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Best Proximity Point Results for Generalized Proximal Z-Contraction Mappings in Metric Spaces and Some Applications

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ABSTRACT: In this paper, we define generalized proximal Z-contraction mappings of first and second kind in a metric space (X, d). The existence of best proximity point is shown for the defined mappings under some specific conditions which generalize and extends some existing results of Olgun *et al.*[23] and Abbas *et al.* [1]. Suitable examples are given to justify the derived results. Some applications are also shown via fixed point formulation for such mappings in variational inequality problems and homotopy result.

Key Words: Best proximity point, proximal simulative contraction, homotopy.

Contents

1. Introduction

In the study of nonlinear functional analysis, fixed point theory plays an important role with different important applications. Fixed point of a self mapping T is a solution of the equation $Tx = x$ on the domain of T. Banach contraction principle, introduced by Banach in 1922, is one of the fundamental results in the study of metric fixed point theory [6]. Since then many authors have established different interesting results related to existence of fixed point for different types of mappings with different applications [5,11,14,15,16,28,29].
On the other hand, the equation $Tx = x$ is not always solvable, in particular, for the non self mappings. In that case, finding an x for which $d(x, Tx)$ is minimum becomes significant, which subsequently leads to a new concept namely, best proximity point. The study of the best proximity point theory has been an interesting area of research since last few decades [7,12,13,23,27].
The aim of this paper is to establish the existence of best proximity point for generalized proximal Z-contraction mappings. Our results extend and improve some known results of [1] and [23]. Some applications are also shown in variational inequality problem and homotopy theory using fixed point formulation.

In 2015, Khojasteh *et al.* [17] introduced simulation function and employed the same to study fixed point results for Z-contraction mappings. After this, several prominent researchers [1,3,9,15] have worked to study fixed point and best proximity point results using simulation function.

Definition 1.1. *[17] Let $\zeta : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ be a function satisfying following conditions:*
(ζ_1) $\zeta(0, 0) = 0$*;*
(ζ_2) $\zeta(a, b) < b - a$*, for all $a, b > 0$;*
(ζ_3) *if $\{a_n\}$, $\{b_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n > 0$, then*

$$\limsup_{n\to\infty} \zeta(a_n, b_n) < 0,$$

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ζ is then called a simulation function. The set of all simulation functions is denoted by τ.

Example 1.2. *[15] Let $\phi : [0,\infty) \longrightarrow [0,\infty)$ be a continuous function such that $\phi(a) = 0$ if and only if $a = 0$. Then $\zeta_1 : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ defined by*

$$\zeta_1(a,b) = b - \phi(b) - a,$$

for all $a,b \in [0,\infty)$ is a simulation function.

Example 1.3. *[15] Let $f,g : [0,\infty)^2 \longrightarrow (0,\infty)$ be two continuous functions such that $f(a,b) > g(a,b)$, for all $a,b > 0$. Then $\zeta_2 : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ defined by*

$$\zeta_2(a,b) = b - \frac{f(a,b)}{g(a,b)}a,$$

for all $a,b \in [0,\infty)$ is a simulation function.

Using the simulation function approach, the notion of Z-contraction was introduced in [17] which is a generalization of Banach contraction. It also unified various existing types of contraction mappings.

Definition 1.4. *[17] Let X be a metric space. A self mapping T on X is called a Z-contraction if for some simulation function $\zeta \in \tau$, T satisfies*

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0,$$

for all $x,y \in X$.

Example 1.5. *Consider $X = \mathbb{R}$, with the usual metric $d(x,y) = |x-y|$, for all $x,y \in X$ and $T : X \longrightarrow X$ be defined by*

$$Tx = \frac{x}{5}, \text{ for all } x \in X.$$

If $\zeta(s,t) = \frac{1}{5}t - s$, for all $s,t \in [0,\infty)$, then T is a Z-contraction.

It can be noted that every Z-contraction mapping is contractive and so continuous. This continuity was relaxed by Olgun *et al.* in [23] where a generalized Z-contraction mapping was defined which is not necessarily continuous.

Definition 1.6. *[23] For a metric space (X,d), a self mapping T on X is called a generalized Z-contraction if there exists $\zeta \in \tau$ such that*

$$\zeta(d(Tx,Ty), M(x,y)) \geq 0, \text{ for all } x,y \in X,$$

where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}.$$

Combining Z-contraction and Suzuki type contraction, a new type of mapping was defined by Kumam *et al.* [19] as described below.

Definition 1.7. *[19] For a metric space (X,d), a self mapping T on X is called a Suzuki type Z-contraction if for some simulation function $\zeta \in \tau$, T satisfies*

$$\frac{1}{2}d(x,Tx) < d(x,y) \text{ implies } \zeta(d(Tx,Ty), d(x,y)) \geq 0,$$

for all $x,y \in X$.

