Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–17.
ISSN-0037-8712 in press
doi:10.5269/bspm.64781

Degree of Convergence of Functions Using Hausdorff-Matrix Operator

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ABSTRACT: In this paper, we review the works of the authors ([10], [23]) etc. and establish two theorems on degree of convergence of a function g and conjugate of a function $\tilde{g}$ in generalized Zygmund $(Z_r^{(\lambda)}, r \geq 1)$ space using Hausdorff-Matrix $(\wedge T)$ operator of its Fourier series and conjugate Fourier series respectively based on the findings of the review. Our results generalize several earlier results. Some important corollaries are also deduced from our main theorems.

Key Words: Degree of convergence, generalized Zygmund spaces, Hausdorff $(\wedge)$ means, Matrix (T) means, Hausdorff-Matrix $(\wedge T)$ product means, Fourier series, conjugate Fourier series.

Contents

1. Introduction

Degree of approximation of a function g and conjugate function $\tilde{g}$ in different function spaces has been of great interest among the researchers in recent past. Recently, Nigam and Sharma [18], Singh and Srivastava [21], Albayrak et. al. [1], Lal [8] and Singh and Singh [24] have studied degree of approximation of a function g in Lipschitz and weighted Lipschitz classes. In last few years, the studies on the degree of approximation of conjugate of a function $\tilde{g}$ in the different Lipschitz and weighted Lipschitz classes have been made by Rhoades [20] Singh and Srivastava [22], Kranz et al. [7], Nigam [14,15] and Nigam and Sharma [16,17].

The investigators like ([10], [23]) etc. have obtained the results on the degree of approximation of a function g in generalized Zygmund space. In each of the papers ([10], [23]) etc., the second theorem has been proved by considering $\frac{\lambda_1(l)}{l\lambda_2(l)}$ as non-increasing function g in addition to the condition $\frac{\lambda_1(l)}{\lambda_2(l)}$ is non-decreasing, which is considered in their first theorems.

But, since the modulus of continuity λ_1 is subadditive function, whence $\frac{\lambda_1(l)}{l}$ is non-increasing function of l, the second theorem in each of the earlier work follows from the first theorem without any additional condition. Thus, the second theorem in each of the earlier work is superfluous. Moreover, since λ_1 and λ_2 are moduli of continuity of second order, then second theorem in each of the earlier work should have been a corollary.

Furthermore, we observe that the degree of approximation of functions of Fourier series and conjugate Fourier series only give the degree of polynomial with respect to the functions but the degree of convergence of functions of Fourier series and conjugate Fourier series give convergence of polynomial with respect to the functions.

2010 *Mathematics Subject Classification:* 42A10.

Submitted August 25, 2022. Published February 06, 2023

Therefore, in this paper, we resolve this issue by dropping second theorem for the condition $\frac{\lambda_1(l)}{l\lambda_2(l)}$ as non-increasing and obtain the degree of convergence of a function g and conjugate function $\tilde{g}$ in generalized Zygmund class $Z_r^{(\lambda)}$, $r \geq 1$ using $\wedge T$ means of its Fourier series and conjugate Fourier series respectively. The results obtained in the paper generalizes the results of ([1], [8], [12]- [15], [18] and [21]).
We denote the d^{th} partial sum of Fourier series as

$$s_d(g;x) - g(x) = \frac{1}{2\pi}\int_0^\pi \alpha_x(l)\frac{\sin(d+\frac{1}{2})l}{\sin(\frac{l}{2})}dl.$$

The d^{th} partial sum of conjugate Fourier series is given by

$$s_d(\tilde{g};x) - \tilde{g}(x) = \frac{1}{2\pi}\int_0^\pi \beta_x(l)\frac{\cos(d+\frac{1}{2})l}{\sin(\frac{l}{2})}dl,$$

where $\tilde{g}$ is the conjugate function of g and is given by

$$\tilde{g} = -\frac{1}{2\pi}\int_0^\pi \beta_x(l)\cot\left(\frac{l}{2}\right)dt.$$

Note 1. *The conjugate Fourier series is not necessarily a Fourier series for example: The series* $\sum_{n=2}^\infty \frac{\sin(nx)}{\log n}$ *conjugate to the Fourier series* $\sum_{n=2}^\infty \frac{\cos(nx)}{\log n}$ *is not a Fourier series ([27], p. 186).*

Hausdorff [5] proved the following theorem:

Theorem 1.1. *Given the sequence* $(\mu_d)_{d=1}^\infty$, *defines*

$$\Delta^p \mu_d = \sum_{i=0}^{p}\left(\begin{array}{c} p \\ i \end{array}\right)(-1)^i \mu_{d+i}.$$

Then the matrix with elements,

$$\lambda_{md} = \begin{cases} \left(\begin{array}{c} m \\ d \end{array}\right)\Delta^{m-d}\mu_d & \text{for } d \leq m \\ 0 & \text{for } d > m \end{cases} \tag{1.1}$$

is regular if and only if μ_d *is the moment sequence*

$$\mu_d = \int_0^1 x^d d\chi(x), \tag{1.2}$$

where, χ, *known as mass function, is a real, bounded variation function defined on the interval [0,1] satisfying the conditions*

$$\chi(0+) = \chi(0) = 0 \text{ and } \chi(1) = 1 \tag{1.3}$$

A sequence μ_n *that satisfies the condition (1.3) is known as a moment sequence, while a sequence that satisfies both the conditions (1.3) and (1.4), is known as a Hausdorff moment sequence. The matrix in (1.2) that satisfies both (1.3) and (1.4) is known as a Hausdorff* ($\wedge$) *matrix (method)*

The Hausdorff means of the Fourier series are defined by

$$\wedge_m(g;x) = \sum_{d=0}^{m}\lambda_{md}s_d(g;x), m = 0,1,2,3,... \tag{1.4}$$

The Fourier series is said to be summable to s by Hausdorff ($\wedge$)method if

$$\wedge_m(g;x) \to s \text{ as } m \to \infty.$$

An infinite matrix $T = [c_{md}]; m, d = 0, 1, ...$ is called a regular matrix (method) if it transforms any convergent sequence into convergent sequence with the same limit.
Toeplitz [26] presented the following equivalent conditions for regularity:

Theorem 1.2. *The matrix $T = [c_{md}]$ is regular if and only if*
(i) $\forall d \geq 0 \lim_{m\to\infty} c_{md} = 0$;
(ii) $\lim_{m\to\infty} \sum_{d=0}^{m} c_{md} = 1$;
(iii) $\exists M > 0 \forall m \geq 0, \sum_{d=0}^{\infty} |c_{md}| < M$.

