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Lie Right Ideals and Homoderivations in 3-Prime Near-Ring

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ABSTRACT: In this article, we study the structure of near-rings involving homoderivations satisfying certain constraints on nonzero Lie right ideals of near-rings.

Key Words: Near-rings, homoderivations, commutativity, Lie ideals.

Contents

1. Introduction

Throughout this paper, $\mathcal{N}$ will be a left near-ring with multiplicative center $Z(\mathcal{N})$, and usually $\mathcal{N}$ will be 3-prime if for $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. A near-ring $\mathcal{N}$ is called zero-symmetric if $0x = 0$, for all $x \in \mathcal{N}$ (recall that left distributivity yields $x0 = 0$). $\mathcal{N}$ is said to be 2-torsion free if whenever $2x = 0$, with $x \in \mathcal{N}$, then $x = 0$. As usual for all $x, y \in \mathcal{N}$, the symbol $[x, y]$ stands for the Lie product (commutator) $xy - yx$ and $x \circ y$ stands for the Jordan product (anticommutator) $xy + yx$. Note that for a near-ring, $-(x + y) = -y - x$.

Some recent results on rings deal with the commutativity of prime and semiprime rings admitting suitably-constrained homoderivations. In [10] El Sofy (2000) defined a homoderivation on prime ring R to be an additive mapping h from R into itself such that $h(xy) = h(x)h(y) + h(x)y + xh(y)$ for all $x, y \in R$. An example of such mapping is to let $h(x) = f(x) - x$ for all $x \in R$ where f is an endomorphism on R. For $S \subseteq \mathcal{N}$, a mapping $f : \mathcal{N} \longrightarrow \mathcal{N}$ is called zero-power valued on S if for every $x \in S$, there exists a positive integer $k(x) > 1$ such that $f^{k(x)}(x) = 0$. A mapping $f : \mathcal{N} \longrightarrow \mathcal{N}$ preserves S if $f(S) \subseteq S$.

Several authors have studied commutativity in prime rings admitting homoderivations satisfying appropriate algebraic conditions on suitable subsets of the rings (see [10] and [1] for references). In [13], S. Mouhssine and A. Boua have studied the 3-prime near rings admitting homoderivations satisfying suitable identities on semigroup ideals of near rings.

The present paper is motivated by the previous results and here we continue this line of investigation to study the structure of near-rings involving homoderivations satisfying certain identities on Lie right ideals of near-rings.

2. Some preliminaries

Lemma 2.1. [2, Lemmas 1.2 (i), 1.2 (iii), and 1.3 (iii)] Let $\mathcal{N}$ be a 3-prime near-ring.

(i) If $z \in Z(\mathcal{N}) \setminus \{0\}$, then z is not a zero divisor.

(ii) If $Z(\mathcal{N})$ contains a nonzero element z for which $z + z \in Z(\mathcal{N})$, then $\mathcal{N}$ is abelian.

(iii) If $z \in Z(\mathcal{N}) \setminus \{0\}$ and $x \in \mathcal{N}$ such that $xz \in Z(\mathcal{N})$ or $zx \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.

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Lemma 2.2. [4, Lemma 2.4 (ii)] Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a homoderivation h such that $h^2(\mathcal{N}) = \{0\}$, then $h = 0$.

Lemma 2.3. [4, Lemma 2.5] Let $\mathcal{N}$ be a near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h, then $h(xy)(h(a)+a) = h(x)h(y)(h(a)+a) + h(x)y(h(a)+a) + xh(y)(h(a)+a)$ for all $x, y, a \in \mathcal{N}$.

Lemma 2.4. [13, Lemma 3.2] Let $\mathcal{N}$ be a 2-torsion free near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$, then $\mathcal{N}$ is zero-symmetric near-ring.

Definition 2.5. *Let $\mathcal{N}$ be a near-ring. An additive subgroup I of $\mathcal{N}$ is said to be a Lie right ideal (resp. left ideal) of $\mathcal{N}$ if $[n, i] \in I$ (resp. $[i, n] \in I$) for all $i \in I$ and $n \in \mathcal{N}$.*

Remark 2.6. *An additive subgroup I of a near-ring $\mathcal{N}$ is said to be a Lie ideal of $\mathcal{N}$ if $[i, n] \in I$ and $[n, i] \in I$ for all $i \in I$ and $n \in \mathcal{N}$.*

Example 2.7. *Let F be a left near-field which is not a division ring, $\mathcal{N} = F \times F^*$ and $\mathcal{I} = F^* \times \{0\}$, where $F^* = F \setminus \{0\}$. Define the addition and the multiplication on $\mathcal{N}$ by $(a, b) + (c, d) = (ac, b + d)$ and $(a, b)(c, d) = (1, bd)$ for all $(a, b), (c, d) \in \mathcal{N}$. Then $(\mathcal{N}, +, .)$ is a zero-symmetric near-ring which is not abelian and $\mathcal{I}$ is a nonzero Lie ideal of $\mathcal{N}$.*

Lemma 2.8. *Let $\mathcal{N}$ be a 3-prime near-ring and $\mathcal{I}$ a nonzero Lie right ideal of $\mathcal{N}$.*

(i) *If $x\mathcal{I} = \{0\}$ for $x \in N$, then $x = 0$.*

(ii) *If $\mathcal{I} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is an abelian near-ring.*

(iii) *If $-\mathcal{I} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is an abelian near-ring.*

(iv) *If $\mathcal{I}^2 \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is an abelian near-ring.*

