Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–7.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.64108

# On the Cotangent Bundle with Vertical Modified Riemannian Extensions

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ABSTRACT: Let $M$ be an n-dimensional differentiable manifold with a torsion-free linear connection $\nabla$ which induces on its cotangent bundle $T^*M$. The main purpose of the present paper is to study some properties of the vertical modified Riemannian extension on $T^*M$ which is given as a new metric in [17]. At first, we investigate a metric connection with torsion on $T^*M$. And then, we present the holomorphy properties with respect to a compatible almost complex structure. Furthermore, we study locally decomposable Golden pseudo-Riemannian structures on the cotangent bundle endowed with vertical modified Riemannian extension.

Key Words: Cotangent bundle, Riemannian extension, almost complex structure, Golden structure.

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## 1. Introduction

The geometry of the tangent bundle of a differentiable manifold is a very useful context in which various research topics can be improved. The cotangent bundle, which is the dual space of the tangent bundle, differs from the construction of lift. A well known metric on the cotangent bundle is the Riemannian extension constructed by Patterson and Walker [22]. The Riemannian extension provides a link between affine and pseudo-Riemannian geometry. For the Riemannian extension, also see [2-5, 18, 19, 29]. The Riemannian extension is modified and extensively studied by several authors and in many different context (see for example [1, 6, 8, 11, 17, 20, 24] ).

Let $(N, \varphi)$ be a $2m$-dimensional almost complex manifold, where $\varphi$ denotes its almost complex structure. A pseudo-Riemannian metric $g$ of neutral signature $(m, m)$ is a Norden metric if $g(\varphi V, W) = g(V, \varphi W)$ for any vector fields $V$ and $W$ on $N$. This metric studied as anti-Hermitian or B-metric [9, 15, 16, 25, 27]. In this case, the triple $(N, \varphi, g)$ is an almost complex Norden manifold. A Kähler-Norden manifold can be described as a triple $(N, \varphi, g)$ which consists of a Norden metric $g$ and an almost complex structure $\varphi$ satisfies $\nabla\varphi = 0$, where $\nabla$ is the Levi-Civita connection of Norden metric $g$. In [15], the condition $\nabla\varphi = 0$ is equivalent to the C-holomorphicity (analiticity) of the Norden metric $g$, i.e., $\phi_\varphi g = 0$, where $\phi_\varphi$ is the Tachibana operator [26, 28]:

$$\begin{aligned}\left(\phi_\varphi g\right)(V, X, Z) = {} & (\varphi V)(g(X, Z)) - V(g(\varphi X, Z)) + g((L_X\varphi)V, Z) \\ & + g(X, (L_Z\varphi)V)\end{aligned} \tag{1.1}$$

for any vector fields $V, X, Z$ on $N$, where $L_Z$ denotes the Lie derivative with respect to $Z$.

Let $J$ be a (1,1)-tensor field on a Riemannian manifold $M$. If the polynomial $X^2 - X - 1$ is the minimal polynomial for a structure $J$ satisfying $J^2 - J - 1 = 0$, then $J$ is defined a Golden structure on $M$ and $(M, J)$ is a Golden manifold [7, 14]. Let $(M, g)$ be a Riemannian manifold endowed with the Golden structure $J$ such that

$$g(JV, Z) = g(V, JZ)$$

2010 *Mathematics Subject Classification:* 53C07, 53C15, 53C25, 53A45.
Submitted June 22, 2022. Published February 02, 2023

for any $V,Z\in\Im_0^1(M)$ [ 7, 12, 14], ( for Golden pseudo-Riemannian manifold [21]). Golden structure has relation with pure Riemannian metrics with respect to the structure.

The rest of this paper is organized as follows: We recall some preliminary details concerning the vertical modified Riemannian extension, in section 2. In section 3, we investigate the metric connection with torsion with respect to the vertical modified Riemannian extension. In section 4, we obtain the case of the cotangent bundle with the vertical modified Riemannian extension being a Kähler-Norden manifold. In section 5, we study locally decomposable Golden Riemannian structure on the cotangent bundle endowed with vertical modified Riemannian extension.

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class $C^\infty$.