Example 1.8. *Consider $X = \{2, 6, 10, 14\}$ with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$. A mapping $T : X \longrightarrow X$ be defined by*

$$T(2) = T(6) = T(10) = 2 \text{ and } T(14) = 6.$$

If $\zeta(s, t) = \frac{4}{5}t - s$, for all $s, t \in [0, \infty)$, then T is a Suzuki type Z-contraction, but not Z-contraction.

Padcharoen *et al.* [24] introduced generalized Suzuki type Z-contraction as follows.

Definition 1.9. *[24] For a metric space (X, d), a self mapping T on X is called generalized Suzuki type Z-contraction if there exists $\zeta \in \tau$ such that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } \zeta(d(Tx, Ty), M(x, y)) \geq 0,$$

for all $x, y \in X$, where

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}.$$

Example 1.10. *For $X = \{0, 2, 4, 6, 8\}$ with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$ and let $T : X \longrightarrow X$ be defined by*

$$T(0) = T(2) = T(4) = 0 \text{ and } T(6) = T(8) = 2.$$

If $\zeta(s, t) = \frac{1}{2}t - s$ for all $s, t \in [0, \infty)$, then T is a generalized Suzuki type Z-contraction.

Remark 1.11. *It is evident that all Suzuki type Z-contraction is generalized Suzuki type Z-contraction for $M(x, y) = d(x, y)$.*

In [1], Abbas *et al.* introduced the concept of proximal simulative contraction of first kind and second kind.

Definition 1.12. *[1] For two non empty subsets A and B of a metric space (X, d), a mapping $T : A \longrightarrow B$ is said to be proximal simulative contraction of first kind if there exists a simulation function $\zeta \in \tau$ such that*

$$d(x_1, Ty_1) = d(A, B) = d(x_2, Ty_2) \text{ implies}$$
$$\zeta(d(x_1, x_2), d(y_1, y_2)) \geq 0, \text{ for all } x_1, x_2, y_1, y_2 \in A.$$

Definition 1.13. *[1] For two non empty subsets A and B of a metric space (X, d), a mapping $T : A \longrightarrow B$ is said to be proximal simulative contraction of second kind if there exists a simulation function $\zeta \in \tau$ such that*

$$d(x_1, Ty_1) = d(A, B) = d(x_2, Ty_2) \text{ implies}$$
$$\zeta(d(Tx_1, Tx_2), d(Ty_1, Ty_2)) \geq 0, \text{ for all } x_1, x_2, y_1, y_2 \in A.$$

2. Main results

Following the works of Olgun *et al.* [23] and Abbas *et al.* [1], we introduce generalized proximal Z-contraction of first kind and second kind as follows:

Definition 2.1. *For two non empty subsets A and B of a metric space (X, d), a mapping $T : A \longrightarrow B$ is said to be a generalized proximal Z-contraction of first kind if there exists a simulation function $\zeta \in \tau$ such that $d(x_1, Ty_1) = d(x_2, Ty_2) = d(A, B)$ implies*

$$\zeta(d(x_1, x_2), M(y_1, y_2, x_1, x_2)) \geq 0, \text{ for all } x_1, x_2, y_1, y_2 \in A,$$

where

$$M(y_1, y_2, x_1, x_2) = max\{d(y_1, y_2), d(y_1, x_1), d(y_2, x_2), \frac{d(y_1, x_2) + d(y_2, x_1)}{2}\}.$$

Here τ denotes the set of all simulation functions.

Remark 2.2. *For $A = B = X$, the above mapping reduces to generalized Z-contraction. Again, for $M(y_1, y_2, x_1, x_2) = d(y_1, y_2)$, it reduces to proximal simulative contraction of first kind. It can be noted here that a generalized proximal Z-contraction of first kind is not necessarily continuous.*

Example 2.3. *Consider two subsets $A = \{3, 4, 5, 6\}$ and $B = \{3.5, 8, 9\}$ of the metric space $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$. The mapping $T : A \longrightarrow B$ defined by*

$$T(5) = T(6) = 3.5 \text{ and } T(4) = T(3) = 8$$

is a generalized proximal Z-contraction of first kind with $\zeta(s, t) = \frac{3}{4}t - s$, for all $s, t \in [0, \infty)$. However, T is not a proximal simulative contraction of first kind.

