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Modified Special Functions: Properties, Integral Transforms and Applications to Fractional Differential Equations

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ABSTRACT: In this paper, we defined the modified gamma and beta functions, which involving the generalized M-series at their kernels. Then by using the modified beta function we also defined the modified Gauss and confluent hypergeometric functions. Furthermore, we presented some of their properties such that, integral representations, summation formulas and derivative formulas. Also, we applied beta, Mellin, Laplace, Sumudu, Elzaki and general integral transforms to these modified special functions. Moreover, we obtained solutions of fractional differential equations involving the modified special functions, as applications. Finally, we gave the relationships between the modified functions with some of the generalized special functions, which can be found in the literature.

Key Words: Caputo fractional derivative, fractional differential equations, integral transforms, beta function, hypergeometric functions, generalized M-series.

Contents

1. Introduction

Special functions, which is one of the important study areas of applied mathematics, are usually defined with the help of generalized integrals or infinite series. Some of the most important of these functions are gamma, beta, Gauss hypergeometric and confluent hypergeometric functions. In recent years, researchers have published extensively on generalizations of special functions (see for example [2,3,5,6,7,9,10,11,12,13,14,16,18,20,22,23,24,25,26,27,29,30,33,34,35] and reference therein). Researchers, in these studies obtained new generalizations of the gamma function by using the confluent hypergeometric, Mittag-Leffler, Wright and Fox-Wright functions instead of the $\exp(-t)$ term in the integral representation of the gamma function. Then, they made new generalizations within the beta function by using similar functions in the integral representation of the beta function without disturbing the symmetry properties. In addition, the researchers also defined generalizations of Gauss hypergeometric and confluent hypergeometric functions with the help of generalized beta functions.

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Firstly in 1994, Chaudhry and Zubair defined the generalized gamma function [9] for $\Re(\xi) > 0$, $\Re(\rho) > 0$ as

$$\Gamma_\rho(\xi) = \int_0^\infty \Delta^{\xi-1} \exp\left(-\Delta - \frac{\rho}{\Delta}\right) d\Delta.$$

In 1997, Chaudhry et al. introduced the generalized beta function [10] for $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\rho) > 0$ as

$$B(\xi, \eta; \rho) = \int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1} \exp\left(-\frac{\rho}{\Delta(1-\Delta)}\right) d\Delta.$$

In 2004, Chaudhry et al. described the generalized Gauss and confluent hypergeometric functions [11] respectively

$$F_\rho(\chi_1, \chi_2; \chi_3; z) = \sum_{n=0}^\infty (\chi_1)_n \frac{B(\chi_2 + n, \chi_3 - \chi_2; \rho)}{B(\chi_2, \chi_3 - \chi_2)} \frac{z^n}{n!},$$

$$\left(|z| < 1; \Re(\rho) > 0, \Re(\chi_3) > \Re(\chi_2) > 0\right),$$

and

$$\Phi_\rho(\chi_2; \chi_3; z) = \sum_{n=0}^\infty \frac{B(\chi_2 + n, \chi_3 - \chi_2; \rho)}{B(\chi_2, \chi_3 - \chi_2)} \frac{z^n}{n!},$$

$$\left(\Re(\rho) > 0, \Re(\chi_3) > \Re(\chi_2) > 0\right).$$

In 2020, Ata and Kıymaz defined the generalized gamma, beta, Gauss hypergeometric and confluent hypergeometric functions [5] respectively

$${}^\Psi\hat{\Gamma}_\rho(\xi) = {}^\Psi\Gamma_\rho\left[\begin{matrix}(\beta_i, \alpha_i)_{1,p} \\ (\mu_j, \kappa_j)_{1,q}\end{matrix}\,\middle|\xi\right] = \int_0^\infty \Delta^{\xi-1} {}_p\Psi_q\left(-\Delta - \frac{\rho}{\Delta}\right) d\Delta,$$

$$\left(\Re(\xi) > 0, \Re(\rho) > 0\right),$$

$${}^\Psi\hat{B}_\rho(\xi, \eta) = {}^\Psi B_\rho\left[\begin{matrix}(\beta_i, \alpha_i)_{1,p} \\ (\mu_j, \kappa_j)_{1,q}\end{matrix}\,\middle|\xi, \eta\right] = \int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1} {}_p\Psi_q\left(-\frac{\rho}{\Delta(1-\Delta)}\right) d\Delta,$$