Proof. The proofs of (i) and (ii) are similar to (i) and (iv) respectively in the paper [3], just replace the right near-ring with the left near-ring.
(iii) Assume that $-\mathcal{I} \subseteq Z(\mathcal{N})$. Then, for all $x, y \in \mathcal{N}$ and $u \in \mathcal{I}$, we have

$$\begin{aligned}
(x+y)((-u)+(-u)) &= ((-u)+(-u))(x+y), \\
(x+y)(-u)+(x+y)(-u) &= ((-u)+(-u))x+((-u)+(-u))y, \\
(-u)(x+y)+(-u)(x+y) &= x((-u)+(-u))+y((-u)+(-u)), \\
(-u)x+(-u)y+(-u)x+(-u)y &= x(-u)+x(-u)+y(-u)+y(-u), \\
(-u)x+(-u)y+(-u)x+(-u)y &= (-u)x+(-u)x+(-u)y+(-u)y.
\end{aligned}$$

This implies that $(-\mathcal{I})(x + y - x - y) = \{0\}$. Since $-\mathcal{I} \subseteq Z(\mathcal{N})$, by part (i), we get $x + y = y + x$ for all $x, y \in \mathcal{N}$, which completes the proof.
(iv) Assume that $\mathcal{I}^2 \subseteq Z(\mathcal{N})$. Then

$$ij \in Z(\mathcal{N}) \text{ for all } i, j \in \mathcal{I}. \tag{2.1}$$

It follows that

$$i[n, j] \in Z(\mathcal{N}) \text{ for all } i, j \in \mathcal{I}, n \in \mathcal{N}. \tag{2.2}$$

Substituting jn for n in (2.2) and using $[jn, j] = j[n, j]$, we get

$$ij[n, j] \in Z(\mathcal{N}) \text{ for all } i, j \in \mathcal{I}, n \in \mathcal{N}. \tag{2.3}$$

By Lemma 2.1 (iii), we arrive at

$$ij = 0 \text{ or } [n, j] \in Z(\mathcal{N}) \text{ for all } i, j \in \mathcal{I}, n \in \mathcal{N}. \tag{2.4}$$

Which implies that

$$ij = 0 \text{ or } j \in Z(\mathcal{N}) \text{ for all } i, j \in \mathcal{I}. \tag{2.5}$$

Suppose there is an element $j_0 \in \mathcal{I}$ such that $ij_0 = 0$ for all $i \in \mathcal{I}$, then $j_0^2 = 0$ and by hypothesis, we have $j_0[n, j_0]m = mj_0[n, j_0]$ for all $m, n \in \mathcal{N}$, which gives $j_0nj_0m = mj_0nj_0$ for all $m, n \in \mathcal{N}$, so that $0 = ij_0nj_0m = imj_0nj_0$ for all $i \in \mathcal{I}, m, n \in \mathcal{N}$, this forces that $i\mathcal{N}j_0\mathcal{N}j_0 = \{0\}$ for all $i \in \mathcal{I}$. By the 3-primeness of $\mathcal{N}$, we get $j_0 = 0$ or $\mathcal{I} = \{0\}$. Since $\mathcal{I} \neq \{0\}$, (2.5) forces that $\mathcal{I} \subseteq Z(\mathcal{N})$, so $\mathcal{N}$ is an abelian near-ring by (ii). □

Example 2.9. *Let $\mathcal{N} = \{0, a, b, c, u, v\}$ and $\mathcal{I} = \{0, a\}$ with the addition and the multiplication tables defined as follows:*

+	0	a	b	c	u	v
0	0	a	b	c	u	v
a	a	0	v	u	c	b
b	b	u	0	v	a	c
c	c	v	u	0	b	a
u	u	b	c	a	v	0
v	v	c	a	b	0	u

∗	0	a	b	c	u	v
0	0	0	0	0	0	0
a	0	a	a	a	0	0
b	0	a	a	a	0	0
c	0	a	a	a	0	0
u	0	0	0	0	0	0
v	0	0	0	0	0	0

Then $(\mathcal{N}, +, .)$ is a commutative zero-symmetric near-ring which is not abelian, and $\mathcal{I}$ is a nonzero Lie ideal of $\mathcal{N}$ such that $\mathcal{I} \subset Z(\mathcal{N})$.

Example 2.10. *Let $\mathcal{N} = R \times R$, where R is a noncommutative integral ring (has no nonzero zero divisors) with identity that has at least three elements. Define the addition and the multiplication on $\mathcal{N}$ by $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac, bc + d)$ if $(c, d) \neq (0, 0)$ and $(a, b)(0, 0) = (0, 0)$. Then $\mathcal{N}$ is a 3-prime zero-symmetric abelian left near-ring with identity $(1, 0)$ which is not a ring. Let $\mathcal{I} = \{(0, r) \mid r \in R\}$. We can easily show that $\mathcal{I}$ is a nonzero Lie ideal of $\mathcal{N}$, and $\mathcal{I}$ is not contained in $Z(\mathcal{N}) = \{(0, 0), (0, 1)\}$. Furthermore, there is no nonzero Lie ideal of $\mathcal{N}$ contained in $Z(\mathcal{N})$.*

Lemma 2.11. *Let $\mathcal{N}$ be a 3-prime near-ring and f be a nonzero additive map on $\mathcal{N}$ which is zero-power valued on $\mathcal{N}$ such that f preserves $\mathcal{I}$. Then $\mathcal{N}$ is abelian, if f has one of the following properties:*

(i) $f(i) + i \in Z(\mathcal{N})$ *for all* $i \in \mathcal{I}$.