Definition 2.4. *Let A and B be two non empty subsets in a metric space (X, d). A mapping $T : A \longrightarrow B$ is said to be a generalized proximal Z-contraction of second kind if there exists a simulation function $\zeta \in \tau$ such that $d(x_1, Ty_1) = d(x_2, Ty_2) = d(A, B)$ implies*

$$\zeta(d(Tx_1, Tx_2), M(Ty_1, Ty_2, Tx_1, Tx_2)) \geq 0, \text{ for all } x_1, x_2, y_1, y_2 \in A,$$

where

$$\begin{aligned} M(Ty_1, Ty_2, Tx_1, Tx_2) = max\{&d(Ty_1, Ty_2), d(Ty_1, Tx_1), d(Ty_2, Tx_2), \\ &\frac{d(Ty_1, Tx_2) + d(Ty_2, Tx_1)}{2}\}. \end{aligned}$$

Example 2.5. *For $X = \mathbb{R}^2$ with the usual metric $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, for all $(x_1, x_2), (y_1, y_2) \in X$, consider $A = A_1 \cup A_2 \cup A_3$ and $B = B_1 \cup B_2$ where $A_1 = \{(x, y) : 0 \leq x \leq 1, y = 1\}$, $A_2 = \{(x, y) : x \geq 2, y = 2\}$, $A_3 = \{(x, y) : x \leq -1, y = 2\}$, $B_1 = \{(x, y) : 0 \leq x \leq \frac{1}{2}, y = \frac{1}{2}\}$ and $B_2 = \{(x, y) : x \geq 2, y = -1\}$. Define $T : A \longrightarrow B$ by*

$$T(x_1, x_2) = \begin{cases} (\frac{1}{2}, \frac{1}{2}) : & (x_1, x_2) \in A_1 \cup A_2 \\ (3, -1) : & (x_1, x_2) \in A_3. \end{cases}$$

Then T is a generalized proximal Z-contraction of second kind with ζ as defined in Example 1.2.

In the next discussion, we derive some results on the existence of best proximity point for the above two types of mappings. By the following two Lemmas, we show that the best proximity point (if exists) is unique for each type of mappings.

Lemma 2.6. *For two non empty subsets A and B of a metric space (X, d), let $T : A \longrightarrow B$ be a generalized proximal Z-contraction of first kind. Then the best proximity point of T is unique, provided it exists.*

Proof. Suppose $x \in A$ be a best proximity point of T. If possible, let $y(\neq x) \in A$ be another best proximity point of T.

Since, $d(x, Tx) = d(A, B)$ and $d(y, Ty) = d(A, B)$, so, by Definition 2.1,

$$0 \leq \zeta(d(x, y), M(x, y, x, y)). \tag{2.1}$$

Now, $d(x, y) > 0$ and $M(x, y, x, y) > 0$.
From the condition (ζ_2) of Definition 1.1, we get,

$$\zeta(d(x, y), M(x, y, x, y)) < M(x, y, x, y) - d(x, y). \tag{2.2}$$

From (2.1) and (2.2)

$$\begin{gathered} M(x, y, x, y) - d(x, y) > 0 \\ i.e., M(x, y, x, y) > d(x, y) \\ i.e., M(x, y, x, y) = d(x, y) < d(x, y), \end{gathered}$$

which is a contradiction. Hence, the best proximity point of T is unique. □

Lemma 2.7. *For two non empty subsets A and B of a metric space (X, d), let $T : A \longrightarrow B$ be a generalized proximal Z-contraction of second kind. Then the best proximity point of T is unique, provided it exists.*

Proof. The result can be obtained in the same way as the previous lemma. □

In [23], Olgun *et al.* derived fixed point result for generalized Z-contraction mappings.

Theorem 2.8. *[23] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a generalized Z-contraction. Then T is a Picard operator.*

Extending this result for generalized proximal Z-contraction mapping of first kind, we establish an existence result of the best proximity point. The followings terminologies (refer to [4]) will be used in the results:.
Let A and B be two non empty subsets of a metric space (X, d). Define

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

Theorem 2.9. *Let A and B be two non empty subsets of a complete metric space (X, d). Suppose $T : A \longrightarrow B$ be a generalized proximal Z-contraction of first kind with $T(A_0) \subseteq B_0$, where A_0, B_0 are non empty and A_0 is closed. Then T has a unique best proximity point in A_0, and for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ with $d(x_n, Tx_{n-1}) = d(A, B)$, for all $n \in \mathbb{N}$, converges to the best proximity point.*

Proof. Suppose $x_0 \in A_0$. Since, $T(A_0) \subseteq B_0$, there exists $x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B).$$

Again, since, $Tx_1 \in B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B).$$

Repeating this process, we get a sequence $\{x_n\}$ in A_0 such that

$$d(x_n, Tx_{n-1}) = d(A, B) = d(x_{n+1}, Tx_n), \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

If for some $n \in \mathbb{N}$, $x_n = x_{n-1}$, then

$$d(x_n, Tx_{n-1}) = d(x_n, Tx_n) = d(A, B),$$

and so, x_n is a best proximity point of T.

We assume that $d(x_n, Tx_{n-1}) > 0$. As T is a generalized proximal Z-contraction of first kind, there exists some $\zeta \in \tau$ such that

$$\zeta(d(x_n, x_{n+1}), M(x_{n-1}, x_n, x_n, x_{n+1})) \geq 0, \text{ for all } n \in \mathbb{N}, \tag{2.4}$$

where

$$\begin{aligned} M(x_{n-1}, x_n, x_n, x_{n+1}) &= max\{d(x_n, x_{n-1}), d(x_n, x_{n-1}), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\} \\ &= max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}. \end{aligned}$$

If $d(x_n, x_{n-1}) < d(x_n, x_{n+1})$ then $M(x_{n-1}, x_n, x_n, x_{n+1}) = d(x_n, x_{n+1})$.

