\diamondsuit_\alpha}\right| &= \left|I'_{\diamondsuit_\alpha} - C_{\diamondsuit_\alpha} + C_{\diamondsuit_\alpha} - I''_{\diamondsuit_\alpha}\right| \\ &\leq \left|I'_{\diamondsuit_\alpha} - C_{\diamondsuit_\alpha}\right| + \left|C_{\diamondsuit_\alpha} - I''_{\diamondsuit_\alpha}\right| < \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, thus we conclude that $I'_{\diamondsuit_\alpha} = I''_{\diamondsuit_\alpha}$. □

Theorem 6.5. *If f is Cauchy Δ-integrable and Cauchy ∇-integrable on $[p,q]$, then it is Cauchy $\diamondsuit_\alpha$-integrable on $[p,q]$ and,*

$$I_{\diamondsuit_\alpha} = \alpha\, \overline{C} \int_p^q f(t)\Delta t + (1-\alpha)\, \underline{C} \int_p^q f(t)\nabla t.$$

Proof. Assume f is Cauchy Δ-integrable and Cauchy ∇-integrable on $[p,q]$, then it is Cauchy $\diamondsuit_\alpha$-integrable on $[p,q]$. Then for each $\epsilon > 0\ \exists\ \delta_1 > 0$ and $\delta_2 > 0$ such that -
Partition $\mathcal{Y}_{\delta_1}$ implies, $\left|\overline{C} - \overline{C}\int_p^q f(t)\Delta t\right| < \epsilon$. Partition $\mathcal{Y}_{\delta_2}$ implies, $\left|\underline{C} - \underline{C}\int_p^q f(t)\nabla t\right| < \epsilon$.
Now, let $\mathcal{Y}_\delta$ be another partition with $\delta = \min\left\{\delta_1, \delta_2\right\}$ then we have,

$$\begin{aligned}&\left|C_{\diamondsuit_\alpha} - \left\{\alpha\, \overline{C}\int_p^q f(t)\Delta t + (1-\alpha)\, \underline{C}\int_p^q f(t)\nabla t\right\}\right| \\ &\qquad = \left|\alpha\overline{C} + (1-\alpha)\underline{C} - \alpha\, \overline{C}\int_p^q f(t)\Delta t - (1-\alpha)\, \underline{C}\int_p^q f(t)\nabla t\right| \\ &\qquad \leq \left|\alpha\overline{C} - \alpha\, \overline{C}\int_p^q f(t)\Delta t\right| + \left|(1-\alpha)\underline{C} - (1-\alpha)\, \underline{C}\int_p^q f(t)\nabla t\right| \leq \epsilon\end{aligned}$$

Thus f is Cauchy $\diamondsuit_\alpha$-integrable on $[p,q]$ and,

$$I_{\diamondsuit_\alpha} = \alpha\, \overline{C} \int_p^q f(t)\Delta t + (1-\alpha)\, \underline{C} \int_p^q f(t)\nabla t.$$

□

Theorem 6.6. *If f is continuous on the interval $[p,q]$ then f is Cauchy $\diamondsuit_\alpha$-integrable on $[p,q]$.*

Proof. Suppose that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are any two partitions of the interval $[p,q]$ whose subintervals have length less than δ.

Together, $\mathcal{Y}_1 \cup \mathcal{Y}_2$ also form a subinterval whose length is less than δ, and $\mathcal{Y}_1, \mathcal{Y}_2 \subset \mathcal{Y}_1 \cup \mathcal{Y}_2$.
Firstly, considering partition $\mathcal{Y}_1$ and $\mathcal{Y}_1 \cup \mathcal{Y}_2$-

$$\begin{aligned}\left|C_{\diamondsuit_\alpha \mathcal{Y}_1} - C_{\diamondsuit_\alpha \mathcal{Y}_1 \cup \mathcal{Y}_2}\right| &= \left|\sum_{h=1}^{n}\Big[\alpha f(t_{h-1}) + (1-\alpha) f(\sigma(t_{h-1}))\Big]\big(t_h - t_{h-1}\big)\right. \\ &\qquad \left. - \sum_{h=1}^{n}\sum_{j=1}^{i_h}\Big[\alpha f(\tilde{t}_{h_{j-1}}) + (1-\alpha) f(\sigma(\tilde{t}_{h_{j-1}}))\Big]\big(\tilde{t}_{h_j} - \tilde{t}_{h_{j-1}}\big)\right| \\ &= \left|\sum_{h=1}^{n}\sum_{j=1}^{i_h}\Big[\alpha f(t_{h-1}) - \alpha f(\tilde{t}_{h_{j-1}}) + (1-\alpha) f(\sigma(t_{h-1})) - (1-\alpha) f(\sigma(\tilde{t}_{h_{j-1}}))\Big]\big(\tilde{t}_{h_j} - \tilde{t}_{h_{j-1}}\big)\right|.\end{aligned}$$