The matrix (T) method of the Fourier series is given by

$$T_m(g;x) = \sum_{d=0}^{m} c_{md} s_d(g;x), m = 0,1,2,3,... \tag{1.5}$$

Fourier series is said to be summable to s by T method if $T_m(g;x) \to s$ as $m \to \infty$.
By superimposing Hausdorff $(\wedge)$ method on Matrix (T) method, Hausdorff-Matrix $(\wedge T)$ method is obtained, which is defined as

$$M_d^{\wedge T}(g;x) = \sum_{k=0}^{d} \lambda_{d,k} \sum_{\nu=0}^{k} c_{k,\nu} s_\nu(g;x).$$

If $M_d^{\wedge T}(g;x) \to s$ as $d \to \infty$, then the Fourier series is said to be summable to s by Hausdorff-Matrix $(\wedge T)$ method.
Similarly, we can define

$$\tilde{M}_d^{\wedge T}(\tilde{g};x) = \sum_{k=0}^{d} \lambda_{d,k} \sum_{\nu=0}^{k} c_{k,\nu} s_\nu(\tilde{g};x).$$

If $\tilde{M}_d^{\wedge T}(\tilde{g};x) \to s$ as $d \to \infty$, then the conjugate Fourier series is said to be summable to s by Hausdorff-Matrix $(\wedge T)$ method.

Remark 1.3. *It is worthwhile to mention here that Hausdorff matrices represent a wider class of summability matrices. Cesàro $(C,1)$ and the Euler matrix $(E,q); q > 0$ are Hausdorff matrices and their products are also Hausdorff matrices. Moreover, Hausdorff-Matrix $(\wedge T)$ product means, which is considered in the present paper, is more powerful than the individual operators such as Hausdorff $(\wedge)$ and Matrix (T) means.*

Remark 1.4. *Particular cases of Hausdorff-Matrix $(\wedge T)$ method:*
Hausdorff-Matrix $(\wedge T)$ means reduces to
(i) $\wedge\left(H, \frac{1}{m+1}\right)$ *or* $\wedge H$ *means if* $c_{md} = \frac{1}{m-d+1}\log(m+1)$.
(ii) $\wedge (C,1)$*or* $\wedge C^1$ *means if* $c_{md} = \frac{1}{m+1}$.
(iii) $\wedge (N,p_m)$ *or* $\wedge N_p$ *means if* $c_{md} = \frac{p_{m-d}}{P_m}$ *where* $P_m = \sum_{d=0}^{m} p_d = 0$.
(iv) $\wedge (N,p,q)$ *or* $\wedge N_{p,q}$ *means if* $c_{md} = \frac{p_{m-d}q_d}{R_m}$ *where* $R_m = \sum_{d=0}^{m} p_m q_{m-d}$.
(v) $\wedge \left(\bar{N},p_m\right)$ *or* $\wedge \bar{N}_p$ *means if* $c_{md} = \frac{p_d}{P_m}$.
(vi) $\wedge (E,q)$ *or* $\wedge E_q$ *means if* $c_{md} = \frac{1}{(1-q)^m}\begin{pmatrix} m \\ n \end{pmatrix} q^{m-d}$.
(vii) Cesàro-Matrix$((C,m)T)$ *or* C_mT *means if the mass function* $\chi(x) = m\int_0^x (1-l)^{m-1}dl$.
(viii) Hölder-Matrix$((H,m)T)$ *or* H_mT *means if the mass function* $\chi(x) = \int_0^x \frac{1}{(m-1)}$
$\left(\log\frac{1}{l}\right)^{m-1} dl$.
(ix) Euler-Matrix$((E,q)T)$ *or* E_qT *means if the mass function* $\chi(x) = \begin{cases} 0, & \text{if } x \in [0,b] \\ 1, & \text{if } x \in [b,1] \end{cases}$,
where $b = \frac{1}{q+1}, q > 0$.

Remark 1.5. *In view of Remark 1.4(vii) to 1.4(ix), Hausdorff-Matrix $(\wedge T)$ means also reduces to*
(i) C_mN_p.*(ii)* $C_mN_{p,q}$. *(iii)* $C_m\bar{N}_p$.*(iv)* H_mN_p.*(v)* $H_mN_{p,q}$.*(vi)* $H_m\bar{N}_p$.*(vii)* E_qN_p. *(viii)* $E_qN_{p,q}$.
(ix) $E_q\bar{N}_p$. *means for* $m,q > 0$.

Remark 1.6. *Our Theorems 2.1 and 2.2 also hold for all the cases mentioned in Remarks 1.2 and 1.3*

Remark 1.7. *Since Cesàro and Euler means and their product are again Hausdorff means ([19]), our Theorems 2.1 and 2.2 also hold for C_mE_q and E_qC_m means for $m, q > 0$.*

The space of all functions (2π-periodic and integrable) be

$$L^r[0,2\pi] = \left\{ g : [0,2\pi] \to R; \int_0^{2\pi} \mid g(x) \mid^r dx < \infty \right\}, r \geq 1.$$

We define $\|\cdot\|$ by

$$\|g\|_r = \begin{cases} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \mid g(x) \mid^r \, dx \right\}^{\frac{1}{r}}, \ 1 \leq r < \infty \\ \text{ess} \sup\limits_{0<x<2\pi} |g(x)|, r = \infty. \end{cases}$$

As defined in Zygmund [28], $\lambda_1 : [0, 2\pi] \to R$ be an arbitrary function with $\lambda_1(l) > 0$ for $0 < l \leq 2\pi$ and $\lim\limits_{l \to 0+} \lambda_1(l) = \lambda_1(0) = 0$.

We also define

$$Z_r^{(\lambda_1)} = \left\{ g \in L^r[0,2\pi] : r \geq 1, \sup_{l \neq 0} \frac{\|g(\cdot+l) + g(\cdot-l) - 2g(\cdot)\|_r}{\lambda_1(l)} < \infty \right\}$$

and

$$\|g\|_r^{(\lambda_1)} = \|g\|_r + \sup_{l \neq 0} \frac{\|g(\cdot+l) + g(\cdot-l) - 2g(\cdot)\|_r}{\lambda_1(l)}, r \geq 1.$$

Hence, the space $Z_r^{(\lambda_1)}$ is a Banach space under the norm $\|\|_r^{(\lambda_1)}$.
The completeness of the space of $L^r, r \geq 1$ implies the completeness of the space $Z_r^{(\lambda_1)}$.

Remark 1.8. *Throughout the paper $\lambda_1(l)$ and $\lambda_2(l)$ denote moduli of continuity of order two ([28]) such that $\frac{\lambda_1(l)}{\lambda_2(l)}$ be positive and non-decreasing in l, then*

$$\|g\|_r^{(\lambda_2)} \leq max\left(1, \frac{\lambda_1(2\pi)}{\lambda_2(2\pi)}\right) \|g\|_r^{(\lambda_1)} < \infty.$$

We also observe that

$$Z_r^{(\lambda_1)} \subset Z_r^{(\lambda_2)} \subset L^r, r \geq 1.$$

Remark 1.9. *In addition to the conditions of moduli of continuity of order two, further condition is defined as*

$$\lambda_1(nl) \leq n^2 \lambda_1(l) \ \text{ for } \ l \geq 0 \text{ and } n \in N,$$

which follows from the condition that for non-negative functions

$$\frac{\lambda_1(l)}{l^2} \text{ is non-increasing on } (0, +\infty) \ ([3]).$$

Thus, in view of Remark 1.7, [6] and [27], in this paper, we drop the second theorem established in the papers ([10], [23]) etc.