Now, from (2.4) using condition (ζ_2), we get,

$$0 \leq \zeta(d(x_n, x_{n+1}), M(x_{n-1}, x_n, x_n, x_{n+1})) < M(x_{n-1}, x_n, x_n, x_{n+1}) - d(x_n, x_{n+1}) = 0,$$

which is a contradiction. Therefore,

$$d(x_n, x_{n+1}) \leq d(x_n, x_{n-1}), \text{ for all } n \in \mathbb{N},$$

i.e., $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of positive real numbers and so there exists a real number $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r$$

If $r > 0$, then condition (ζ_3) gives

$$0 \leq \limsup_{n\to\infty} \zeta(d(x_n, x_{n+1}), M(x_{n-1}, x_n, x_n, x_{n+1})) < 0,$$

which is a contradiction. Therefore, $r = 0$, i.e., $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.
Now we show that $\{x_n\}$ is a Cauchy sequence. If not, then for given $\varepsilon > 0$, there exist subsequences $\{x_{n_l}\}$ and $\{x_{m_l}\}$ of $\{x_n\}$ such that for any $n_l > m_l \geq l$

$$d(x_{m_l}, x_{n_l}) > \varepsilon.$$

Without loss of generality, we can choose n_l as the smallest positive integer greater than m_l such that the above inequality holds. Then

$$d(x_{m_l}, x_{n_l-1}) \leq \varepsilon, \text{ for all } l \in \mathbb{N}.$$

Now,

$$\begin{aligned} \varepsilon < d(x_{m_l}, x_{n_l}) &\leq d(x_{m_l}, x_{n_l-1}) + d(x_{n_l-1}, x_{n_l}) \\ &\leq \varepsilon + d(x_{n_l-1}, x_{n_l}). \end{aligned} \tag{2.5}$$

Taking limit as $l \to \infty$ on both sides of (2.5), we get,

$$\lim_{l\to\infty} d(x_{m_l}, x_{n_l}) = \varepsilon. \tag{2.6}$$

Again, from (2.3),

$$d(x_{m_l}, Tx_{m_l-1}) = d(A, B) \text{ and } d(x_{n_l}, Tx_{n_l-1}) = d(A, B),$$

and so,

$$\zeta(d(x_{m_l}, x_{n_l}), M(x_{m_l-1}, x_{n_l-1}, x_{m_l}, x_{n_l})) \geq 0, \tag{2.7}$$

where

$$\begin{aligned} M(x_{m_l-1}, x_{n_l-1}, x_{m_l}, x_{n_l}) = max\{&d(x_{m_l-1}, x_{n_l-1}), d(x_{m_l-1}, x_{m_l}), d(x_{n_l-1}, x_{n_l}), \\ &\frac{d(x_{m_l-1}, x_{n_l}) + d(x_{n_l-1}, x_{m_l})}{2}\}. \end{aligned} \tag{2.8}$$

Therefore,

$$\lim_{l\to\infty} M(x_{m_l-1}, x_{n_l-1}, x_{m_l}, x_{n_l}) = \varepsilon. \tag{2.9}$$

Using condition (ζ_3), we get,

$$0 \leq \limsup_{l\to\infty}\{\zeta(d(x_{m_l}, x_{n_l}), M(x_{m_l-1}, x_{n_l-1}, x_{m_l}, x_{n_l})) < 0,$$

which is a contradiction.

Therefore, $\{x_n\}$ is a Cauchy sequence in A_0. Since, A_0 is closed, there exists some $x \in A_0$ such that

$$x_n \to x \text{ as } n \to \infty.$$

Since, $Tx \in B_0$, there exists $z \in A_0$ such that

$$d(z, Tx) = d(A, B). \tag{2.10}$$

Now, we show that $d(x, z) = 0$. If possible, let $d(x, z) > 0$.