(ii) $f(i) - i \in Z(\mathcal{N})$ *for all* $i \in \mathcal{I}$.

(iii) $-i + f(-i) \in Z(\mathcal{N})$ *for all* $i \in \mathcal{I}$.

(iv) $f(-i) - i \in Z(\mathcal{N})$ *for all* $i \in \mathcal{I}$.

Proof. (i) Suppose that

$$f(i) + i \in Z(\mathcal{N}) \text{ for all } i \in \mathcal{I}. \tag{2.6}$$

If $f(i) \neq 0$ for all $i \in \mathcal{I} \setminus \{0\}$. By recurrence we have $f^n(i) \neq 0$ for all $i \in \mathcal{I} \setminus \{0\}$ and $n \in \mathbb{N}^*$. Since f is zero-power valued on $\mathcal{N}$, for every $i \in \mathcal{I}$, there exists a positive integer $k(i) > 1$ such that $f^{k(i)}(i) = 0$, it follows that for $z = f^{k(i)-1}(i) \neq 0$, $f(z) = f^{k(i)}(i) = 0$ which is a contradiction. So, there exists $j \in \mathcal{I} \setminus \{0\}$ such that $f(j) = 0$, so we get $j = f(j) + j \in Z(\mathcal{N}) \setminus \{0\}$ and $j + j = f(j + j) + j + j \in Z(\mathcal{N})$ which forces that $\mathcal{N}$ is abelian.
Using similar arguments as used in the proof of (i), we can easily prove (ii), (iii), and (vi). □

3. Some results for homoderivations and Lie ideals in 3-prime near-rings

Theorem 3.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime zero-symmetric near-ring and $\mathcal{I}$ be a nonzero Lie right ideal. If $\mathcal{N}$ admits a nonzero homoderivation h such that $h(\mathcal{I}) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is abelian.*

Proof. If $\{0\} \neq h(\mathcal{I}) \subseteq Z(\mathcal{N})$, then there exists $i \in \mathcal{I}$ such that $h(i) \in Z(\mathcal{N}) \setminus \{0\}$. Since $h(i) + h(i) = h(i + i) \in Z(\mathcal{N})$, $(\mathcal{N}, +)$ is abelian by Lemma 1 (ii) and the 2-torsion freeness of $\mathcal{N}$. Now suppose that $h(i) = 0$ for all $i \in \mathcal{I}$. Replacing i by $[n, i]$, where $n \in \mathcal{N}$, and using the last equation, we get

$$h(n)i = ih(n) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.1}$$

Substituting $h(n)t$ for n in (3.1) and applying (3.1), we give
$$h^2(n)ti = ih^2(n)t \quad \text{for all} \quad i \in \mathcal{I}, n, t \in \mathcal{N}. \tag{3.2}$$
Putting tm instead of t in (3.2) and using (3.2), we have
$$\begin{aligned} h^2(n)tmi &= ih^2(n)tm \\ &= h^2(n)tim \quad \text{for all} \quad i \in \mathcal{I}, m, n, t \in \mathcal{N}. \end{aligned}$$
The last expression implies that $h^2(n)\mathcal{N}[i, m] = \{0\}$ for all $i \in \mathcal{I}, n, m \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we arrive at
$$h^2(\mathcal{N}) = \{0\} \text{ or } \mathcal{I} \subseteq Z(\mathcal{N}).$$
If $h^2(\mathcal{N}) = \{0\}$, then we get $h = 0$ by Lemma 2.2; a contradiction. So, $\mathcal{I} \subseteq Z(\mathcal{N})$, and $\mathcal{N}$ is abelian by Lemma 2.8 (ii). □

Theorem 3.2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime zero-symmetric near-ring and $\mathcal{I}$ be a nonzero Lie right ideal. If $\mathcal{N}$ admits a nonzero homoderivation h such that $h([n, i]) = 0$ for all $i \in \mathcal{I}, n \in \mathcal{N}$, then $\mathcal{N}$ is abelian.*

Proof. Suppose that
$$h([n, i]) = 0 \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.3}$$
Replacing n by in in (3.3), and applying the definition of h, we get
$$\begin{aligned} 0 &= h(i[n, i]) \\ &= h(i)h([n, i]) + h(i)[n, i] + ih([n, i]) \\ &= h(i)[n, i] \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \end{aligned}$$
Thus
$$h(i)ni = h(i)in \text{ for all } i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.4}$$
Taking nm instead of n in (3.4), we find
$$\begin{aligned} h(i)nmi &= h(i)inm \\ &= h(i)nim \text{ for all } i \in \mathcal{I}, n, m \in \mathcal{N}, \end{aligned}$$
which implies that
$$h(i)\mathcal{N}[m, i] = \{0\} \text{ for all } i \in \mathcal{I}, m \in \mathcal{N}. \tag{3.5}$$
By the 3-primeness of $\mathcal{N}$, we have
$$h(i) = 0 \quad \text{or} \quad i \in Z(\mathcal{N}) \text{ for all} \quad i \in \mathcal{I}. \tag{3.6}$$
Suppose there is an element $i_0 \in \mathcal{I}$ such that $h(i_0) = 0$. Thus by (3.3), and applying the definition of h we get
$$h(n)i_0 = i_0h(n) \quad \text{for all} \quad n \in \mathcal{N}. \tag{3.7}$$
Substituting $h(n)t$ for n in (3.7) and applying (3.7), we get
$$h^2(n)ti_0 = i_0h^2(n)t \quad \text{for all} \quad n, t \in \mathcal{N}. \tag{3.8}$$
Putting tm in place of t in (3.8) and using (3.7), we have
$$\begin{aligned} h^2(n)tmi_0 &= i_0h^2(n)tm \\ &= h^2(n)ti_0m \quad \text{for all} \quad m, n, t \in \mathcal{N}. \end{aligned}$$
The last expression implies that $h^2(n)\mathcal{N}[m, i_0] = \{0\}$ for all $m, n \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we have $h^2(\mathcal{N}) = \{0\}$ or $i_0 \in Z(\mathcal{N})$. If $h^2(\mathcal{N}) = \{0\}$, then we get $h = 0$ by Lemma 2.2; a contradiction. So, we must have $i_0 \in Z(\mathcal{N})$. Thus (3.6) becomes $\mathcal{I} \subseteq Z(\mathcal{N})$, and by Lemma 2.8 (ii), we conclude that $\mathcal{N}$ is abelian. □