## 2. The Vertical Modified Riemannian Extension $\breve{G}$ on $T^*M$

Let $(M,g)$ be an n-dimensional differentiable manifold, $T^*M$ its cotangent bundle and $\pi$ the natural projection $\pi : T^*M \to M$. For any local coordinates $\left(U,x^k\right)$, $k=1,...,n$ on $M$, we denote by $\left(\pi^{-1}(U),x^k,x^{\bar{k}}=p_k\right)$, $\bar{k}=n+k=n+1,...,2n$ the corresponding coordinates on $T^*M$. Also, we denote by $\Im_s^r(M)\,(\Im_s^r(T^*M))$ the set of all tensor fields of type $(r,s)$ on $M(T^*M)$.

Let $\vartheta=\vartheta_k dx^k$ and $V=V^k\frac{\partial}{\partial x^k}$ be the local statements in $U\subset M$ of a covector field (1-form) $\vartheta\in\Im_1^0(M)$ and a vector field $V\in\Im_0^1(M)$, respectively. Then the horizontal lift ${}^HV$ of $V$, the vertical lift ${}^V\vartheta$ of $\vartheta$ are defined by

$$ {}^HV=V^k\partial_k+\sum_k p_h\Gamma_{kj}^h V^j\partial_{\bar{k}}, \tag{2.1}$$

$$ {}^V\vartheta=\sum_k \vartheta_k\partial_{\bar{k}}, \tag{2.2}$$

where $\frac{\partial}{\partial x^k}=\partial_k$, $\frac{\partial}{\partial x^{\bar{k}}}=\partial_{\bar{k}}$ and $\Gamma_{kj}^h$ are the components of $\nabla$ on $M$ (for details, see [29]).

In [29], the adapted frame $\left\{\tilde{f}_{(\beta)}\right\}=\left\{\tilde{f}_{(t)},\tilde{f}_{(\bar{t})}\right\}$ is defined by

$$\begin{cases}\tilde{f}_{(\bar{t})}=\partial_{\bar{t}},\\ \tilde{f}_{(t)}=\partial_t+\sum\limits_h p_a\Gamma_{ht}^a\partial_{\bar{h}}.\end{cases} \tag{2.3}$$

The Lie bracket operation of the adapted frame $\left\{\tilde{f}_{(\beta)}\right\}$ on the cotangent bunle $T^*M$ is given by

$$\begin{aligned}\left[\tilde{f}_{(t)},\tilde{f}_{(l)}\right]&=p_aR_{tlk}{}^a\tilde{f}_{(\bar{k})},\\ \left[\tilde{f}_{(\bar{t})},\tilde{f}_{(\bar{l})}\right]&=0,\\ \left[\tilde{f}_{(t)},\tilde{f}_{(\bar{l})}\right]&=-\Gamma_{tk}^l\tilde{f}_{(\bar{k})},\end{aligned} \tag{2.4}$$

where $R_{tlk}{}^a$ being local components of the curvature tensor $R$ of $\nabla$ on $M$.

Hence we have the undermentioned components for vector fields ${}^HV$ and ${}^V\vartheta$ on $T^*M$

$$ {}^HV=\begin{pmatrix}V^k\\0\end{pmatrix}\quad\text{and}\quad {}^V\vartheta=\begin{pmatrix}0\\ \vartheta_k\end{pmatrix} \tag{2.5}$$

in the adapted frame $\left\{\tilde{f}_{(\beta)}\right\}$.

In [17] by using Riemannian extension and quadratic differential form $\sum\limits_{k,l=1}^n a^{kl}\delta p_k\delta p_l$, where $\delta p_k=dp_k-p_h\Gamma_{jk}^h dx^k$ and $a^{kl}$ denote the components of a symmetric (2,0)-tensor fiels on $M$, one can define a new metric

$$\breve{G}=2dx^k\delta p_l+\sum_{k,l=1}^n a^{kl}\delta p_k\delta p_l$$

on $T^*M$. Troughout this paper we call this metric the vertical modified Riemannian extension.