From (2.3), (2.10) and definition of T we get,

$$\zeta(d(x_{n+1}, z), M(x_n, x, x_{n+1}, z)) \geq 0, \tag{2.11}$$

where

$$M(x_n, x, x_{n+1}, z) = max\{d(x, x_n), d(x, z), d(x_{n+1}, x_n), \frac{d(x_{n+1}, x) + d(x_n, z)}{2}\}.$$

So,

$$\begin{aligned}\lim_{n\to\infty} M(x_n, x, x_{n+1}, z) &= max\{d(x, z), \frac{d(x, z)}{2}\}\\ &= d(x, z) > 0\end{aligned}$$

and

$$\lim_{n\to\infty} d(x_{n+1}, z) = d(x, z) > 0.$$

Using condition (ζ_3), we get,

$$0 \leq \limsup_{n\to\infty} \zeta(d(x_{n+1}, z), M(x_n, x, x_{n+1}, z)) < 0,$$

a contradiction. Hence, $x = z$ and therefore $d(x, Tx) = d(A, B)$ i.e., x is a best proximity point of T. The uniqueness of best proximity point is evident from Lemma 2.6. □

Remark 2.10. *Taking $A = B = X$ in the above Theorem, we get Theorem 2 of [23] as a particular case.*

Remark 2.11. *In Theorem 2.9, the mapping T is not necessarily continuous. Moreover, the sets A and B are not required to be closed. Thus, for $M(y_1, y_2, x_1, x_2) = d(y_1, y_2)$,(when the mapping T reduces to proximal simulative contraction of first kind) Theorem 2.9 improves Theorem 1 of [1].*

The following example exhibits the contents of Theorem 2.9.

Example 2.12. *Consider $X = \ell^\infty$ with metric d defined by*

$$d(x, y) = \sup_{i\in\mathbb{N}} |x_i - y_i|, \text{ for all } x = \{x_i\}, y = \{y_i\} \in X.$$

Consider two subsets A and B of X such that $A = \{e_{2n-1}, n \in \mathbb{N}\} \cup \{\frac{1}{3}e_1\}$ and $B = \{e_{2n}, n \in \mathbb{N}\} \cup \{e_0\} \cup \{\frac{1}{4}e_{2n}, n \in \mathbb{N}\}$.

It is evident that $d(A, B) = \frac{1}{3}$, $A_0 = \{\frac{1}{3}e_1\}$ and $B_0 = \{e_0\} \cup \{\frac{1}{4}e_{2n}\}$.
Define $T : A \longrightarrow B$ by

$$T(e_{2n-1}) = e_0 \text{ and } T(\frac{1}{3}e_1) = \frac{1}{4}e_2.$$

Then T is a generalized proximal Z-contraction with $\zeta(s, t) = \frac{1}{2}t - s$, for all $s, t \in [0, \infty)$. It is seen that $T(A_0) \subseteq B_0$. Hence, all the conditions of Theorem 2.9 are satisfied and T has a best proximity point. Clearly, $\frac{1}{3}e_1$ is the best proximity point of T.

Corollary 2.13. *Let A and B be two non-empty subsets of a complete metric space (X, d). Suppose $T : A \longrightarrow B$ be a mapping with $T(A_0) \subseteq B_0$ where A_0, B_0 are non-empty and A_0 is closed subset such that $d(x_1, Ty_1) = d(A, B) = d(x_2, Ty_2)$ implies*

$$\zeta(d(x_1, x_2), max\{d(y_1, y_2), \frac{d(y_1, x_1) + d(y_2, x_2)}{2}, \frac{d(y_1, x_2) + d(y_2, x_1)}{2}\}) \geq 0,$$

for all $x_1, x_2, y_1, y_2 \in A$. Then T has a unique best proximity point.

In the following, it is shown that similar result as in Theorem 2.9 holds for generalized proximal Z-contraction of second kind.

Theorem 2.14. *Let A and B be two non empty subsets of a complete metric space (X,d). Suppose $T : A \longrightarrow B$ be a generalized proximal Z-contraction of second kind with $T(A_0) \subseteq B_0$ where A_0 and B_0 are non empty and B_0 is a closed subset of B. Then T has a unique best proximity point in A_0.*

Proof. Proceeding similar to Theorem 2.9, and using the condition of generalized proximal Z-contraction of second kind, we can show that $\{d(Tx_n, Tx_{n+1})\}$ is a decreasing sequence and

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0.$$

Following the technique of Theorem 2.9, it can be shown that $\{Tx_n\}$ is a Cauchy sequence in B_0 and so it converges to some $y = Tu \in B_0$, where $u \in A_0$. Since, $Tu \in B_0$, there exists $z \in A_0$ such that

$$d(z, Tu) = d(A,B).$$

Therefore,

$$\zeta(d(Tz, Tx_{n+1}), M(Tu, Tx_n, Tz, Tx_{n+1})) \geq 0,$$

where

$$\begin{aligned} M(Tu, Tx_n, Tz, Tx_{n+1}) = max\{&d(Tu, Tx_n), d(Tz, Tu), d(Tx_{n+1}, Tx_n), \\ &\frac{d(Tx_n, Tz) + d(Tu, Tx_{n+1})}{2}\}. \end{aligned} \tag{2.12}$$