Theorem 3.3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime zero-symmetric near-ring and $\mathcal{I}$ be a nonzero Lie right ideal. If $\mathcal{N}$ admits a nonzero homoderivation $\mathcal{I}$ which is invariant by h, then $\mathcal{N}$ is abelian.*

Proof. Since $\mathcal{I}$ is invariant by h, we have

$$h([n,i]) = [n,i] \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.9}$$

Replacing n by in in (3.9), and applying the definition of h, we get

$$\begin{aligned} i[n,i] &= h(i[n,i]) \\ &= h(i)h([n,i]) + h(i)[n,i] + ih([n,i]) \\ &= 3h(i)[n,i] \\ &= 3i[n,i] \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \end{aligned}$$

Thus

$$2i[n,i] = 0 \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}.$$

Using the 2-torsion freeness of $\mathcal{N}$, we obtain

$$i[n,i] = 0 \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}.$$

So

$$ini = i^2n \text{ for all } i \in \mathcal{I}, n \in \mathcal{N}.$$

Substituting nm for n in the last equation, we get

$$inmi = i^2nm = inim \text{ for all } i \in \mathcal{I}, n, m \in \mathcal{N}.$$

Thus

$$i\mathcal{N}[m,i] = \{0\} \text{ for all } i \in \mathcal{I}, m \in \mathcal{N}.$$

The 3-primeness of $\mathcal{N}$ implies that $\mathcal{I} \subseteq Z(\mathcal{N})$, and so $\mathcal{N}$ is abelian by Lemma 2.8 (ii). □

Theorem 3.4. *Let $\mathcal{N}$ be a 2-torsion free 3-prime zero-symmetric near-ring and $\mathcal{I}$ be a nonzero Lie right ideal. If $\mathcal{N}$ admits a nonzero homoderivation h such that $h(n \circ i) = 0$ for all $i \in \mathcal{I}, n \in \mathcal{N}$, then $\mathcal{N}$ is abelian.*

Proof. Suppose that

$$h(n \circ i) = 0 \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.10}$$

Replacing n by in in (3.10), and applying the definition of h we get

$$\begin{aligned} 0 &= h(in \circ i) \\ &= h(i(n \circ i)) \\ &= h(i)h(n \circ i) + h(i)(n \circ i) + ih(n \circ i) \\ &= h(i)(n \circ i) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \end{aligned}$$

Which implies that

$$h(i)ni = -h(i)in \text{ for all } i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.11}$$

Replacing n by nm in (3.11), we find

$$\begin{aligned} h(i)nmi &= -h(i)inm \\ &= -(-h(i)ni)m \\ &= (-h(i)ni)(-m) \\ &= h(i)n(-i)(-m) \text{ for all } i \in \mathcal{I}, n, m \in \mathcal{N}. \end{aligned}$$

Thus

$$h(i)\mathcal{N}(mi-(-i)(-m))=\{0\} \text{ for all } i\in\mathcal{I}, m\in\mathcal{N}. \tag{3.12}$$

Taking $-i$ instead of i in (3.12) and by the 3-primeness of $\mathcal{N}$, we arrive at

$$h(-i)=0 \quad \text{or} \quad i\in Z(\mathcal{N}) \text{ for all } \quad i\in\mathcal{I}. \tag{3.13}$$

Assume that there exists $i_0\in\mathcal{I}$ such that $i_0\in Z(\mathcal{N})$. Using our hyphotesis, we obtain

$$2h(ni_0)=0 \quad \text{for all} \quad n\in\mathcal{N}.$$

The 2-torsion freeness of $\mathcal{N}$ implies that

$$h(ni_0)=0 \quad \text{for all} \quad n\in\mathcal{N}. \tag{3.14}$$

Replacing n by i_0n in (3.14), we get

$$\begin{aligned} h(i_0ni_0) &= h(i_0)h(ni_0)+h(i_0)ni_0+i_0h(ni_0) \\ &= h(i_0)ni_0 \\ &= 0 \quad \text{for all} \quad n\in\mathcal{N}. \end{aligned}$$

So $h(i_0)\mathcal{N}i_0=\{0\}$ and by the 3-primeness of $\mathcal{N}$, we arrive at $h(i_0)=0$. Hence, (3.13) becomes $h(\mathcal{I})=\{0\}$, so $\mathcal{N}$ is abelian by Theorem 3.1. □