In the adapted frame $\{\tilde{f}_{(\beta)}\}$ , the vertical modified Riemannian extension $\breve{G}$ has the components

$$\breve{G} = \begin{pmatrix} \breve{G}_{ji} & \breve{G}_{j\bar{i}} \\ \breve{G}_{\bar{j}i} & \breve{G}_{\bar{j}\bar{i}} \end{pmatrix} = \begin{pmatrix} 0 & \delta_j^i \\ \delta_i^j & a^{ji} \end{pmatrix} \tag{2.6}$$

Moreover, the vector fields ${}^HV$ and ${}^V\vartheta$ span the module $\Im_0^1(T^*M)$. Thus tensor field $\breve{G}$ is given by its action of ${}^HV$ and ${}^V\vartheta$. By using (2.1), (2.2) and (2.6) we have

$$\begin{aligned} &\breve{G}\left({}^HV, {}^HW\right) = 0, \\ &\breve{G}\left({}^HV, {}^V\alpha\right) = {}^V(\alpha(V)) = \alpha(V) \circ \pi, \\ &\breve{G}({}^V\vartheta, {}^V\alpha) = {}^V\left(\tilde{a}\left(\vartheta, \alpha\right)\right) = \tilde{a}\left(\vartheta, \alpha\right) \circ \pi \end{aligned} \tag{2.7}$$

for any $V, W \in \Im_0^1(M)$, $\vartheta, \beta \in \Im_1^0(M)$, where $\tilde{a}$ is a symmetric (2,0)-tensor field on $M$.

For the Levi-Civita connection $\breve{\nabla}$ of the vertical modified Riemannian extension $\breve{G}$, we find the following formulas:

**Theorem 2.1.** *[17] In adapted frame $\{\tilde{f}_{(\beta)}\}$, the Levi-Civita connection $\breve{\nabla}$ of the vertical modified Riemannian extension $\breve{G}$ on $T^*M$ satisfies the following equations:*

$$\begin{aligned} &i)\breve{\nabla}_{\tilde{f}_i}\tilde{f}_j = \left(\Gamma_{ij}^l + \tfrac{1}{2}p_s R_{ijt}{}^s a^{lt}\right)\tilde{f}_l + \left(p_s R_{lji}{}^s\right)\tilde{f}_{\bar{l}}, \\ &ii)\breve{\nabla}_{\tilde{f}_i}\tilde{f}_{\bar{j}} = \left(\tfrac{1}{2}\nabla_i a^{jl} - \Gamma_{it}^l a^{lt}\right)\tilde{f}_l + \left(-\Gamma_{il}^j + p_s R_{lit}{}^s a^{jt}\right)\tilde{f}_{\bar{l}}, \\ &iii)\breve{\nabla}_{\tilde{f}_{\bar{i}}}\tilde{f}_j = \left(\tfrac{1}{2}\nabla_j a^{li}\right)\tilde{f}_l + \left(\tfrac{1}{2}p_s R_{ljt}{}^s a^{it}\right)\tilde{f}_{\bar{l}}, \\ &iv)\breve{\nabla}_{\tilde{f}_{\bar{i}}}\tilde{f}_{\bar{j}} = \left(-\tfrac{1}{2}\nabla_l a^{ij}\right)\tilde{f}_{\bar{l}}, \end{aligned} \tag{2.8}$$

*where $R_{lji}{}^s$, $\Gamma_{ij}^l$ are respectively the components of the curvature tensor $R$ and coefficients of $\nabla$.*

Then we write $\breve{\nabla}_{\tilde{f}_\alpha}\tilde{f}_\beta = \breve{\Gamma}_{\alpha\beta}^\delta \tilde{f}_\delta$ with respect to the adapted frame $\{\tilde{f}_{(\alpha)}\}$ of $T^*M$ , where $\breve{\Gamma}_{\alpha\beta}^\delta$ denote the Christoffel symbols constructed by $\breve{G}$.