Taking limit as $n \to \infty$ on both sides of (2.12), we get,

$$\lim_{n\to\infty} M(Tu, Tx_n, Tz, Tx_{n+1}) = d(Tz, Tu).$$

Also,

$$\lim_{n\to\infty} d(Tz, Tx_{n+1}) = d(Tz, Tu).$$

If $d(Tz, Tu) > 0$ then using condition (ζ_3),

$$0 \leq \limsup_{n\to\infty} \zeta(d(Tz, Tx_{n+1}), M(Tu, Tx_n, Tz, Tx_{n+1})) < 0,$$

a contradiction. Therefore, $d(Tz, Tu) = 0$ i.e., $Tz = Tu$. Hence, z is a best proximity point of T. The uniqueness of best proximity point of T follows from Lemma 2.7. □

Remark 2.15. *Assuming $M(Ty_1, Ty_2, Tx_1, Tx_2) = d(Ty_1, Ty_2)$ in Theorem 2.14, we get an improvement of Theorem 2 of [1].*

In 2015, Su *et al* [25] established a fixed point result for the self mapping T on a metric space (X,d) satisfying the conditions:

$$\phi(d(Tx, Ty)) \leq \psi(d(x,y)), \text{ for all } x, y \in X,$$

where the two mappings $\phi, \psi : [0, \infty) \longrightarrow [0, \infty)$ are such that

(1) $\phi(a) \leq \psi(b) \Rightarrow a \leq b$,

(2) $a_n \to r, b_n \to r$ and $\phi(a_n) \leq \psi(b_n) \Rightarrow r = 0$.

Using the above mappings ϕ and ψ, now we obtain the following version of Theorem 2.9. It is worth mentioning that this result is obtained in a much simpler way without using the condition (ζ_3) of the Definition 1.1.

Theorem 2.16. *For two non-empty subsets A and B of a complete metric space (X, d), let $T : A \longrightarrow B$ be a mapping with $T(A_0) \subseteq B_0$, where A_0, B_0 are non-empty and A_0 is closed in A such that $d(x, Ty) = d(A, B) = d(x', Ty')$ implies*

$$\zeta(\phi(d(x, x')), \psi(M(y, y', x, x'))) \geq 0, \text{ for all } x, x', y, y' \in A,$$

where $M(y, y', x, x') = max\{d(y, y'), d(x, y), d(x', y'), \frac{d(x,y')+d(x',y)}{2}\}$ and ζ is a function satisfying the conditions (ζ_1) and (ζ_2) of Definition 1.1.
Then T has a unique best proximity point in A.

Proof. For $x_0 \in A_0$, we get a sequence $\{x_n\}$ in A_0 such that

$$d(x_n, Tx_{n-1}) = d(A, B) \text{ and } d(x_{n+1}, Tx_n) = d(A, B), \text{ for all } n \in \mathbb{N}. \tag{2.13}$$

From the definition of the mapping T, we get,

$$\zeta(\phi(d(x_{n+1}, x_n)), \psi(M(x_n, x_{n-1}, x_{n+1}, x_n))) \geq 0,$$

where

$$M(x_n, x_{n-1}, x_{n+1}, x_n) = d(x_{n-1}, x_n).$$

Using the condition (1) and (ζ_2), it is easy to see that $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence. Therefore there exists an element $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r = \lim_{n\to\infty} M(x_n, x_{n-1}, x_{n+1}, x_n).$$

Since, $\phi(d(x_{n+1}, x_n)) < \psi(M(x_n, x_{n-1}, x_{n+1}, x_n))$, so condition (2) implies that $r = 0$.

As in Theorem 2.9, it can be shown that $\{x_n\}$ is a Cauchy sequence in A_0 and so, it converges to some $x \in A_0$. Since, $Tx \in B_0$, there exists $z \in A_0$ such that

$$d(z, Tx) = d(A, B),$$

and so,

$$\zeta(\phi(d(x_{n+1}, z)), \psi(M(x_n, x, x_{n+1}, z))) \geq 0,$$

where $\lim_{n\to\infty} M(x_n, x, x_{n+1}, z) = d(x, z)$ and $\lim_{n\to\infty} d(x_{n+1}, z) = d(x, z)$.

Therefore, condition (2) and (ζ_2) imply that $d(x, z) = 0$. Hence, $d(x, Tx) = d(A, B)$ i.e., x is a best proximity of T. The uniqueness of the best proximity point can be easily seen. □

Example 2.17. *Consider $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$. Let A and B be two subsets of X such that $A = \{3, 4, 6\}$ and $B = \{2, 4.5, 6.5\}$. Define the mappings $T : A \longrightarrow B$ and $\phi, \psi : [0, \infty) \longrightarrow [0, \infty)$ by*

$$T(3) = 2, T(4) = T(6) = 6.5$$

and

$$\phi(m') = m', \psi(m') = \frac{4}{5}m', m' \in [0, \infty).$$

It is clear that $d(A, B) = 0.5$, $A_0 = \{4, 6\}$, $B_0 = \{4.5, 6.5\}$, $T(A_0) \subseteq B_0$ and T satisfies the conditions of Theorem 2.16 with $\zeta(s, t) = \frac{2}{3}t - s$, for all $s, t \in [0, \infty)$. Hence, T has a unique best proximity point, which is clearly 6 here.