Since given f is continuous on $[p,q] \Rightarrow f$ is uniformly continuous on $[p,q]$, and given $\epsilon^* > 0$ we have $\delta > 0$ so that $\big|f(r) - f(s)\big| < \epsilon^*$ for $r, s \in [p,q]$ within δ of each other.
Hence,

$$\left|C_{\diamondsuit_\alpha \mathcal{Y}_1} - C_{\diamondsuit_\alpha \mathcal{Y}_1 \cup \mathcal{Y}_2}\right| < \epsilon^*\big(q-p\big).$$

Similarly,

$$\left|C_{\diamondsuit_\alpha \mathcal{Y}_2} - C_{\diamondsuit_\alpha \mathcal{Y}_1 \cup \mathcal{Y}_2}\right| < \epsilon^*\big(q-p\big).$$

Thus the associated sums, $\sum_{\mathcal{Y}_1} f(t)\diamondsuit_\alpha t$ and $\sum_{\mathcal{Y}_2} f(t)\diamondsuit_\alpha t$ are within $\epsilon^*\big(q-p\big)$ of $\sum_{\mathcal{Y}_1 \cup \mathcal{Y}_2} f(t)\diamondsuit_\alpha t$ and hence within $2\epsilon^*\big(q-p\big)$ of each other.
Let us choose $\epsilon = \frac{\epsilon^*}{2\big(q-p\big)}$, then we get,

$$\begin{aligned}\left|C_{\diamondsuit_\alpha \mathcal{Y}_1} - C_{\diamondsuit_\alpha \mathcal{Y}_2}\right| &< 2\epsilon^*\big(q-p\big) \\ &= \epsilon,\end{aligned}$$

which is the desired result. □

Theorem 6.7. *Cauchy Criterion for Cauchy $\diamondsuit_\alpha$-integrability: A continuous function f on the interval $[p,q]$ is Cauchy $\diamondsuit_\alpha$-integrable if and only if for each $\epsilon > 0$ there exists a positive constant δ so that, if $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are any Cauchy partitions of $[p,q]$ whose subintervals have length less than δ, the associated Cauchy $\diamondsuit_\alpha$-sums are within in ϵ of each other:*

$$\left|C_{\diamondsuit_\alpha \mathcal{Y}_1} - C_{\diamondsuit_\alpha \mathcal{Y}_2}\right| < \epsilon.$$

Proof. Proof is similar to Theorem 6.6. □

Theorem 6.8. *Convergence for Cauchy $\diamondsuit_\alpha$-integrable functions: If $\{f_h\}$ is a sequence of continuous functions converging uniformly to the function f on $[p,q]$, then f is Cauchy $\diamondsuit_\alpha$-integrable on $[p,q]$ and $C_{\diamondsuit_\alpha}\int_p^q f(t)\diamondsuit_\alpha t = \lim C_{\diamondsuit_\alpha}\int_p^q f_h(t)\diamondsuit_\alpha t$.*

Proof. Given $\{f_h\}$ is a sequence of continuous function converging uniformly to function f, thus from Weierstrass's Theorem f is continuous,

$$\big|f(t) - f(\tilde{t})\big| \le \big|f(t) - f_h(t)\big| + \big|f_h(t) - f_h(\tilde{t})\big| + \big|f_h(\tilde{t}) - f(\tilde{t})\big|.$$

Thus, from uniform convergence and by continuity of f_h we conclude that f is Cauchy $\diamondsuit_\alpha$-integrable on $[p,q]$.
As for the second conclusion,

$$\left|C_{\diamondsuit_\alpha}\int_p^q f_h(t)\diamondsuit_\alpha t - C_{\diamondsuit_\alpha}\int_p^q f(t)\diamondsuit_\alpha t\right| \le C_{\diamondsuit_\alpha}\int_p^q \big|f_h(t) - f(t)\big|\diamondsuit_\alpha t,$$

here $C_{\diamondsuit_\alpha}\int_p^q \big|f_h(t) - f(t)\big|\diamondsuit_\alpha t$ can be made arbitrarily small by uniform convergence: Given $\epsilon > 0$, we have a number K so that $\big|f_h - f\big| < \epsilon$ whenever $h \ge K$, throughout the interval $[p,q]$. □

7. Conclusion

In this article we provide constructive definitions of the Cauchy Δ-integral, the Cauchy ∇-integral and the Cauchy $\Diamond_\alpha$-integral on time scales. We also establish the Cauchy criterion of integrability for these integrals and discuss a few results.

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