From Theorem 2.1, we immediately have

**Corollary 2.2.** *[17] In adapted frame $\{\tilde{f}_{(\alpha)}\}$, the components of the Christoffel symbols $\breve{\Gamma}_{\alpha\beta}^\delta$ of $\breve{\nabla}$ on $\left(T^*M, \breve{G}\right)$ are found as follows:*

$$\begin{aligned} &\breve{\Gamma}_{ij}^k = \Gamma_{ij}^k + \tfrac{1}{2}p_s R_{ijt}{}^s a^{tk}, & &\breve{\Gamma}_{ij}^{\bar{k}} = p_s R_{kji}{}^s, \\ &\breve{\Gamma}_{i\bar{j}}^k = \tfrac{1}{2}\nabla_i a^{jk} - \Gamma_{it}^j a^{tk}, & &\breve{\Gamma}_{\bar{i}j}^k = \tfrac{1}{2}\nabla_j a^{ik}, \\ &\breve{\Gamma}_{\bar{i}\bar{j}}^{\bar{k}} = -\tfrac{1}{2}\nabla_k a^{ij}, & &\breve{\Gamma}_{\bar{i}\bar{j}}^k = 0, \\ &\breve{\Gamma}_{i\bar{j}}^{\bar{k}} = -\Gamma_{ik}^j + \tfrac{1}{2}p_s R_{kit}{}^s a^{jt}, & &\breve{\Gamma}_{\bar{i}j}^{\bar{k}} = \tfrac{1}{2}p_s R_{kjt}{}^s a^{ti}. \end{aligned} \tag{2.9}$$

## 3. The metric connection with torsion on $\left(T^*M, \breve{G}\right)$

In this section, we deal with the metric connection on the cotangent bundle $T^*M$ endowed with the vertical modified Riemannian extension $\breve{G}$.

From [17], the Levi-Civita connection $\breve{\nabla}$ of the vertical modified Riemannian extension $\breve{G}$ on $T^*M$ was given. This connection is the unique connection which obtains $\breve{\nabla}\breve{G} = 0$ and has no torsion. On the other hand, it may be mentioned a connection which satisfies $\bar{\nabla}\breve{G} = 0$ and has non-trivial torsion tensor. Such connection is defined as the metric connection of the vertical modified Riemannian extension $\breve{G}$.

In [13], it was defined the metric connection with nonvanishing torsion tensor on Riemannian manifolds. The metric connection $\bar{\nabla}$ satisfies the followings:

$$\bar{\nabla}_I \breve{G}_{JK} = 0 \quad \text{and} \quad \bar{\Gamma}^K_{IJ} - \bar{\Gamma}^K_{JI} = \bar{\mathrm{T}}^K_{IJ}, \tag{3.1}$$

where $\bar{\Gamma}^K_{IJ}$ are components of the metric connection $\bar{\nabla}$ and torsion tensor $\bar{\mathrm{T}}^K_{IJ}$ is skew-symmetric in the indices $I$ and $J$. Then the above equation (3.1), one shows the following solution [13]

$$\bar{\Gamma}^K_{IJ} = \breve{\Gamma}^K_{IJ} + \bar{U}^K_{IJ}, \tag{3.2}$$

where $\breve{\Gamma}^{K}_{IJ}$ being the components of the Levi-Civita connection $\breve{\nabla}$ of the vertical modified Riemannian extension $\breve{G}$,

$$\bar{U}_{IJK} = \frac{1}{2}\left(\bar{\mathrm{T}}_{IJK} + \bar{\mathrm{T}}_{KIJ} + \bar{\mathrm{T}}_{KJI}\right) \tag{3.3}$$

and

$$\bar{U}_{IJK} = \bar{U}^{\varepsilon}_{IJ}\breve{G}_{\varepsilon K}, \quad \bar{\mathrm{T}}_{IJK} = \bar{\mathrm{T}}^{\varepsilon}_{IJ}\breve{G}_{\varepsilon K}. \tag{3.4}$$