3. Application on variational inequality problem

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and $\|.\|$ be the corresponding norm on H. Let Y be a closed and convex subset of H and T be an operator on H. A variational inequality problem is to find $x \in Y$ such that

$$\langle Tx, y - x \rangle \geq 0, \text{ for all } y \in Y.$$

Variational inequalities are in fact, equivalent to the fixed point problems. There is an extensive study on the variational inequality theory ([8], [22]) and it is now an important tool in the study of equilibrium problems. For the origin of variational inequality, one can refer to the works of Stampacchia [26]. Here we obtain solution of variational inequality problem via fixed point formulation of our established type of mappings which reduces to generalized Z-contraction in this case. For this, the following two well known Lemmas will be used.

Lemma 3.1. *[10] Let H be a real Hilbert space and Y be a closed and convex subset of H. For $z \in H$, $x \in Y$ the inequality*

$$\langle x - z, y - x \rangle \geq 0, \text{ for all } y \in Y$$

is satisfied if and only if $x = P_Y z$, where P_Y is the projection of H onto Y.

Lemma 3.2. *[10] Let T be an operator on a real Hilbert space H and Y be a closed and convex subset of H. Then $x \in Y$ is a solution of $\langle Tx, y - x \rangle \geq 0$, for all $y \in Y$ if and only if $x = P_Y(x - \lambda Tx)$, with $\lambda > 0$.*

Theorem 3.3. *Let Y be a closed and convex subset of a real Hilbert space H and $T : H \longrightarrow H$ be a mapping such that $T(Y) \subseteq Y$ and $I - \lambda T$ is a generalized Z-contraction for some $\lambda > 0$ (where I denotes the identity mapping on H). Then the variational inequality problem*

$$\langle Tx, y - x \rangle \geq 0, \text{ for all } y \in Y \tag{3.1}$$

has a solution in Y.

Proof. We define a mapping $f : Y \longrightarrow Y$ such that

$$fy = y - \lambda Ty, \text{ for all } y \in Y.$$

Since, $T(Y) \subseteq Y$, so, $fy = P_Y(I - \lambda T)y$, for all $y \in Y$, where P_Y is the projection of H on Y.
From Lemma 3.2, it is evident that $x \in Y$ is a solution of $\langle Tx, y - x \rangle \geq 0$, for all $y \in Y$ if and only if $x = P_Y(x - \lambda Tx)$ i.e., $fx = x$. Using the hypothesis of the theorem, f is a generalized Z-contraction and so, it has a fixed point, say x. Hence x is a solution of (3.1). $\square$

4. Application on homotopy result

The invariance of fixed points for contraction mappings in metric spaces under homotopy has been an important topic of discussion by several prominent researchers (refer to [2],[20]). In [9], Butt *et al.* derived a homotopy result applying fixed point theorem on ordered metric spaces. Motivated by this, here we establish a homotopy result considering a particular case of Definition 2.1 with

$$M(y_1, y_2, x_1, x_2) = max\{d(y_1, x_1), d(y_2, x_2)\}.$$

we call the above type of mappings as generalized proximal Z-contraction of first kind with restricted M.

Theorem 4.1. *Let V be an open subset of a complete metric space (X, d) and $T : \overline{V} \times [0, 1] \longrightarrow X$ be a generalized proximal Z-contraction of first kind in first variable with restricted M satisfying the following conditions:*

(i) $y \neq T(y, s)$, for all $y \in \partial V$, $s \in [0, 1]$, where ∂V and $\overline{V}$ denote the boundary of V and closure of V respectively,

(ii) there exists a constant $L > 0$ such that

$$d(T(y,s),T(y,s')) \leq L|s-s'|,$$

for all $y \in \overline{V}$ and $s, s' \in [0,1]$,

(iii) there exists a constant $\alpha \in [0,1)$ such that $d(y,y')d(y,T(y,t))d(y',T(y',t)) \neq 0$ implies that

$$d(y,T(y',t)) \leq \alpha d(y,y'),$$

for all $t \in [0,1]$.

If $T(.,0)$ has a fixed point then $T(.,1)$ also has a fixed point in V.

Proof. Suppose

$$A = \{s \in [0,1] : y = T(y,s) \text{ for some } y \in V\}.$$

Since, $T(.,0)$ has a fixed point in V, so, A is non-empty.