Remark 1.10. *(i) If we take $r \to \infty$ in $Z_r^{(\lambda_1)}$ then $Z_r^{(\lambda_1)}$ reduces to $Z^{(\lambda_1)}$.*
(ii) If we take $\lambda_1(l) = l^\alpha$ in $Z^{(\lambda_1)}$ then $Z^{(\lambda_1)}$ reduces to Z_α.
(iii) If we take $\lambda_1(l) = l^\alpha$ in $Z_r^{(\lambda_1)}$ then $Z_r^{(\lambda_1)}$ reduces to $Z_{\alpha,r}$.
(iv) If we take $r \to \infty$ in $Z_{\alpha,r}$ then $Z_{\alpha,r}$ reduces to Z_α.
(v) Let $0 \leq \delta_2 < \delta_1 < 1$, if $\lambda_1(l) = l^{\delta_1}$ and $\lambda_2(l) = l^{\delta_2}$ then $\frac{\lambda_1(l)}{\lambda_2(l)}$ is increasing, while $\frac{\lambda_1(l)}{l\lambda_2(l)}$ is decreasing.

Degree of Convergence:

The degree of convergence of a summation method to a given function g is a measure that how fast l_d converges to g and is given by

$$||g - l_d|| = \mathcal{O}\left(\frac{1}{k_a}\right) \;([11]),$$

where $k_a \to \infty$ as $a \to \infty$.

We use the following notations:

$$\phi_{(x)}(l) = g(x+l) + g(x-l) - 2g(x);$$

$$\psi_{(x)}(l) = g(x+l) - g(x-l);$$

$$D_d(l) = \frac{1}{2\pi}\int_0^1 \left[\left\{\sum_{\nu=0}^{d}\binom{d}{\nu} u^{\nu}(1-u)^{d-\nu}\right\}\left\{\sum_{k=0}^{\nu} c_{\nu,k}\frac{\sin\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right\}\right] d\gamma(u);$$

$$\tilde{D}_d(l) = \frac{1}{2\pi}\int_0^1 \left[\left\{\sum_{\nu=0}^{d}\binom{d}{\nu} u^{\nu}(1-u)^{d-\nu}\right\}\left\{\sum_{k=0}^{\nu} c_{\nu,k}\frac{\cos\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right\}\right] d\gamma(u);$$

$$\tau\left(\text{Integral part of } \frac{1}{l}\right) = \left[\frac{1}{l}\right].$$

2. Main Results

Theorem 2.1. *Error estimation of a function g (2π-periodic) in generalized Zygmund class $Z_r^{(\lambda_1)}, r \geq 1$, by $\wedge T$ means of its Fourier series is given by*

$$\inf_{M_d^{\wedge T}} \|M_d^{\wedge T}(x) - g(x)\|_r^{(\lambda_2)} = O\left[\frac{\log \pi e(d+1)}{\log \pi(d+1)}\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)} dl\right],$$

where $\lambda_1(l)$ and $\lambda_2(l)$ are moduli of continuity of order two such that $\frac{\lambda_1(l)}{\lambda_2(l)}$ is positive and non-decreasing.

Theorem 2.2. *Error estimation of conjugate of a function $\tilde{g}$ (2π-periodic) in generalized Zygmund class $Z_r^{(\lambda_1)}, r \geq 1$, by $\wedge T$ means of its conjugate Fourier series is given by*

$$\inf_{\tilde{M}_d^{\wedge T}} \|\tilde{M}_d^{\wedge T}(x) - \tilde{g}(x)\|_r^{(\lambda_2)} = O\left[\frac{\log \pi(d+1)^2}{\log \pi(d+1)}\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)} dl\right],$$

where $\lambda_1(l)$ and $\lambda_2(l)$ are moduli of continuity of order two such that $\frac{\lambda_1(l)}{\lambda_2(l)}$ is positive and non-decreasing.

3. Lemmas

Lemma 3.1. *For $l \in \left(0, \frac{1}{d+1}\right)$, $|D_d(l)| = O(d+1)$*

Proof. For $l \in \left(0, \frac{1}{d+1}\right)$, $\sin dl \leq dl$, $\sin(l/2) \geq l/\pi$ and $\sup_{0\leq u\leq 1} |\gamma'(u)| = N$

$$|D_d(l)| = \frac{1}{2\pi}\left|\int_0^1 \left[\left\{\sum_{\nu=0}^{d}\binom{d}{\nu} u^{\nu}(1-u)^{d-\nu}\right\}\left\{\sum_{k=0}^{\nu} c_{\nu,k}\frac{\sin\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right\}\right] d\gamma(u)\right| \tag{3.1}$$

First, we solve the following

$$\begin{aligned}\sum_{k=0}^{\nu} c_{\nu,k}\frac{\sin\left(k+\frac{1}{2}\right)l}{\sin(l/2)} \leq & \sum_{k=0}^{\nu} c_{\nu,k}\frac{\sin\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\\
\leq & \sum_{k=0}^{\nu} c_{\nu,k}\frac{\left(k+\frac{1}{2}\right)l}{(l/\pi)}\\
= & \frac{\pi}{2}\left\{\sum_{k=0}^{\nu} c_{\nu,k}(2k+1)\right\}\\
= & \frac{\pi}{2}\left\{\sum_{k=0}^{\nu}\ c_{\nu,k}+2\sum_{k=0}^{\nu} k\ c_{\nu,k}\right\}\\
= & \frac{\pi}{2}\left\{1+2(c_{\nu,1}+2c_{\nu,2}+c_{\nu,1}+....\nu c_{\nu,\nu})\right\}\\
\leq & \frac{\pi}{2}\left\{1+2(\nu c_{\nu,1}+\nu c_{\nu,2}+\nu c_{\nu,3}+....\nu c_{\nu,\nu})\right\}\\
\leq & \frac{\pi}{2}\left\{1+2\nu(c_{\nu,1}+c_{\nu,2}+c_{\nu,3}+....c_{\nu,\nu})\right\}\\
\leq & \frac{\pi}{2}\left\{1+2\nu(c_{\nu,0}+c_{\nu,1}+c_{\nu,2}+....c_{\nu,\nu})-2\nu c_{\nu,0}\right\}\\
\leq & \frac{\pi}{2}\left\{1+2\nu(1-c_{\nu,0}\right\}\\
\leq & \frac{\pi}{2}(1+2\nu)\\
= & O(\nu+1) \end{aligned} \tag{3.2}$$

Using (3.2) in (3.1), we have

$$\begin{aligned}|D_d(l)| = & \frac{1}{2\pi}\left|\int_0^1\sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu\end{array}\right)u^{\nu}(1-u)^{d-\nu}(\nu+1)d\gamma(u)\right|\\
= & \frac{1}{2\pi}\left|\int_0^1 h(u)d\gamma(u)\right|, \end{aligned} \tag{3.3}$$

where $h(u)=\sum\limits_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu\end{array}\right)u^{\nu}(1-u)^{d-\nu}(\nu+1)$.