If we take

$$\bar{\mathrm{T}}^{\bar{k}}_{ij} = -p_a R_{ijk}{}^a \tag{3.5}$$

and all the others being assumed to be zero. Using (3.3), (3.4) and (3.5), we find the following components for $\left(T^*M, \breve{G}\right)$

$$\begin{array}{ll}
\bar{U}^k_{ij} = \frac{1}{2}p_s a^{lk}\left(R_{lij}{}^s - R_{jli}{}^s\right), & \bar{U}^k_{\bar{i}j} = \frac{1}{2}p_s R_{ljt}{}^s a^{ti} a^{lk}, \\
\bar{U}^{\bar{k}}_{ij} = p_s R_{jki}{}^s, & \bar{U}^{\bar{k}}_{\bar{i}j} = -\frac{1}{2}p_s R_{kjl}{}^s a^{li}, \\
\bar{U}^{\bar{k}}_{i\bar{j}} = -\frac{1}{2}p_s R_{kil}{}^s a^{lj}, & \bar{U}^k_{i\bar{j}} = \frac{1}{2}p_s R_{lit}{}^s a^{tj} a^{lk}, \\
\bar{U}^k_{\bar{i}\bar{j}} = \bar{U}^{\bar{k}}_{\bar{i}\bar{j}} = 0. &
\end{array}$$

Taking account of (2.9) and (3.2), we find the following proposition.

**Proposition 3.1.** *In adapted frame* $\left\{\tilde{f}_{(\beta)}\right\}$*, the metric connection* $\bar{\nabla}$ *on* $\left(T^*M, \breve{G}\right)$ *satisfies*

$i)\bar{\nabla}_{\tilde{f}_i}\tilde{f}_j = \left(\Gamma^k_{ij} - p_s R_{jli}{}^s a^{lk}\right)\tilde{f}_k$

$ii)\bar{\nabla}_{\tilde{f}_i}\tilde{f}_{\bar{j}} = \left(\frac{1}{2}\nabla_i a^{jk} - \Gamma^j_{it}a^{tk} + \frac{1}{2}p_s R_{lit}{}^s a^{tj}a^{lk}\right)\tilde{f}_k - \Gamma^j_{ik}\tilde{f}_{\bar{k}},$

$iii)\bar{\nabla}_{\tilde{f}_{\bar{i}}}\tilde{f}_j = \left(\frac{1}{2}\nabla_j a^{ik} + \frac{1}{2}p_s R_{ljt}{}^s a^{ti}a^{lk}\right)\tilde{f}_k,$

$iv)\bar{\nabla}_{\tilde{f}_{\bar{i}}}\tilde{f}_{\bar{j}} = \left(-\frac{1}{2}\nabla_k a^{ij}\right)\tilde{f}_{\bar{k}}.$

On the other hand, the horizontal lift ${}^H\nabla$ of any connection $\nabla$ on $T^*M$, is given by

$$\left\{\begin{array}{ll}
{}^H\nabla_{{}^V\theta}{}^V\alpha = 0, & {}^H\nabla_{{}^V\theta}{}^H V = 0, \\
{}^H\nabla_{{}^H V}{}^V\alpha = {}^V\left(\nabla_V \alpha\right), & {}^H\nabla_{{}^H V}{}^H W = {}^H\left(\nabla_V W\right)
\end{array}\right. \tag{3.6}$$

for any $V, W \in \Im^1_0(M)$ and $\alpha, \theta \in \Im^0_1(M)$ in [29].

Now, we recall the following theorem.

**Theorem 3.2.** *[17] The horizontal lift* ${}^H\nabla$ *of* $\nabla$ *is a metric connection of the vertical modified Riemannian extension* $\breve{G}$ *if and only if the symmetric (2,0)-tensor field* $\tilde{a}$ *on* $(M, g)$ *is parallel with respect to* $\nabla$.

Then using Proposition 3.1 and Theorem 3.2, we get the following:

**Theorem 3.3.** *The metric connection* $\bar{\nabla}$ *on* $\left(T^*M, \breve{G}\right)$ *coincides with the horizontal lift* ${}^H\nabla$ *of the torsion-free linear connection* $\nabla$ *on* $M$ *if and only if* $M$ *is flat.*

## 4. Almost complex structure on $T^*M$

This section deals with the holomorphy properties of $\left(T^*M, \breve{G}\right)$ with respect to a compatible almost complex structure.