We show that A is a closed set. Let $\{s_n\} \subseteq A$ be such that $s_n \to s \in [0,1]$ as $n \to \infty$. Since, $s_n \in A$, so, for each $n \in \mathbb{N}$, there exists $y_n \in V$ such that

$$y_n = T(y_n, s_n).$$

Now,

$$\begin{aligned} d(y_m,y_n) &= d(T(y_m,s_m),T(y_n,s_n)) \\ &\leq d(T(y_m,s_m),T(y_m,s_n)) + d(T(y_m,s_n),T(y_n,s_n)) \\ &\leq L|s_m-s_n| + d(T(y_m,s_n),T(y_n,s_n)) \text{ (using condition (ii))} \end{aligned} \tag{4.1}$$

Since, T is a generalized proximal Z-contraction of first kind in first variable with restricted M, so, for some simulation function $\zeta \in \tau$,

$$\zeta(d(T(y_m,s_n),T(y_n,s_n)), M(y_m,y_n,T(y_m,s_n),T(y_n,s_n))) \geq 0$$

$$\begin{aligned} &\Rightarrow d(T(y_m,s_n),T(y_n,s_n)) < M(y_m,y_n,T(y_m,s_n),T(y_n,s_n)), \\ &\qquad\qquad \text{(using condition } (\zeta_2) \text{ of Definition (1.1))} \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} M(y_m,y_n,T(y_m,s_n),T(y_n,s_n)) &= max\{d(y_m,T(y_m,s_n)), d(y_n,T(y_n,s_n))\} \\ &= d(y_m,T(y_m,s_n)). \end{aligned}$$

Therefore,

$$\begin{aligned} &d(T(y_n,s_n),T(y_m,s_n)) < d(y_m,T(y_m,s_n)) \\ \Rightarrow\ &d(T(y_n,s_n),T(y_m,s_n)) < L|s_m-s_n| \text{ (using condition (ii) of the theorem) }. \end{aligned} \tag{4.3}$$

Using (4.3) in the inequality (4.1), we get,

$$d(y_m,y_n) < 2L|s_m-s_n|$$

So, $\{y_n\}$ is a Cauchy sequence in X, since, $\{s_n\}$ is a Cauchy sequence in $[0,1]$. Therefore, there exists $y \in X$ such that

$$y_n \to y \text{ as } n \to \infty.$$

Next we show that $d(y, T(y,s)) = 0$. If possible, let $d(y, T(y,s)) > 0$.

Now,

$$\begin{aligned}0 &\le \zeta(d(T(y_n, s_n), T(y,s)), M(y_n, y, T(y_n, s_n), T(y,s)))\\ &= \zeta(d(y_n, T(y,s)), M(y_n, y, T(y_n, s_n), T(y,s))),\end{aligned}$$

where

$$M(y_n, y, T(y_n, s_n), T(y,s)) = max\{d(y_n, T(y_n, s_n)), d(y, T(y,s))\}$$

So, $\lim_{n\to\infty} M(y_n, y, T(y_n, s_n), T(y,s)) = d(y, T(y,s)) > 0$.

Also, $\lim_{n\to\infty} d(y_n, T(y,s)) = d(y, T(y,s)) > 0$.

So, condition (ζ_3) gives

$$\limsup_{n\to\infty}\{\zeta(d(y_n, T(y,s)), M(y_n, y, T(y_n, s_n), T(y,s)))\} < 0,$$

which is a contradiction.

Therefore, $d(y, T(y,s)) = 0$ i.e., $y = T(y,s)$. This shows that $s \in A$. Hence A is a closed set.
Finally, we show that A is also an open set in $[0,1]$. Suppose $s^* \in A$. Then there exists $y^* \in V$ such that $y^* = T(y^*, s^*)$. As V is an open subset of X, there exists $r > 0$ such that the open ball $B(y^*, r) \subseteq V$. Consider $s \in (s^* - r, s^* + r)$ and $y' \in \overline{B(y^*, r)} = \{y' \in V : d(y', y^*) \le r\}$.
Using condition (iii) of the theorem, we have,

$$\begin{aligned}d(T(y', s), y^*) &\le \alpha d(y', y^*)\\ &\le \alpha r.\end{aligned}$$

So, $T(y', s) \in \overline{B(y^*, r)}$. Thus $T : \overline{B(y^*, r)} \times [0,1] \to \overline{B(y^*, r)}$ and $T(., s)$ satisfies all the conditions of Theorem 2.8. So, $T(., s)$ has a fixed point in V. Therefore, $(s^* - r, s^* + r) \subseteq A$ and hence A is open set in $[0,1]$. Thus A is open as well as closed set in $[0,1]$ and so by connectedness of [0,1], $A = [0,1]$. Hence $T(., 1)$ has a fixed point in V. □

5. Conclusion

In this paper, we have defined generalized proximal Z-contraction mappings in metric spaces and established the existence of best proximity point for such mappings. Significantly, our established results generalize and improve some existing results in the literature [1,23].In addition, we have discussed variational inequality problem and homotopy result applying our main theorem via fixed point formulation. The obtained results may also be applicable in solving matrix equations through best proximity point via fixed point formulation in line with the work done by Zhenhua *et al.* [21]. To deal with mixed equilibrium problem applying our results is another scope for future discussion.

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