Theorem 3.5. *Let $\mathcal{N}$ be a 2-torsion free 3-prime zero-symmetric near-ring and $\mathcal{I}$ be a nonzero Lie right ideal. If $\mathcal{N}$ admits a nonzero homoderivation h such that $h(n\circ i)=n\circ i$ for all $i\in\mathcal{I}, n\in\mathcal{N}$, then $\mathcal{N}$ is abelian.*

Proof. Suppose that

$$h(n\circ i)=n\circ i \quad \text{for all} \quad i\in\mathcal{I}, n\in\mathcal{N}. \tag{3.15}$$

Replacing n by in in (3.15), and applying the definition of h we get

$$\begin{aligned} i(n\circ i) &= h(in\circ i) \\ &= h(i(n\circ i)) \\ &= h(i)h(n\circ i)+h(i)(n\circ i)+ih(n\circ i) \\ &= 2h(i)(n\circ i)+i(n\circ i) \quad \text{for all} \quad i\in\mathcal{I}, n\in\mathcal{N}. \end{aligned}$$

Thus

$$2h(i)(n\circ i)=0 \quad \text{for all} \quad i\in\mathcal{I}, n\in\mathcal{N}. \tag{3.16}$$

Using the 2-torsion freeness of $\mathcal{N}$, we find

$$h(i)(n\circ i)=0 \quad \text{for all} \quad i\in\mathcal{I}, n\in\mathcal{N}. \tag{3.17}$$

So

$$h(i)ni=-h(i)in \text{ for all } i\in\mathcal{I}, n\in\mathcal{N}. \tag{3.18}$$

Replacing n by nm in (3.18), we obtain

$$\begin{aligned} h(i)nmi &= -h(i)inm \\ &= -(-h(i)ni)m \\ &= (-h(i)ni)(-m) \\ &= h(i)n(-i)(-m) \text{ for all } i\in\mathcal{I}, n,m\in\mathcal{N}. \end{aligned}$$

Which implies that

$$h(i)\mathcal{N}(mi-(-i)(-m))=\{0\} \text{ for all } i\in\mathcal{I}, m\in\mathcal{N}. \tag{3.19}$$

Taking $-i$ instead of i in (3.19) and using the 3-primeness of $\mathcal{N}$, we arrive at

$$h(-i) = 0 \quad \text{or} \quad i \in Z(\mathcal{N}) \text{ for all } \quad i \in \mathcal{I}. \tag{3.20}$$

Suppose there is an element $i_0 \in \mathcal{I}$ such that $i_0 \in Z(\mathcal{N})$. Taking i_0 instead of i in (3.15), we obtain

$$h(n(2i_0)) = n(2i_0) \text{ for all } \quad n \in \mathcal{N}. \tag{3.21}$$

Placing tn instead of n in (3.21), and using the definition of h, we find

$$\begin{aligned} tn(2i_0) &= h(tn(2i_0))) \\ &= h(t)h(n(2i_0)) + h(t)n(2i_0) + th(n(2i_0)) \\ &= h(t)n(2i_0) + h(t)n(2i_0) + tn(2i_0) \text{ for all } \quad t, n \in \mathcal{N}. \end{aligned}$$

So, $2h(t)n(2i_0) = 0$ for all $t, n \in \mathcal{N}$. By the 2-torsion freeness of $\mathcal{N}$ we arrive at $h(t)ni_0 = 0$ for all $t, n \in \mathcal{N}$. Thus, $h(t)\mathcal{N}i_0 = \{0\}$ for all $t \in \mathcal{N}$. Since $\mathcal{N}$ is 3-prime and $h \neq 0$, we conclude that $i_0 = 0$. So, (3.20) implies $h(\mathcal{I}) = \{0\}$, hence, $\mathcal{N}$ is abelian by Theorem 3.1. □

Theorem 3.6. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{I}$ be a nonzero Lie right ideal. There is no nonzero homoderivation h such that $h([n,i]) + [n,i] = n \circ i$ for all $i \in \mathcal{I}$, $n \in \mathcal{N}$.*

Proof. Suppose that

$$h([n,i]) + [n,i] = n \circ i \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.22}$$

Replacing n by in in (3.22), we find

$$\begin{aligned} i(n \circ i) &= (in \circ i) \\ &= h([in,i]) + [in,i] \\ &= h(i[n,i]) + i[n,i] \\ &= h(i)h([n,i]) + h(i)[n,i] + ih([n,i]) + i[n,i] \\ &= h(i)(h([n,i]) + [n,i]) + i(h([n,i]) + [n,i]) \\ &= h(i)(n \circ i) + i(n \circ i) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \end{aligned}$$

This expression gives us $h(i)(n \circ i) = 0$ for all $i \in \mathcal{I}, n \in \mathcal{N}$, that is

$$h(i)ni = -h(i)in \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.23}$$

Substituting nm in place of n in (3.23), we get

$$\begin{aligned} h(i)nmi &= -h(i)inm \\ &= h(i)in(-m) \\ &= h(i)n(-i)(-m) \quad \text{for all} \quad i \in \mathcal{I}, n, m \in \mathcal{N}, \end{aligned}$$

which can be rewritten as $h(i)\mathcal{N}(-m(-i) + (-i)m) = \{0\}$ for all $i \in \mathcal{I}, m \in \mathcal{N}$. Equivalently,

$$h(-i)\mathcal{N}(-mi + im) = \{0\} \quad \text{for all} \quad i \in \mathcal{I}, m \in \mathcal{N}. \tag{3.24}$$

By the 3-primeness of $\mathcal{N}$, we have

$$h(-i) = 0 \quad \text{or} \quad i \in Z(\mathcal{N}) \quad \text{for all} \quad i \in \mathcal{I}. \tag{3.25}$$