Now, solving

$$\begin{aligned}h(u) = & \sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu\end{array}\right)u^{\nu}(1-u)^{d-\nu}(\nu+1)\\
= & (1-u)^d\sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu\end{array}\right)\left\{\frac{u}{1-u}\right\}^{\nu}(\nu+1)\\
= & (1-u)^d\left[\sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu\end{array}\right)\nu\left\{\frac{u}{1-u}\right\}^{\nu}+\sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu\end{array}\right)\left\{\frac{u}{1-u}\right\}^{\nu}\right]\\
= & (1-u)^d\left[\sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu\end{array}\right)\nu\ p^{\nu}+\sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu\end{array}\right)p^{\nu}\right], \text{where } p=\frac{u}{1-u}\\
= & (1-u)^d\left[\sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu\end{array}\right)\nu\ p^{\nu}+(1+p)^d\right] \end{aligned} \tag{3.4}$$

Now, solving

$$\begin{aligned}\sum_{\nu=0}^{d}\left(\begin{array}{c}d\\ \nu\end{array}\right)\nu\ p^{\nu} =&0\left(\begin{array}{c}d\\ 0\end{array}\right)p^{0}+1\left(\begin{array}{c}d\\ 1\end{array}\right)p^{1}+2\left(\begin{array}{c}d\\ 2\end{array}\right)p^{2}+....+d\left(\begin{array}{c}d\\ d\end{array}\right)p^{d}\\ =&p\left[\left(\begin{array}{c}d\\ 1\end{array}\right)+2\left(\begin{array}{c}d\\ 2\end{array}\right)p+.......+d\left(\begin{array}{c}d\\ d\end{array}\right)p^{d-1}\right]\end{aligned} \tag{3.5}$$

We know that

$$(1+p)^{d}=\left[\left(\begin{array}{c}d\\ 1\end{array}\right)+\left(\begin{array}{c}d\\ 2\end{array}\right)p+.....+\left(\begin{array}{c}d\\ d\end{array}\right)p^{d}\right] \tag{3.6}$$

Differentiating (3.6) with respect to p on both sides,

$$d(1+p)^{d-1}=\left[\left(\begin{array}{c}d\\ 1\end{array}\right)+....+d\left(\begin{array}{c}d\\ d\end{array}\right)p^{d-1}\right] \tag{3.7}$$

Using (3.7) in (3.5), we get

$$\sum_{\nu=0}^{d}\left(\begin{array}{c}d\\ \nu\end{array}\right)\nu\ p^{\nu}=pd(1+p)^{d-1} \tag{3.8}$$

Using (3.8) in (3.4), we get

$$\begin{aligned}h(u)=&(1-u)^{d}\left[pd(1+p)^{d-1}+(1+p)^{d}\right]\\ =&(1-u)^{d}\left\{d\ \left(\frac{u}{1-u}\right)\left(\frac{1}{1-u}\right)^{d-1}+\left(\frac{1}{1-u}\right)^{d}\right\}\\ =&ud+1\end{aligned} \tag{3.9}$$

From (3.3) and (3.9), we get

$$\begin{aligned}|D_d(l)|=&\frac{N}{2\pi}\left[\left\{\int_0^1(ud+1)du\right\}\right]\\ =&O(d+1)\end{aligned} \tag{3.10}$$

□

Lemma 3.2. *For* $l\in\left[\frac{1}{d+1},\pi\right],|D_d(l)|=O\left(\frac{1}{l}\right)$

Proof. For $l\in\left[\frac{1}{d+1},\pi\right]$, $\sin(l/2)\geq l/\pi$ [28] and $\sup\limits_{0\leq u\leq 1}|\gamma^{'}(u)|=N$

$$|D_d(l)|\leq\frac{1}{2\pi}\int_0^1\left[\left|\sum_{\nu=0}^{d}\left(\begin{array}{c}d\\ \nu\end{array}\right)u^{\nu}(1-u)^{d-\nu}\right|\left|\sum_{k=0}^{\nu}c_{\nu,k}\frac{\sin\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right|\right]d\gamma(u) \tag{3.11}$$

First, we solve

$$\begin{aligned}\left|\sum_{k=0}^{\nu}c_{\nu,k}\frac{\sin\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right|\leq&\frac{\pi}{l}\left|\sum_{k=0}^{\nu}c_{\nu,k}\ \mathrm{Im}\ e^{i\left(k+\frac{1}{2}\right)l}\right|\\ \leq&\frac{\pi}{l}\left|\sum_{k=0}^{\nu}c_{\nu,k}\ \mathrm{Im}\ e^{ikl}\right|\left|e^{i\frac{l}{2}}\right|\\ \leq&\frac{\pi}{l}\left|\sum_{k=0}^{\tau-1}c_{\nu,k}\ \mathrm{Im}\ e^{ikl}\right|+\frac{\pi}{l}\left|\sum_{k=\tau}^{\nu}c_{\nu,k}\ \mathrm{Im}\ e^{ikl}\right|\end{aligned} \tag{3.12}$$

Now, considering the first term of (3.12), we get

$$\begin{aligned}\frac{\pi}{l}\left|\sum_{k=0}^{\tau-1} c_{\nu,k}\ \mathrm{Im}\ e^{ikl}\right| \leq &\frac{\pi}{l}\left|\sum_{k=0}^{\tau-1} c_{\nu,k}\right|\left|e^{ikl}\right| \\ \leq &\frac{\pi}{l}\left|\sum_{k=0}^{\tau-1} c_{\nu,k}\right| \end{aligned}\tag{3.13}$$

Now, considering the second term of (3.12) and using Abel's lemma, we get

$$\begin{aligned}\frac{\pi}{l}\left|\sum_{k=\tau}^{\nu} c_{\nu,k}\ \mathrm{Im}\ e^{ikl}\right| \leq &\frac{\pi}{l}\sum_{k=\tau}^{\nu} c_{\nu,k} \max_{0\leq m\leq k}\left|e^{iml}\right| \\ \leq &\frac{\pi}{l}\sum_{k=\tau}^{\nu} c_{\nu,k} \end{aligned}\tag{3.14}$$

Combining (3.12), (3.13) and (3.14), we get

$$\begin{aligned}\left|\sum_{k=0}^{\nu} c_{\nu,k}\frac{\sin\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right| \leq &\frac{\pi}{l}\sum_{k=0}^{\tau-1} c_{\nu,k} + \frac{\pi}{l}\sum_{k=\tau}^{\nu} c_{\nu,k} \\ = &O\left(\frac{1}{l}\right) \end{aligned}\tag{3.15}$$

From (3.11) and (3.15), we get

$$\begin{aligned}|D_d(l)| = &O\left[\frac{N}{2\pi l}\left\{\int_0^1 \sum_{\nu=0}^{d}\binom{d}{\nu} u^{\nu}(1-u)^{d-\nu}du\right\}\right] \\ = &O\left[\frac{N}{2\pi l}\left\{\int_0^1 (u+1-u)^d du\right\}\right] \\ = &O\left(\frac{1}{l}\right)\end{aligned}$$

□

Lemma 3.3. *For* $l \in \left(0, \frac{1}{d+1}\right)$, $|D_d(l)| = O\left(\frac{1}{l}\right)$