$$\left(\Re(\xi) > 0, \Re(\eta) > 0, \Re(\rho) > 0\right),$$

$${}^\Psi\hat{F}_\rho(\chi_1, \chi_2; \chi_3; z) = {}^\Psi F_\rho\left[\begin{matrix}(\beta_i, \alpha_i)_{1,p} \\ (\mu_j, \kappa_j)_{1,q}\end{matrix}\,\middle|\,\chi_1, \chi_2; \chi_3; z\right] = \sum_{n=0}^\infty (\chi_1)_n \frac{{}^\Psi\hat{B}_\rho(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{z^n}{n!},$$

$$\left(|z| < 1; \Re(\rho) > 0, \Re(\chi_3) > \Re(\chi_2) > 0\right),$$

$${}^\Psi\hat{\Phi}_\rho(\chi_2; \chi_3; z) = {}^\Psi\Phi_\rho\left[\begin{matrix}(\beta_i, \alpha_i)_{1,p} \\ (\mu_j, \kappa_j)_{1,q}\end{matrix}\,\middle|\chi_2; \chi_3; z\right] = \sum_{n=0}^\infty \frac{{}^\Psi\hat{B}_\rho(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{z^n}{n!},$$

$$\left(\Re(\rho) > 0, \Re(\chi_3) > \Re(\chi_2) > 0\right),$$

where ${}_p\Psi_q(z)$ is the Fox-Wright function [21], which defined as:

$${}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(\beta_i, \alpha_i)_{1,p} \\ (\mu_j, \kappa_j)_{1,q}\end{matrix}\,\middle|z\right] = \sum_{n=0}^\infty \frac{\prod_{i=1}^p \Gamma(\alpha_i n + \beta_i)}{\prod_{j=1}^q \Gamma(\kappa_j n + \mu_j)} \frac{z^n}{n!},$$

$$\left(z, \beta_i, \mu_j \in \mathbb{C}; \alpha_i, \kappa_j \in \mathbb{R}; i = 1, \ldots, p \text{ and } j = 1, \ldots, q\right).$$

Motivated by all these studies, in this paper, we describe new modified gamma, beta, Gauss hypergeometric and confluent hypergeometric functions by using the generalized M-series [31], which defined by

$$ {}_{p}^{\alpha}M_{q}^{\beta}(z) = {}_{p}^{\alpha}M_{q}^{\beta}(\Lambda_1, \ldots, \Lambda_p; \Omega_1, \ldots, \Omega_q; z) = \sum_{n=0}^{\infty} \frac{(\Lambda_1)_n \ldots (\Lambda_p)_n}{(\Omega_1)_n \ldots (\Omega_q)_n} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.1} $$

where $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$ and $\Re(\alpha) > 0$.

We aimed to use the generalized M-series because it has a more general form than most of the functions mentioned above. Since the generalized M-series contains more parameters, it is thought that the application areas of special functions will expand with the results obtained here.

The remainder of this paper is planned as follows: In Section 2, basic definitions are given. In Section 3, new generalizations of special functions are defined. Some properties of the defined functions were examined, in Section 4. In Section 5, various integral transformations were calculated and as applications of the study, solutions of fractional differential equations involving new generalizations of special functions were obtained using the Laplace transform, in Section 6. Finally, some concluding remarks and further directions of research are discussed in Section 7.

2. Preliminaries

In this section, we have given some basic definitions that are needed throughout this paper, such as special functions, fractional derivative and integral transformations.

The gamma function [4] for $\Re(\xi) > 0$ is defined by

$$ \Gamma(\xi) = \int_0^{\infty} \Delta^{\xi-1} \exp(-\Delta) d\Delta. $$

The beta function [4] for $\Re(\xi) > 0$ and $\Re(\eta) > 0$ is given by

$$ B(\xi, \eta) = \int_0^1 \Delta^{\xi-1} (1-\Delta)^{\eta-1} d\Delta. $$

The Pochhammer symbol [4] for $\Re(\lambda) > -n$, $n \in \mathbb{N}$ and $\lambda \neq 0, -1, -2, \ldots$ is defined by

$$ (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \quad (\lambda)_0 \equiv 1. $$