Suppose there is an element $i_0 \in \mathcal{I}$ such that $i_0 \in Z(\mathcal{N})$. Then (3.22) becomes $2i_0 n = 0$, for all $n \in \mathcal{N}$. Using the 2-torsion freeness of $\mathcal{N}$, gives $i_0 n = 0$, for all $n \in \mathcal{N}$. Replacing n by ni_0 in the last expression, we arrive at $i_0\mathcal{N}i_0 = 0$. By the 3-primeness of $\mathcal{N}$, we have $i_0 = 0$. In this case, (3.25) implies that $h(\mathcal{I}) = \{0\}$. So, (3.22) implies

$$[n,i] = n \circ i \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.26}$$

Thus, $2in = 0$ for all $n \in \mathcal{N}, i \in \mathcal{I}$. The 2-torsion freeness of $\mathcal{N}$, implies that $in = 0$ for all $n \in \mathcal{N}, i \in \mathcal{I}$. Replacing n by ni in the last expression, we arrive at $i\mathcal{N}i = \{0\}$. By the 3-primeness of $\mathcal{N}$, we have $\mathcal{I} = \{0\}$; a contradiction. □

Theorem 3.7. *Let $\mathcal{N}$ be a 3-prime zero-symmetric near-ring and $\mathcal{I}$ be a nonzero Lie ideal. If $\mathcal{N}$ admits nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ such that $h(ni)+ni=[n,i]$ for all $i \in \mathcal{I}, n \in \mathcal{N}$, then $\mathcal{N}$ is abelian.*

Proof. Suppose that

$$h(ni)+ni=[n,i] \quad \text{for all} \quad i \in \mathcal{I} \quad \text{and} \quad n \in \mathcal{N}. \tag{3.27}$$

Replacing n by in in (3.27), it follows that

$$\begin{aligned} i[n,i] &= [in,i] \\ &= h(ini)+ini \\ &= h(i(ni))+i(ni) \\ &= h(i)h(ni)+h(i)ni+ih(ni)+i(ni) \\ &= h(i)(h(ni)+ni)+i(h(ni)+ni) \\ &= h(i)[n,i]+i[n,i] \ \text{ for all } \ i \in \mathcal{I} \ \text{ and } \ n \in \mathcal{N}. \end{aligned}$$

This expression gives us $h(i)[n,i]=0$ for all $i \in \mathcal{I}, n \in \mathcal{N}$, which means that

$$h(i)ni=h(i)in \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.28}$$

Substituting nm in place of n in (3.28), we get

$$\begin{aligned} h(i)nmi &= h(i)inm \\ &= h(i)nim \quad \text{for all} \quad i \in \mathcal{I}, n, m \in \mathcal{N}. \end{aligned}$$

Which implies that $h(i)\mathcal{N}[m,i]=0$ for all $i \in \mathcal{I}, m \in \mathcal{N}$. By the 3-primeness of $\mathcal{N}$, we have

$$h(i)=0 \quad \text{or} \quad i \in Z(\mathcal{N}) \text{ for all } \quad i \in \mathcal{I} \tag{3.29}$$

Suppose there is an element $i_0 \in \mathcal{I}$ such that $h(i_0)=0$. From (3.27) we have

$$h(ni_0)+ni_0=[n,i_0] \quad \text{for all} \quad n \in \mathcal{N}. \tag{3.30}$$

Replacing n by i_0 in (3.30), we arrive at $(i_0)^2=0$. Substituting ni_0 in place of n in (3.30), we get

$$\begin{aligned} -i_0ni_0 &= n(i_0)^2-i_0ni_0 \\ &= [ni_0,i_0] \\ &= h(n(i_0)^2)+n(i_0)^2 \\ &= 0 \quad \text{for all} \quad n \in \mathcal{N}. \end{aligned}$$

That is $i_0\mathcal{N}i_0=0$. The 3-primeness of $\mathcal{N}$, gives $i_0=0$. In this case, (3.29) implies that $\mathcal{I} \subseteq Z(\mathcal{N})$, so $\mathcal{N}$ is abelian by Lemma 2.8 (ii). □

Theorem 3.8. *Let $\mathcal{N}$ be a 2-torsion free 3-prime zero-symmetric near-ring and $\mathcal{I}$ be a nonzero Lie right ideal of $\mathcal{N}$. If $\mathcal{N}$ admits nonzero homoderivation h such that $h(ni)+ni=n\circ i$ for all $i \in \mathcal{I}, n \in \mathcal{N}$, then $\mathcal{N}$ is abelian.*

Proof. Suppose that

$$h(ni)+ni=n\circ i \quad \text{for all} \quad i \in \mathcal{I} \quad \text{and} \quad n \in \mathcal{N}. \tag{3.31}$$

Replacing n by in in (3.31), it follows that

$$\begin{aligned} i(n\circ i) &= (in\circ i) \\ &= h(ini)+ini \\ &= h(i(ni))+i(ni) \\ &= h(i)h(ni)+h(i)ni+ih(ni)+i(ni) \\ &= h(i)(h(ni)+ni)+i(h(ni)+ni) \\ &= h(i)(n\circ i)+i(n\circ i) \ \text{ for all } \ i \in \mathcal{I}, n \in \mathcal{N} \end{aligned}$$

This expression gives us $h(i)(n \circ i) = 0$ for all $i \in \mathcal{I}, n \in \mathcal{N}$. That is,

$$h(i)ni = -h(i)in \quad \text{for all} \quad i \in \mathcal{I} \quad \text{and} \quad n \in \mathcal{N}. \tag{3.32}$$