Proof. For $l \in \left(0, \frac{1}{d+1}\right)$, $\sin(l/2) \geq l/\pi$, $|\cos dl| \leq 1$ and $\sup\limits_{0\leq u\leq 1} |\gamma^{'}(u)| = N$

$$\begin{aligned}|D_d(l)| \leq &\frac{1}{2\pi}\int_0^1 \left[\left|\sum_{\nu=0}^{d}\binom{d}{\nu} u^{\nu}(1-u)^{d-\nu}\right|\left|\sum_{k=0}^{\nu} c_{\nu,k}\frac{\cos\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right|\right] d\gamma(u) \\ \leq &\frac{1}{2l}\left|\left\{\int_0^1 \sum_{\nu=0}^{d}\binom{d}{\nu} u^{\nu}(1-u)^{d-\nu}d\gamma(u)\right\}\right| \\ \leq &\frac{1}{2l}\left\{\int_0^1 \sum_{\nu=0}^{d}\binom{d}{\nu} u^{\nu}(1-u)^{d-\nu}|d\gamma(u)|\right\} \\ = &\frac{N}{2l}\left\{\int_0^1 \sum_{\nu=0}^{d}\binom{d}{\nu} u^{\nu}(1-u)^{d-\nu}du\right\} \\ = &\frac{N}{2l}\left\{\int_0^1 j(u)du\right\}, \end{aligned}\tag{3.16}$$

where $j(u)=\sum\limits_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu \end{array}\right)u^{\nu}(1-u)^{d-\nu}$.
Now, considering

$$\begin{aligned} j(u)=&\sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu \end{array}\right)u^{\nu}(1-u)^{d-\nu}\\ =&(1-u)^{d}\sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu \end{array}\right)\left\{\frac{u}{1-u}\right\}^{\nu}\\ =&(1-u)^{d}\left[\sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu \end{array}\right)\ p^{\nu}\right], \text{where } p=\frac{u}{1-u}\\ =&(1-u)^{d}\left[(1+p)^{d}\right]\\ =&(1-u)^{d}\left[\left(\frac{1}{1-u}\right)^{d}\right]\\ =&1 \end{aligned} \tag{3.17}$$

From (3.16) and (3.17), we get

$$\begin{aligned} |\tilde{D}_d(l)|=&O\left[\frac{N}{2l}\left\{\int_0^1 1du\right\}\right]\\ =&O\left(\frac{1}{l}\right) \end{aligned}$$

□

Lemma 3.4. *For* $l\in\left[\frac{1}{d+1},\pi\right], |\tilde{D}_d(l)|=O\left(\frac{1}{l}\right)$

Proof. For $l\in\left[\frac{1}{d+1},\pi\right]$, $\sin(l/2)\geq l/\pi$ and $\sup\limits_{0\leq u\leq 1}|\gamma^{'}(u)|=N$

$$|\tilde{D}_d(l)|\leq\frac{1}{2\pi}\int_0^1\left[\left|\sum_{\nu=0}^{d}\left(\begin{array}{c} d\\ \nu \end{array}\right)u^{\nu}(1-u)^{d-\nu}\right|\left|\sum_{k=0}^{\nu}c_{\nu,k}\frac{\cos\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right|\right]d\gamma(u) \tag{3.18}$$

First, we solve

$$\begin{aligned} \left|\sum_{k=0}^{\nu}c_{\nu,k}\frac{\cos\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right|\leq&\sum_{k=0}^{\nu}c_{\nu,k}\frac{\cos\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\\ \leq&\frac{\pi}{l}\left|\sum_{k=0}^{\nu}c_{\nu,k}\ \mathrm{Re}\ e^{i\left(k+\frac{1}{2}\right)l}\right|\\ \leq&\frac{\pi}{l}\left|\sum_{k=0}^{\nu}c_{\nu,k}\ \mathrm{Re}\ e^{ikl}\right|\left|e^{i\frac{l}{2}}\right|\\ \leq&\frac{\pi}{l}\left|\sum_{k=0}^{\tau-1}c_{\nu,k}\ \mathrm{Re}\ e^{ikl}\right|+\frac{\pi}{l}\left|\sum_{k=\tau}^{\nu}c_{\nu,k}\ \mathrm{Re}\ e^{ikl}\right| \end{aligned} \tag{3.19}$$

Now, considering the first term of (3.19), we get

$$\begin{aligned} \frac{\pi}{l}\left|\sum_{k=0}^{\tau-1}c_{\nu,k}\ \mathrm{Re}\ e^{ikl}\right|\leq&\frac{\pi}{l}\left|\sum_{k=0}^{\tau-1}c_{\nu,k}\right|\left|e^{ikl}\right|\\ \leq&\frac{\pi}{l}\left|\sum_{k=0}^{\tau-1}c_{\nu,k}\right| \end{aligned} \tag{3.20}$$

Now, considering the second term of (3.19) and using Abel's lemma, we get

$$\begin{aligned}\frac{\pi}{l}\left|\sum_{k=\tau}^{\nu} c_{\nu,k} \text{ Re } e^{ikl}\right| \leq &\frac{\pi}{l}\sum_{k=\tau}^{\nu} c_{\nu,k} \max_{0\leq m\leq k}\left|e^{iml}\right| \\ \leq &\frac{\pi}{l}\sum_{k=\tau}^{\nu} c_{\nu,k} \end{aligned} \tag{3.21}$$

Combining (3.19), (3.20) and (3.21), we get

$$\begin{aligned}\left|\sum_{k=0}^{\nu} c_{\nu,k}\frac{\cos\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right| \leq &\frac{\pi}{l}\sum_{k=0}^{\tau-1} c_{\nu,k} + \frac{\pi}{l}\sum_{k=\tau}^{\nu} c_{\nu,k} \\ = &O\left(\frac{1}{l}\right)\end{aligned} \tag{3.22}$$

From (3.18) and (3.22), we get

$$|\tilde{D}_d(l)| = O\left[\frac{N}{2l}\left\{\int_0^1 \sum_{\nu=0}^{d}\left(\begin{array}{c} d \\ \nu \end{array}\right) u^{\nu}(1-u)^{d-\nu}du\right\}\right]$$

Further solving along the same lines of Lemma 3.3, we get

$$|\tilde{D}_d(l)| = O\left(\frac{1}{l}\right)$$

□

Lemma 3.5. *(i) Let $g \in Z_r^{(\lambda_1)}$, then for $0 < l \leq \pi$,*
$\| \phi_{(\cdot+z)}(l) + \phi_{(\cdot-z)}(l) - 2\phi_{(\cdot)}(l) \|_r = O\left(\lambda_2(|z|)\frac{\lambda_1(l)}{\lambda_2(l)}\right)$,
where $\lambda_1(l)$ and $\lambda_2(l)$ are moduli of continuity of order two.

(ii) Let $\tilde{g} \in Z_r^{(\lambda_1)}$, then for $0 < l \leq \pi$,
$\| \psi_{(\cdot+z)}(l) - \psi_{(\cdot-z)}(l) \|_r = O\left(\lambda_2(|z|)\frac{\lambda_1(l)}{\lambda_2(l)}\right)$,
where $\lambda_1(l)$ and $\lambda_2(l)$ are moduli of continuity of order two.