The Gauss hypergeometric function [21] is given by

$$ {}_2F_1(\chi_1, \chi_2; \chi_3; z) = \sum_{n=0}^{\infty} \frac{(\chi_1)_n (\chi_2)_n}{(\chi_3)_n} \frac{z^n}{n!}, \quad (|z| < 1). $$

The confluent hypergeometric function [21] is defined by

$$ \Phi(\chi_2; \chi_3; z) = \sum_{n=0}^{\infty} \frac{(\chi_2)_n}{(\chi_3)_n} \frac{z^n}{n!}. $$

The Caputo fractional derivative operator [21] of order $\varepsilon \in \mathbb{C}$ for $m-1 < \Re(\varepsilon) < m$, $m \in \mathbb{N}$ is given by

$$ {}^{c}D_{\rho}^{\varepsilon}\{f(\rho)\} = \frac{1}{\Gamma(m-\varepsilon)} \int_0^{\rho} (\rho-\omega)^{m-\varepsilon-1} f^{(m)}(\omega) d\omega, \quad (\Re(\varepsilon) > 0; \rho > 0). $$

The beta transform [32] for $\Re(\omega) > 0$ and $\Re(w) > 0$ is defined by

$$ \mathfrak{B}\{f(\rho); \omega, w\} = \int_0^1 \rho^{\omega-1} (1-\rho)^{w-1} f(\rho) d\rho. $$

The Mellin and inverse Mellin transforms [15] for $s \in \mathbb{C}$ respectively are given by

$$\mathfrak{M}\left\{f(\rho);s\right\} = \hat{f}(s) = \int_0^\infty \rho^{s-1} f(\rho)d\rho,$$

and

$$\mathfrak{M}^{-1}\left\{\hat{f}(s)\right\} = f(\rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \rho^{-s}\hat{f}(s)ds, \quad (c>0).$$

The Laplace and inverse Laplace transforms [15] for $\Re(s) > 0$ respectively are defined by

$$\mathfrak{L}\left\{f(\rho);s\right\} = F(s) = \int_0^\infty \exp(-s\rho)f(\rho)d\rho,$$

and

$$\mathfrak{L}^{-1}\left\{F(s)\right\} = f(\rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(s\rho)F(s)ds, \quad (c>0).$$

Note that the Laplace transform of the Caputo fractional derivative for $m-1 < \Re(\varepsilon) \le m$ as follows [28]:

$$\mathfrak{L}\left\{{}^{c}D_\rho^\varepsilon\left\{f(\rho)\right\};s\right\} = s^\varepsilon F(s) - \sum_{k=0}^{m-1} s^{\varepsilon-k-1} f^{(k)}(0). \tag{2.1}$$

We consider functions in the set A as follows:

$$A = \left\{f(\rho) : \exists M, k_1, k_2 > 0, |f(\rho)| < M\exp\left(\frac{|\rho|}{k_j}\right), \text{ if } \rho \in (-1)^j \times [0,\infty)\right\}.$$

Then, the Sumudu transform [36] depending on the set A is given by

$$S\left\{f(\rho);s\right\} = \frac{1}{s}\int_0^\infty \exp\left(\frac{-\rho}{s}\right)f(\rho)d\rho, \quad \big(s \in (-k_1, k_2)\big),$$

and the Elzaki transform [17] depending on the set A is defined by

$$E\left\{f(\rho);s\right\} = s\int_0^\infty \exp\left(\frac{-\rho}{s}\right)f(\rho)d\rho, \quad \big(s \in [k_1, k_2]\big).$$

The general integral transform [19] for $p(s) \neq 0$ and $q(s)$ are positive real functions is given by

$$T\left\{f(\rho);s\right\} = p(s)\int_0^\infty \exp\big(-q(s)\rho\big)f(\rho)d\rho.$$

3. Modified Special Functions

In this section, we have introduced modified gamma, beta, Gauss hypergeometric and confluent hypergeometric functions involving the generalized M-series (1.1) in their kernels. If $p \le q$ then (1.1) is convergent for all z. If $p = q+1$ it is also convergent for $|z| < \delta = \alpha^\alpha$, but if $p > q+1$ it is divergent. If $p = q+1$, (1.1) can be convergent for $|z| = \delta$ depending on the conditions of the parameters [31].