On the other hand, the horizontal lift ${}^{H}\varphi \in \Im_1^1(T^*M)$ is given by

$$\begin{aligned} {}^{H}\varphi {}^{H}V &= {}^{H}(\varphi V), \\ {}^{H}\varphi {}^{V}\vartheta &= {}^{V}(\vartheta \circ \varphi) \end{aligned} \tag{4.1}$$

for any $V \in \Im_0^1(M)$ and $\vartheta \in \Im_1^0(M)$. We recall from [29, p.283], if $\varphi$ is an almost complex structure on $(M, g)$, then ${}^{H}\varphi$ is an almost complex structure on $T^*M$.

Now, we are ready to give the next theorem as follows:

**Theorem 4.1.** *Given an almost complex Norden manifold $(M, \varphi, g)$. Then $\left(T^*M, {}^{H}\varphi, \breve{G}\right)$ is an almost complex Norden manifold if and only if the symmetric (2,0)-tensor field $\tilde{a}$ on $M$ is pure with respect to $\varphi$.*

*Proof.* Let $(M, \varphi, g)$ be an almost complex Norden manifold. We put

$$A(V, Z) = \breve{G}\left({}^{H}\varphi V, Z\right) - \breve{G}\left(V, {}^{H}\varphi Z\right)$$

for any $V, Z \in \Im_0^1(T^*M)$. From (2.7) and (1.1)

$$\begin{aligned} A\left({}^{H}V, {}^{H}Z\right) &= \breve{G}\left({}^{H}\varphi V, Z\right) - \breve{G}\left({}^{H}V, {}^{H}\varphi {}^{H}Z\right) \\ &= \breve{G}\left({}^{H}(\varphi V), {}^{H}Z\right) + \breve{G}\left({}^{H}V, {}^{H}(\varphi Z)\right) = 0, \\ A\left({}^{H}V, {}^{V}\vartheta\right) &= \breve{G}\left({}^{H}\varphi {}^{H}V, {}^{V}\vartheta\right) - \breve{G}\left({}^{H}V, {}^{H}\varphi {}^{V}\vartheta\right) \\ &= \breve{G}\left({}^{H}(\varphi V), {}^{V}\vartheta\right) - \breve{G}\left({}^{H}V, {}^{V}(\vartheta \circ \varphi)\right) \\ &= {}^{V}(\vartheta(\varphi V)) - {}^{V}((\vartheta \circ \varphi) V) = 0, \\ A\left({}^{V}\vartheta, {}^{H}V\right) &= -A\left({}^{H}V, {}^{V}\vartheta\right) = 0, \\ A\left({}^{V}\vartheta, {}^{V}\theta\right) &= \breve{G}\left({}^{H}\varphi {}^{V}\vartheta, {}^{V}\theta\right) - \breve{G}\left({}^{V}\vartheta, {}^{H}\varphi {}^{V}\theta\right) \\ &= \breve{G}\left({}^{V}(\vartheta \circ \varphi), {}^{V}\theta\right) - \breve{G}\left({}^{V}\vartheta, {}^{V}(\theta \circ \varphi)\right) \\ &= \tilde{a}((\vartheta \circ \varphi), \theta) - \tilde{a}(\vartheta, (\theta \circ \varphi)). \end{aligned}$$

In the last equations, if symmetric tensor field $\tilde{a} \in \Im_0^2(M)$ is pure with respect to $\varphi$, then we obtain $A(V, Z) = 0$, i.e., the vertical modified Riemannian extension $\breve{G}$ is pure with respect to ${}^{H}\varphi$. $\square$

Here we discuss the holomorphy property of the vertical modified Riemannian extension $\breve{G}$ by using the ${}^{H}\varphi$. We obtain from (1.1)