Proof. This Lemma can be proved along the same lines of the proof of Lemma 3 of ([9], p.93). □

4. Proof of the Main Results

4.1. Proof of Theorem 2.1

The integral representation of $s_d(g;x)$ is given by [25] in the following form:

$$s_d(g;x) - g(x) = \frac{1}{2\pi}\int_0^{\pi} \phi_x(l)\frac{\sin(d+\frac{1}{2})l}{\sin\frac{l}{2}}dl.$$

Denoting $\wedge T$ transform of $s_d(g;x)$ by $M^{\wedge T}$, we get

$$\begin{aligned}&M_d^{\wedge T}(x) - g(x) \\ &= \int_0^{\pi}\frac{\phi_{(x)}(l)}{2\pi}\left[\int_0^1\left\{\left\{\sum_{\nu=0}^{d}\left(\begin{array}{c} d \\ \nu \end{array}\right) u^{\nu}(1-u)^{d-\nu}d\gamma(u)\right\}\left\{\sum_{k=0}^{\nu} c_{\nu,k}\frac{\sin\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right\}\right\}\right]dl \\ &= \int_0^{\pi}\phi_{(x)}(l)D_d(l) = \rho_d(l) \text{ (say).}\end{aligned} \tag{4.1}$$

Let

$$\rho_d(x) = M_d^{\wedge T}(x) - g(x) = \int_0^{\pi} \phi_{(x)}(l) D_d(l).$$

Then,

$$\rho_d(x+z) + \rho_d(x-z) - 2\rho_d(x) = \int_0^{\pi} \left\{ \phi_{(x+z)}(l) - \phi_{(x-z)}(l) - 2\phi_{(x)}(l) \right\} D_d(l) dl$$

Using generalized Minkowski's inequality ([2]), we can write

$$\begin{aligned}
&\| \rho_d(\cdot + z) + \rho_d(\cdot - z) - 2\rho_d(\cdot) \|_r \\
\leq &\int_0^{\frac{1}{d+1}} \| \phi_{(\cdot+z)}(l) - \phi_{(\cdot-z)}(l) - 2\phi_{(\cdot)}(l) \|_r | D_d(l) | dl \\
&+ \int_{\frac{1}{d+1}}^{\pi} \| \phi_{(\cdot+z)}(l) - \phi_{(\cdot-z)}(l) - 2\phi_{(\cdot)}(l) \|_r | D_d(l) | dl \\
= &I_1 + I_2
\end{aligned} \tag{4.2}$$

Using Lemmas 3.1 and 3.5(i),

$$\begin{aligned}
I_1 = &O\left[\int_0^{\frac{1}{d+1}} \lambda_2(|z|) \frac{\lambda_1(l)}{\lambda_2(l)} (d+1) dl \right] \\
= &O\left[(d+1)\lambda_2(|z|) \int_0^{\frac{1}{d+1}} \frac{\lambda_1(l)}{\lambda_2(l)} dl \right] \\
= &O\left[(d+1)\lambda_2(|z|) \frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)} \int_0^{\frac{1}{d+1}} dl \right] \\
= &O\left[\lambda_2(|z|) \frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)} \right]
\end{aligned} \tag{4.3}$$

Using Lemmas 3.2 and 3.5(i),

$$\begin{aligned}
I_2 = &O\left[\int_{\frac{1}{d+1}}^{\pi} \lambda_2(|z|) \frac{\lambda_1(l)}{\lambda_2(l)} \frac{1}{l} dl \right] \\
= &O\left[\lambda_2(|z|) \int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{\lambda_2(l)} \frac{1}{l} dl \right]
\end{aligned} \tag{4.4}$$

Combining (4.2) to (4.4), we have

$$\| \rho_d(\cdot + z) + \rho_d(\cdot - z) - 2\rho_d(\cdot) \|_r = O\left[\lambda_2(|z|) \frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)} \right] + O\left[\lambda_2(|z|) \int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{\lambda_2(l)} \frac{1}{l} dl \right]$$

$$\sup_{z \neq 0} \frac{\| \rho_d(\cdot + z) + \rho_d(\cdot - z) - 2\rho_d(\cdot) \|_r}{\lambda_2(|z|)} = O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)} \right] + O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{\lambda_2(l)} \frac{1}{l} dl \right] \tag{4.5}$$

Again, using Lemmas 3.1 and 3.2; and $\| \phi_{(\cdot)}(l) \|_r = O\left(\lambda_1(l)\right)$, we have

$$\begin{aligned}\| \rho_d(\cdot) \|_r \leq & \left[\int_0^{\frac{1}{d+1}} + \int_{\frac{1}{d+1}}^{\pi}\right] \| \phi_{(\cdot)}(l) \|_r | D_d(l) | \, dl \\ =& O\left[(d+1)\int_0^{\frac{1}{d+1}} \lambda_1(l)dl\right] + O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l}dl\right] \\ =& O\left[\lambda_1\left(\frac{1}{d+1}\right)\right] + O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l}dl\right] \end{aligned} \tag{4.6}$$

We know that

$$\| \rho_d(\cdot) \|_r^{(\lambda_2)} = \| \rho_d(\cdot) \|_r + \sup_{z\neq 0} \frac{\| \rho_d(\cdot + z) + \rho_d(\cdot - z) - 2\rho_d(\cdot) \|_r}{\lambda_2(z)} \tag{4.7}$$

Combining (4.5) to (4.7), we get

$$\| \rho_d(\cdot) \|_r^{(\lambda_2)} = O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\right] + O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{\lambda_2(l)}\frac{1}{l}dl\right] + O\left[\lambda_1\left(\frac{1}{d+1}\right)\right] + O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l}dl\right]$$

In view of monotonicity of $\lambda_2(l)$, we have

$\lambda_1(l) = \frac{\lambda_1(l)}{\lambda_2(l)}\lambda_2(l) \leq \lambda_2(\pi)\frac{\lambda_1(l)}{\lambda_2(l)} = O\left(\frac{\lambda_1(l)}{\lambda_2(l)}\right)$ for $0 < l \leq \pi$. Hence

$$\| \rho_d(\cdot) \|_r^{(\lambda_2)} = O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\right] + O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{\lambda_2(l)}\frac{1}{l}dl\right] \tag{4.8}$$

Since λ_1 and λ_2 are moduli of continuity of order two such that $\frac{\lambda_1(l)}{\lambda_2(l)}$ is positive and non-decreasing, then

$$\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)}dl \geq \frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\int_{\frac{1}{d+1}}^{\pi} \frac{1}{l}dl = \frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\log \pi(d+1).$$

i.e.

$$\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)} = O\left[\frac{1}{\log \pi(d+1)}\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)}dl\right] \tag{4.9}$$

From (4.8) and (4.9), we get

$$\begin{aligned}\| \rho_d(\cdot) \|_r^{(\lambda_2)} &= O\left[\frac{1}{\log \pi(d+1)}\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)}dl\right] + O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)}dl\right] \\ &= O\left[\left(\frac{1}{\log \pi(d+1)} + 1\right)\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)}dl\right] \\ &= O\left[\frac{\log \pi e(d+1)}{\log \pi(d+1)}\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)}dl\right]\end{aligned}$$

This completes the proof of the theorem 2.1.