Definition 3.1. *Let $\Re(\xi) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$ and $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. Then, modified gamma function is defined as:*

$$\begin{aligned} {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\xi;\rho) &= {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(\Lambda_1,\ldots,\Lambda_p;\Omega_1,\ldots,\Omega_q;\xi;\rho) \\ &:= \int_0^\infty \Delta^{\xi-1}\ {}_{p}^{\alpha}M_q^\beta\left(\Lambda_1,\ldots,\Lambda_p;\Omega_1,\ldots,\Omega_q;-\Delta-\frac{\rho}{\Delta}\right)d\Delta. \end{aligned} \tag{3.1}$$

Definition 3.2. *Let $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$ and $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. Then, modified beta function is defined as:*

$$
\begin{aligned}
{}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho) &= {}^{M}B_{p,q}^{(\alpha,\beta)}(\Lambda_1,\ldots,\Lambda_p;\Omega_1,\ldots,\Omega_q;\xi,\eta;\rho) \\
&:= \int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\Lambda_1,\ldots,\Lambda_p;\Omega_1,\ldots,\Omega_q;\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta. \qquad (3.2)
\end{aligned}
$$

Definition 3.3. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$ and $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. Then, modified Gauss hypergeometric function is defined as:*

$$
\begin{aligned}
{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) &= {}^{M}F_{p,q}^{(\alpha,\beta)}(\Lambda_1,\ldots,\Lambda_p;\Omega_1,\ldots,\Omega_q;\chi_1,\chi_2;\chi_3;z;\rho) \\
&:= \sum_{n=0}^{\infty}(\chi_1)_n \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\chi_2+n,\chi_3-\chi_2;\rho)}{B(\chi_2,\chi_3-\chi_2)}\frac{z^n}{n!}, \quad (|z|<1).
\end{aligned}
$$

Definition 3.4. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$, $\Re(\rho) > 0$ and $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. Then, modified confluent hypergeometric function is defined as:*

$$
\begin{aligned}
{}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) &= {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\Lambda_1,\ldots,\Lambda_p;\Omega_1,\ldots,\Omega_q;\chi_2;\chi_3;z;\rho) \\
&:= \sum_{n=0}^{\infty} \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\chi_2+n,\chi_3-\chi_2;\rho)}{B(\chi_2,\chi_3-\chi_2)}\frac{z^n}{n!}.
\end{aligned}
$$

For the sake of shortness, we call modified gamma, modified beta, modified Gauss hypergeometric and modified confluent hypergeometric functions as M-gamma, M-beta, M-Gauss hypergeometric, and M-confluent hypergeometric functions, respectively.

Remark 3.5. *If we put $\rho = 0$ and $p = q = \Lambda_1 = \Omega_1 = \alpha = \beta = 1$ to the M-gamma, M-beta, M-Gauss hypergeometric and M-confluent hypergeometric functions, we get the classical gamma, beta, Gauss hypergeometric and confluent hypergeometric functions [4], respectively.*

4. Properties of Modified Special Functions

In this section, we have presented some properties of the modified gamma, beta, Gauss hypergeometric, and confluent hypergeometric functions.

Theorem 4.1. *Let $\Re(u) > 0$, $\Re(v) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. The relationships of M-gamma functions is obtained as:*

$$
\begin{aligned}
{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(u;\rho)\,{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(v;\rho) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} & r^{2(u+v)-1}\cos^{2\xi-1}(\theta)\sin^{2\eta-1}(\theta) \\
&\times {}_{p}^{\alpha}M_q^{\beta}\left(-r^2\cos^2(\theta)-\frac{\rho}{r^2\cos^2(\theta)}\right) \\
&\times {}_{p}^{\alpha}M_q^{\beta}\left(-r^2\sin^2(\theta)-\frac{\rho}{r^2\sin^2(\theta)}\right) dr d\theta.
\end{aligned}
$$

Proof. Taking $\Delta = \eta^2$ in (3.1), we have

$$
{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(u;\rho) = 2\int_0^{\infty} \eta^{2u-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(-\eta^2-\frac{\rho}{\eta^2}\right) d\eta.
$$