Substituting nm in place of n in (3.32), we get

$$\begin{aligned} h(i)nmi &= -h(i)inm \\ &= h(i)in(-m) \\ &= h(i)n(-i)(-m) \quad \text{for all} \quad i \in \mathcal{I}, m, n \in \mathcal{N}, \end{aligned}$$

which can be rewritten as $h(i)\mathcal{N}(-m(-i) + (-i)m) = \{0\}$ for all $i \in \mathcal{I}$, $m \in \mathcal{N}$. Equivalently,

$$h(-i)\mathcal{N}(-mi + im) = \{0\} \quad \text{for all} \quad i \in \mathcal{I} \quad \text{and} \quad m \in \mathcal{N}. \tag{3.33}$$

By the 3-primeness of $\mathcal{N}$, we have

$$h(-i) = 0 \quad \text{or} \quad i \in Z(\mathcal{N}) \text{ for all} \quad i \in \mathcal{I}. \tag{3.34}$$

Suppose there is an element $i_0 \in \mathcal{I}$ such that $h(i_0) = 0$. Equation (3.31) gives

$$h(ni_0) + ni_0 = n \circ i_0 \quad \text{for all} \quad n \in \mathcal{N}. \tag{3.35}$$

Taking i_0 instead of n in (3.35), we arrive at $(i_0)^2 = 0$. Substituting ni_0 in place of n in (3.31), we find

$$\begin{aligned} i_0ni_0 &= n(i_0)^2 + i_0ni_0 \\ &= ni_0 \circ i_0 \\ &= h(n(i_0)^2) + n(i_0)^2 \\ &= 0 \quad \text{for all} \quad n \in \mathcal{N} \end{aligned}$$

That is, $i_0\mathcal{N}i_0 = 0$. By the 3-primeness of $\mathcal{N}$, we have $i_0 = 0$. In virtue of equation (3.34), we have $\mathcal{I} \subseteq Z(\mathcal{N})$, and $\mathcal{N}$ is abelian by Lemma 2.8 (ii). □

Theorem 3.9. *Let $\mathcal{N}$ be a 3-prime near-ring, $\mathcal{I}$ a nonzero Lie right ideal of $\mathcal{N}$ and h be a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ and preserves $\mathcal{I}$. Then $\mathcal{N}$ is abelian if h has one the following properties:*

(i) $[h(i) + i, n] \in Z(\mathcal{N})$ *for all* $i \in \mathcal{I}, n \in \mathcal{N}$.

(ii) $h\,(in) + in \in Z(\mathcal{N})$ *for all* $i \in \mathcal{I}, n \in \mathcal{N}$.

Proof. (i) Suppose that

$$[h(i) + i, n] \in Z(\mathcal{N}) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.36}$$

Replacing n by $(h(i) + i)n$ in (3.36), we get

$$(h(i) + i)[h(i) + i, n] \in Z(\mathcal{N}) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.37}$$

By Lemma 2.1(iii), we obtain

$$h(i) + i \in Z(\mathcal{N}) \text{ or } [h(i) + i, n] = 0 \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.38}$$

Both cases force that $h(i) + i \in Z(\mathcal{N})$ for all $i \in \mathcal{I}$. By Lemma 2.11 we conclude that $\mathcal{N}$ is abelian.
(ii) Now assume that

$$h\,(in) + in \in Z(\mathcal{N}) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.39}$$

Replacing n by jn in (3.39), we get

$$\begin{aligned} h\,(ijn) + ijn &= h(i)h(jn) + h(i)jn + ih(jn) + ijn \\ &= h(i)(h(jn) + jn) + i(h(jn) + jn) \\ &= (h(jn) + jn)(h(i) + i) \in Z(\mathcal{N}) \quad \text{for all} \quad i, j \in \mathcal{I}, n \in \mathcal{N}. \end{aligned}$$

Using Lemma 2.1 (iii) implies that

$$h(jn) + jn = 0 \text{ for all } j \in \mathcal{I}, n \in \mathcal{N} \text{ or } h(i) + i \in Z(\mathcal{N}) \text{ for all } i \in \mathcal{I}.$$

If $h(jn) + jn = 0$ for all $j \in \mathcal{I}, n \in \mathcal{N}$, then by recurrence we have $h^k(jn) + (-1)^{k+1}jn = 0$ for all $j \in \mathcal{I}, n \in \mathcal{N}, k \in \mathbb{N}^*$. Since h is zero-power valued on $\mathcal{N}$, there exists an integer $k(jn) > 1$ such that $h^{k(jn)}(jn) = 0$. Replacing k by $k(jn)$ in the above expression we get $jn = 0$ for all $j \in \mathcal{I}, n \in \mathcal{N}$. Substituting nj for n in the last equation, we get $j\mathcal{N}j = \{0\}$ for all $j \in \mathcal{I}$. Thus, by the 3-primeness of $\mathcal{N}$ we get $\mathcal{I} = \{0\}$; a contradiction. Hence, $h(i) + i \in Z(\mathcal{N})$ for all $i \in \mathcal{I}$, and by Lemma 2.11 we conclude that $\mathcal{N}$ is abelian. □

Theorem 3.10. *Let $\mathcal{N}$ be a 3-prime near-ring and $\mathcal{I}$ be a nonzero Lie right ideal of $\mathcal{N}$. If $\mathcal{N}$ admits nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ and preserves $\mathcal{I}$ such that $(h(i)+i) \circ n \in Z(\mathcal{N})$ for all $i \in \mathcal{I}$ and $n \in \mathcal{N}$, then $\mathcal{N}$ is abelian.*