4.2. Proof of Theorem 2.2

The integral representation of $s_d(\tilde{g};x)$ is given by

$$s_d(\tilde{g};x)-\tilde{g}(x)=\frac{1}{2\pi}\int_0^\pi \psi_x(l)\frac{\cos(d+\frac{1}{2})l}{\sin\frac{l}{2}}dl.$$

Denoting $\wedge T$ transform of $s_d(\tilde{g};x)$ by $\tilde{M}^{\wedge T}$, we get

$$\begin{aligned}
&\tilde{M}_d^{\wedge T}(x)-\tilde{g}(x)\\
&=\int_0^\pi \frac{\psi_{(x)}(l)}{2\pi}\left[\int_0^1\left\{\sum_{\nu=0}^{d}\binom{d}{\nu}u^\nu(1-u)^{d-\nu}\left(\sum_{k=0}^{\nu}c_{\nu,k}\frac{\cos\left(k+\frac{1}{2}\right)l}{\sin(l/2)}\right)\right\}d\gamma(u)\right]dl\\
&=\int_0^\pi \psi_{(x)}(l)\tilde{D}_d(l). \qquad (4.10)
\end{aligned}$$

Let

$$\tilde{\rho}_d(x)=\tilde{M}_d^{\wedge T}(x)-\tilde{g}(x)=\int_0^\pi \psi_{(x)}(l)\tilde{D}_d(l).$$

Then,

$$\tilde{\rho}_d(x+z)-\tilde{\rho}_d(x-z)=\int_0^\pi\left\{\psi_{(x+z)}(l)-\psi_{(x-z)}(l)\right\}\tilde{D}_d(l)dl$$

Using generalized Minkowski inequality ([2]), we can write

$$\begin{aligned}
\|\tilde{\rho}_d(\cdot+z)-\tilde{\rho}_d(\cdot-z)\|_r\le&\int_0^\pi\|\psi_{(x+z)}(l)-\psi_{(x-z)}(l)\|_r\,\tilde{D}_d(l)dl\\
\le&\int_0^{\frac{1}{d+1}}\|\psi_{(\cdot+z)}(l)-\psi_{(\cdot-z)}(l)\|_r|\,\tilde{D}_d(l)\,|\,dl\\
&+\int_{\frac{1}{d+1}}^{\pi}\|\psi_{(\cdot+z)}(l)-\psi_{(\cdot-z)}(l)\|_r|\,\tilde{D}_d(l)\,|\,dl\\
=&J_1+J_2. \qquad (4.11)
\end{aligned}$$

Using Lemmas 3.3 and 3.5(ii),

$$\begin{aligned}
J_1=&O\left[\int_0^{\frac{1}{d+1}}\lambda_2(|z|)\frac{\lambda_1(l)}{\lambda_2(l)}\left(\frac{1}{l}\right)dl\right]\\
=&O\left[\lambda_2(|z|)\int_0^{\frac{1}{d+1}}\frac{\lambda_1(l)}{l\,\lambda_2(l)}dl\right]\\
=&O\left[\lambda_2(|z|)\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\int_0^{\frac{1}{d+1}}\frac{1}{l}dl\right]\\
=&O\left[\lambda_2(|z|)\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\log(d+1)\right]. \qquad (4.12)
\end{aligned}$$

Using Lemmas 3.4 and 3.5(ii),

$$\begin{aligned}
J_2=&O\left[\int_{\frac{1}{d+1}}^{\pi}\lambda_2(|z|)\frac{\lambda_1(l)}{\lambda_2(l)}\frac{1}{l}dl\right]\\
=&O\left[\lambda_2(|z|)\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{\lambda_2(l)}\frac{1}{l}dl\right]. \qquad (4.13)
\end{aligned}$$

Combining (4.11) to (4.13), we have

$$\sup_{z\neq 0}\frac{\parallel \rho_d(\cdot+z)-\rho_d(\cdot-z)\parallel_r}{\lambda_2(|z|)}=O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\log(d+1)\right]+O\left[\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{\lambda_2(l)}\frac{1}{l}dl\right]. \tag{4.14}$$

Again, using Lemmas 3.3 and 3.4; and $\parallel \psi_{(\cdot)}(l)\parallel_r = O\left(\lambda_1(l)\right)$, we have

$$\begin{aligned}\parallel \tilde{\rho}_d(\cdot)\parallel_r \leq &\left[\int_0^{\frac{1}{d+1}}+\int_{\frac{1}{d+1}}^{\pi}\right]\parallel \psi_{(\cdot)}(l)\parallel_r |\ \tilde{D}_d(l)\ |\ dl\\ =&O\left[\int_0^{\frac{1}{d+1}}\frac{\lambda_1(l)}{l}dl\right]+O\left[\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l}dl\right]\\ =&O\left[\lambda_1\left(\frac{1}{d+1}\right)\log(d+1)\right]+O\left[\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l}dl\right].\end{aligned} \tag{4.15}$$

We know that

$$\parallel \tilde{\rho}_d(\cdot)\parallel_r^{(\lambda_2)}=\parallel \tilde{\rho}_d(\cdot)\parallel_r+\sup_{z\neq 0}\frac{\parallel \tilde{\rho}_d(\cdot+z)-\tilde{\rho}_d(\cdot-z)\parallel_r}{\lambda_2(z)}. \tag{4.16}$$

Combining (4.14) and (4.16), we get

$$\begin{aligned}\parallel \tilde{\rho}_d(\cdot)\parallel_r^{(\lambda_2)}=&O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\log(d+1)\right]+O\left[\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{\lambda_2(l)}\frac{1}{l}dl\right]\\ &+O\left[\lambda_1\left(\frac{1}{d+1}\right)\log(d+1)\right]+O\left[\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l}dl\right].\end{aligned}$$

In view of monotonicity of $\lambda_2(l)$, we have

$\lambda_1(l)=\frac{\lambda_1(l)}{\lambda_2(l)}\lambda_2(l)\leq\lambda_2(\pi)\frac{\lambda_1(l)}{\lambda_2(l)}=O\left(\frac{\lambda_1(l)}{\lambda_2(l)}\right)$ for $0<l\leq\pi$. Hence

$$\parallel \tilde{\rho}_d(\cdot)\parallel_r^{(\lambda_2)}=O\left[\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\log(d+1)\right]+O\left[\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{\lambda_2(l)}\frac{1}{l}dl\right]. \tag{4.17}$$

Since λ_1 and λ_2 are moduli of continuity of order two such that $\frac{\lambda_1(l)}{\lambda_2(l)}$ is positive and non-decreasing, then

$$\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l\lambda_2(l)}dl\geq\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\int_{\frac{1}{d+1}}^{\pi}\frac{1}{l}dl=\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}\log\pi(d+1).$$

i.e.

$$\frac{\lambda_1\left(\frac{1}{d+1}\right)}{\lambda_2\left(\frac{1}{d+1}\right)}=O\left[\frac{1}{\log\pi(d+1)}\int_{\frac{1}{d+1}}^{\pi}\frac{\lambda_1(l)}{l\lambda_2(l)}dl\right]. \tag{4.18}$$

From (4.17) and (4.18), we get

$$\begin{aligned}\| \tilde{\rho}_d(\cdot) \|_r^{(\lambda_2)} &= O\left[\frac{\log(d+1)}{\log \pi(d+1)} \int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)} dl\right] + O\left[\int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)} dl\right] \\ E_d(\tilde{g}) &= O\left[\left(\frac{\log(d+1)}{\log \pi(d+1)} + 1\right) \int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)} dl\right] \\ &= O\left[\frac{\log \pi(d+1)^2}{\log \pi(d+1)} \int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)} dl\right].\end{aligned}$$

This completes the proof of the Theorem 2.2.