$$\begin{aligned} \left(\phi_{{}^{H}\varphi}\breve{G}\right)\left({}^{H}V, {}^{H}X, {}^{H}Z\right) &= {}^{V}((pR(X, \varphi V)) Z) \\ &\quad - {}^{V}((pR(X, V) \circ \varphi) Z) + {}^{V}((pR(Z, \varphi V)) X) \\ &\quad - {}^{V}((pR(Z, V) \circ \varphi) X), \\ \left(\phi_{{}^{H}\varphi}\breve{G}\right)\left({}^{H}V, {}^{H}X, {}^{V}\vartheta\right) &= (\vartheta(\nabla_X \varphi))(V) - (\vartheta(\nabla_V \varphi))(X) \\ &\quad + \tilde{a}(pR(X, \varphi V), \vartheta) - \tilde{a}(\vartheta, pR(X, V)\varphi), \\ \left(\phi_{{}^{H}\varphi}\breve{G}\right)\left({}^{H}V, {}^{V}\vartheta, {}^{H}X\right) &= (\vartheta(\nabla_X \varphi))(V) - (\vartheta(\nabla_V \varphi))(X) \\ &\quad + \tilde{a}(\vartheta, pR(X, \varphi V)) - \tilde{a}(\vartheta, pR(X, V)\varphi),, \\ \left(\phi_{{}^{H}\varphi}\breve{G}\right)\left({}^{V}\vartheta, {}^{H}X, {}^{H}Z\right) &= (\vartheta(\nabla_X \varphi))(Z) + (\vartheta(\nabla_Z \varphi))(X), \\ \left(\phi_{{}^{H}\varphi}\breve{G}\right)\left({}^{V}\vartheta, {}^{V}\theta, {}^{H}Z\right) &= {}^{V}(\tilde{a}(\vartheta, \theta(\nabla_Z \varphi))), \\ \left(\phi_{{}^{H}\varphi}\breve{G}\right)\left({}^{V}\vartheta, {}^{H}X, {}^{V}\theta\right) &= {}^{V}(\tilde{a}(\vartheta(\nabla_X \varphi), \theta)), \\ \left(\phi_{{}^{H}\varphi}\breve{G}\right)\left({}^{H}V, {}^{V}\vartheta, {}^{V}\theta\right) &= \left(\phi_{\varphi}\tilde{a}\right)(V, \vartheta, \theta) - \tilde{a}((\nabla_\vartheta \varphi) V, \theta) \\ &\quad - \tilde{a}(\vartheta, (\nabla_\theta \varphi) V), \\ \left(\phi_{{}^{H}\varphi}\breve{G}\right)\left({}^{V}\vartheta, {}^{V}\theta, {}^{V}\xi\right) &= 0. \end{aligned} \tag{4.2}$$

Next, we have the following theorem:

**Theorem 4.2.** *Let $M$ be a Kähler-Norden manifold which consists of an almost complex structure $\varphi$ and a Norden metric $g$. Then $\left(T^*M, {}^H\varphi, \breve{G}\right)$ is a Kähler-Norden manifold if and only if the symmetric tensor field $\tilde{a}$ on $M$ is a holomorphic tensor field with respect to almost complex structure $\varphi$.*

## 5. Locally Decompasable Golden Riemannian structure on $T^*M$

In this section, we study locally decomposable Golden Riemannian structure on $T^*M$ endowed with the vertical modified Riemannian extension $\breve{G}$.

The horizontal lift ${}^H\varphi \in \Im_1^1(T^*M)$ given by (4.1) holds the following

$$ {}^H I = I, \left({}^H\varphi\right)^2 = {}^H\left(\varphi^2\right) ... $$

where $I \in \Im_1^1(M)$ is the unit tensor field [29]. From (4.3), $\varphi^2 - \varphi - I = 0$ implies $\left({}^H\varphi\right)^2 - {}^H\varphi - I = 0$. Hence, we see that if $\varphi$ is a Golden structure on $M$, then ${}^H\varphi$ is also a Golden structure on $T^*M$ [10]. onsidering Theorem 4.1, we get the following theorem:

**Theorem 5.1.** *Given a Golden Riemannian manifold $(M, J, g)$. Then $\left(T^*M, {}^H\varphi, \breve{G}\right)$ is a Golden pseudo-Riemannian manifold if and only if the symmetric (2,0)-tensor field $\tilde{a}$ on $M$ is pure with respect to $\varphi$.*

Taking account of (4.2), we get

**Theorem 5.2.** *Given a locally decomposable Golden manifold $(M, J, g)$. Then the cotangent bundle $T^*M$ is a locally decomposable Golden manifold equipped with the vertical modified Riemannian extension $\breve{G}$ and Golden structure ${}^H\varphi$ if and only if $\phi_\varphi \tilde{a} = 0$.*

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