Proof. Assume that

$$(h(i) + i) \circ n \in Z(\mathcal{N}) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.40}$$

Replacing n by $(h(i) + i)n$ in (3.40), we get

$$(h(i) + i)((h(i) + i) \circ n) \in Z(\mathcal{N}) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.41}$$

By Lemma 2.1, it follows that

$$(h(i) + i) \circ n = 0 \text{ or } h(i) + i \in Z(\mathcal{N}) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.42}$$

If there exists $i_0 \in \mathcal{I}$ such that $h(i_0) + i_0 \in Z(\mathcal{N}) \setminus \{0\}$, by Lemma 2.1 (iii), we can conclude that

$$n + n \in Z(\mathcal{N}) \text{ for all } \quad n \in \mathcal{N}. \tag{3.43}$$

Therefore, $\mathcal{N}$ is abelian by Lemma 2.1 (ii). In view of (3.42), we can now assume that

$$(h(i) + i) \circ n = 0 \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}.$$

i.e.

$$n(h(i) + i) = -(h(i) + i)n \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}.$$

Replacing n by nt, where $t \in \mathcal{N}$ in the last equation, we obtain

$$\begin{aligned} nt(h(i) + i) &= -((h(i) + i))nt \\ &= ((h(i) + i))n(-t) \\ &= (-n((h(i) + i)))(-t) \\ &= n(-(h(i) + i))(-t) \quad \text{for all} \quad i \in \mathcal{I}, t, n \in \mathcal{N}, \end{aligned}$$

which leads to

$$n(t(h(i) + i) - (-(h(i) + i))(-t)) = 0 \quad \text{for all} \quad i \in \mathcal{I}, t, n \in \mathcal{N}, \tag{3.44}$$

i.e.

$$\mathcal{N}(-t(-(h(i) + i)) + (-(h(i) + i))t) = \{0\} \quad \text{for all} \quad i \in \mathcal{I}, t \in \mathcal{N}. \tag{3.45}$$

From the 3-primeness of $\mathcal{N}$, we conclude that $-i + h(-i) \in Z(\mathcal{N})$ for all $i \in \mathcal{I}$. By to Lemma 2.11, it follows that $\mathcal{N}$ is abelian. □

Theorem 3.11. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{I}$ be a nonzero Lie right ideal of $\mathcal{N}$. If $\mathcal{N}$ admits nonzero homoderivation h which is zero-power valued on $\mathcal{N}$ and preserves $\mathcal{I}$ such that $h(i \circ n) + i \circ n \in Z(\mathcal{N})$ for all $i \in \mathcal{I}, n \in \mathcal{N}$, then $\mathcal{N}$ is abelian.*

Proof. Suppose that

$$h\,(i \circ n) + i \circ n \in Z(\mathcal{N}) \text{ for all } i \in \mathcal{I}, n \in \mathcal{N}, \tag{3.46}$$

Since $i \circ in = i(i \circ n)$ for all $i \in \mathcal{I}, n \in \mathcal{N}$, replacing n by in in (3.46), we obtain

$$h(i)h(i \circ n) + h(i)(i \circ n) + ih(i \circ n) + i(i \circ n) \in Z(\mathcal{N}) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.47}$$

Thus

$$h(i)(h(i \circ n) + i \circ n) + i(h(i \circ n) + i \circ n) \in Z(\mathcal{N}) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.48}$$

Using (3.46), we get

$$(h(i \circ n) + i \circ n)(h(i) + i) \in Z(\mathcal{N}) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.49}$$

By Lemma 2.1 (iii) we have

$$h(i \circ n) + i \circ n = 0 \quad \text{or} \quad h(i) + i \in Z(\mathcal{N}) \quad \text{for all} \quad i \in \mathcal{I}, n \in \mathcal{N}. \tag{3.50}$$

Suppose there is an element $i_0 \in I$ such that $h(i_0 \circ n) + i_0 \circ n = 0$ for all $n \in \mathcal{N}$. Then, by recurrence we prove that

$$h^k(i_0 \circ n) + (-1)^{k+1} i_0 \circ n = 0 \quad \text{for all} \quad n \in \mathcal{N}, k \in \mathbb{N}^*. \tag{3.51}$$

Since h is zero-power valued on $\mathcal{N}$, there exists an integer $k(i_0 \circ n) > 1$ such that

$$h^{k(i_0 \circ n)}(i_0 \circ n) = 0 \quad \text{for all} \quad n \in \mathcal{N}.$$

Replacing k by $k(i_0 \circ n)$ in (3.51), it follows that

$$i_0 \circ n = 0 \quad \text{for all} \quad n \in \mathcal{N}. \tag{3.52}$$

Substituting i_0 instead of n in (3.52), we get $2(i_0)^2 = 0$. Using the 2-torsion freeness of $\mathcal{N}$, we obtain $(i_0)^2 = 0$. Substituting $i_0 n$ in place of n in (3.52), we get $i_0 n i_0 = 0$ for all $n \in \mathcal{N}$, so $i_0 \mathcal{N} i_0 = \{0\}$. Thus, $i_0 = 0$ by the 3-primenesse of $\mathcal{N}$. In this case, (3.50) implies that $h(i) + i \in Z(\mathcal{N})$ for all $i \in \mathcal{I}$. By Lemma 2.11, we conclude that $\mathcal{N}$ is abelian. $\square$

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