5. Corollaries

Corollary 5.1. *Error estimates of the function g (2π-periodic) in the class $Z_{\alpha,r}, r \geq 1$, using $\wedge T$ means of Fourier Series is given by*

$$\underset{M_d^{\wedge T}}{inf} \|M_d^{\wedge T}(x) - g(x)\|_r^{(\lambda_2)} = \begin{cases} O\left\{\frac{\log \pi e(d+1)}{\log \pi(d+1)}(d+1)^{\delta_2 - \delta_1}\right\}, 0 \leq \delta_2 < \delta_1 < 1 \\ O\left\{\frac{\log \pi e(d+1)}{(d+1)\log \pi(d+1)}\right\}, \delta_2 = 0, \delta_1 = 1. \end{cases}$$

Proof. Putting $\lambda_1(l) = l^{\delta_1}$ and $\lambda_2(l) = l^{\delta_2}$ in Theorems 2.1, the result follows. □

Corollary 5.2. *If $a_{mn} = \frac{1}{m-n+1}\log(m+1)$ in Theorem 2.1, then error estimates of the function g (2π-periodic) in generalized Zygmund class $Z_r^{(\lambda_1)}, r \geq 1$, using $\wedge\left(H, \frac{1}{m+1}\right)$ or $\wedge H$ means of Fourier Series is given by*

$$\underset{M_d^{\wedge H}}{inf} \|M_d^{\wedge H}(x) - g(x)\|_r^{(\lambda_2)} = O\left[\frac{\log \pi e(d+1)}{\log \pi(d+1)} \int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)} dl\right],$$

where $\lambda_1(l)$ and $\lambda_2(l)$ are moduli of continuity of order two such that $\frac{\lambda_1(l)}{\lambda_2(l)}$ is positive and non-decreasing.

Remark 5.3. *Other corollaries for obtaining error estimates of the function g in the classes $Z_r^{(\lambda_1)}, r \geq 1$, and $Z_{\alpha,r}, r \geq 1$, can be deduced for the particular cases of $\wedge T$ defined in Remark 1.2 ((ii) to (ix)), Remark 1.3 ((i) to (ix)) and Remark 1.5.*

Corollary 5.4. *Error estimates of the function $\tilde{g}$ (2π-periodic) in the class $Z_{\alpha,r}, r \geq 1$, using $\wedge T$ means of conjugate Fourier Series is given by*

$$\underset{\tilde{M}_d^{\wedge T}}{inf} \|M_d^{\wedge T}(x) - \tilde{g}(x)\|_r^{(\lambda_2)} = \begin{cases} O\left\{\frac{\log \pi(d+1)^2}{\log \pi(d+1)}(d+1)^{\delta_2 - \delta_1}\right\}, 0 \leq \delta_2 < \delta_1 < 1 \\ O\left\{\frac{\log \pi(d+1)^2}{(d+1)\log \pi(d+1)}\right\}, \delta_2 = 0, \delta_1 = 1. \end{cases}$$

Proof. Putting $\lambda_1(l) = l^{\delta_1}$ and $\lambda_2(l) = l^{\delta_2}$ in Theorems 2.2, the result follows. □

Corollary 5.5. *If $a_{mn} = \frac{1}{m-n+1}\log(m+1)$ in Theorem 2.2, then error estimates of the function g (2π-periodic) in generalized Zygmund class $Z_r^{(\lambda_1)}, r \geq 1$, using $\wedge\left(H, \frac{1}{m+1}\right)$ or $\wedge H$ means of conjugate Fourier Series is given by*

$$\underset{\tilde{M}_d^{\wedge H}}{inf} \|\tilde{M}_d^{\wedge H}(x) - \tilde{g}(x)\|_r^{(\lambda_2)} = O\left[\frac{\log \pi(d+1)^2}{\log \pi(d+1)} \int_{\frac{1}{d+1}}^{\pi} \frac{\lambda_1(l)}{l\lambda_2(l)} dl\right],$$

where $\lambda_1(l)$ and $\lambda_2(l)$ are moduli of continuity of order two such that $\frac{\lambda_1(l)}{\lambda_2(l)}$ is positive and non-decreasing.

Remark 5.6. *Other corollaries for obtaining error estimates of the function $\tilde{g}$ in the classes $Z_r^{(\lambda_1)}, r \geq 1$, and $Z_{\alpha,r}, r \geq 1$, can be deduced for the particular cases of $\wedge T$ defined in Remark 1.2 ((ii) to (ix)), Remark 1.3 ((i) to (ix)) and Remark 1.5.*

Remark 5.7. *(i) In our Theorems 2.1 and 2.2, if $r \to \infty$ in $Z_r^{(\lambda_1)}$ class then $Z_r^{(\lambda_1)}$ class reduces to $Z^{(\lambda_1)}$ class. Also putting $\lambda_1(l) = l^\alpha$ and $\lambda_2(l) = l^\beta$ in our Theorem 2.1, $Z^{(\lambda_1)}$ class reduces to Z_α class then by putting $\beta = 0$ in Z_α class, Z_α class reduces to Lip α class.*

6. Particular Cases

(i) Using Remark 5.3(i) and $c_{m,d} = \frac{1}{(m+1)}$ then in view of Remark 1.2 (case (ix)) for $q = 1$, Theorem 1 of [12] becomes a particular case of our main Theorem 2.1.

(ii) Using Remark 5.3(i) and $c_{m,d} = \frac{1}{(m+1)}$ then in view of Remark 1.3 (case (viii)) for $q = 1$, the result of [18] becomes a particular case of our main Theorem 2.1.

(iii)) Using Remark 5.3(i) and in view of Remark 1.2 (case (viii)) for $m = 1$, the result of [13] becomes a particular case of our main Theorem 2.1.

(iv) Using Remark 5.3(i) and in view of Remark 1.2 (case (vii)) for $m = 1$, Theorem 1 of [21] becomes a particular case of our main Theorem 2.1.

(v) Using Remark 5.3(i) and $c_{m,d} = \frac{1}{(q+1)^m} q^{(m-d)}$ then in view of Remark 1.2 (case (vii)) for $m = 2$, the result of [1] becomes a particular case of our main Theorem 2.1.

(vi) Using Remark 5.3(i) and $c_{m,d} = p_{m-d}/P_m$, where $P_m = \sum_{d=0}^{m} p_m$ then in view of Remark 1.2 (case (vii)) for $m = 1$, Theorem 1 of [8] becomes a particular case of our main Theorem 2.1.

(vii)) Using Remark 5.3(i) and in view of Remark 1.2 (case (viii)) for $m = 1$, the result of [14] becomes a particular case of our main Theorem 2.2.

(viii)) Using Remark 5.3(i) and in view of Remark 1.2 (case (viii)) for $m = 1$ and $q = 1$, the result of [15] becomes a particular case of our main Theorem 2.2.

Acknowledgments

The author is thankful to the Council of Scientific and Industrial Research, Government of India for support through the project 25/(0225)/13/EMR-II.

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