Therefore, we get

$$
{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(u;\rho)\,{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(v;\rho) = 4\int_0^{\infty}\int_0^{\infty} \eta^{2u-1}\xi^{2v-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(-\eta^2-\frac{\rho}{\eta^2}\right){}_{p}^{\alpha}M_q^{\beta}\left(-\xi^2-\frac{\rho}{\xi^2}\right) d\eta d\xi.
$$

Taking $\eta = r\cos(\theta)$ and $\xi = r\sin(\theta)$ in above equation, we obtain

$$\begin{aligned} {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(u;\rho)\,{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(v;\rho) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} & r^{2(u+v)-1}\cos^{2\xi-1}(\theta)\sin^{2\eta-1}(\theta) \\ & \times {}_{p}^{\alpha}M_q^{\beta}\left(-r^2\cos^2(\theta) - \frac{\rho}{r^2\cos^2(\theta)}\right) \\ & \times {}_{p}^{\alpha}M_q^{\beta}\left(-r^2\sin^2(\theta) - \frac{\rho}{r^2\sin^2(\theta)}\right) dr d\theta, \end{aligned}$$

which completes the proof. □

Theorem 4.2. *The integral representations of the M-beta function are obtained as:*

(a) Let $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$${}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho) = 2\int_0^{\frac{\pi}{2}} \sin^{2\xi-1}(\theta)\cos^{2\eta-1}(\theta)\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\sin^2(\theta)\cos^2(\theta)}\right) d\theta.$$

(b) Let $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$${}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho) = \int_0^{\infty} \frac{u^{\xi-1}}{(1+u)^{\xi+\eta}}\ {}_{p}^{\alpha}M_q^{\beta}\left(-2\rho - \rho\left(u + \frac{1}{u}\right)\right) du.$$

(c) Let $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$${}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho) = (b-a)^{1-\xi-\eta}\int_a^b (u-a)^{\xi-1}(b-u)^{\eta-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho(b-a)^2}{(u-a)(b-u)}\right) du.$$

Proof. Taking $\Delta = \sin^2(\theta)$ in (3.2), we get

$$\begin{aligned} {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho) &= \int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta \\ &= 2\int_0^{\frac{\pi}{2}} \sin^{2\xi-1}(\theta)\cos^{2\eta-1}(\theta)\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\sin^2(\theta)\cos^2(\theta)}\right) d\theta. \end{aligned}$$

Taking $\Delta = \frac{u}{1+u}$ in (3.2), we have

$$\begin{aligned} {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho) &= \int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta \\ &= \int_0^{\infty} \frac{u^{\xi-1}}{(1+u)^{\xi+\eta}}\ {}_{p}^{\alpha}M_q^{\beta}\left(-2\rho - \rho\left(u + \frac{1}{u}\right)\right) du. \end{aligned}$$

Taking $\Delta = \frac{u-a}{b-a}$ in (3.2), we obtain

$$\begin{aligned} {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho) &= \int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta \\ &= (b-a)^{1-\xi-\eta}\int_a^b (u-a)^{\xi-1}(b-u)^{\eta-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho(b-a)^2}{(u-a)(b-u)}\right) du, \end{aligned}$$

which completes the proofs. □

Theorem 4.3. *Let $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. The relationships of M-beta functions is obtained as:*

$$ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta+1;\rho) + {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi+1,\eta;\rho) = {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho). $$

Proof. Using equation (3.2), we have

$$ \begin{aligned} &{}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta+1;\rho) + {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi+1,\eta;\rho) \\ &\qquad = \int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta + \int_0^1 \Delta^{\xi}(1-\Delta)^{\eta-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta \\ &\qquad = \int_0^1 \left(\Delta^{\xi-1}(1-\Delta)^{\eta} + \Delta^{\xi}(1-\Delta)^{\eta-1}\right) {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta \\ &\qquad = \int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta \\ &\qquad = {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho), \end{aligned} $$

which completes the proof. □

Theorem 4.4. *Let $\Re(\xi) > 0$, $\Re(1-\eta) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. The summation formula of the M-beta function is obtained as:*

$$ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,1-\eta;\rho) = \sum_{n=0}^{\infty} \frac{(\eta)_n}{n!}\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi+n,1;\rho). $$

Proof. From equation (3.2), we get

$$ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,1-\eta;\rho) = \int_0^1 \Delta^{\xi-1}(1-\Delta)^{-\eta}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta. $$

The binomial series [4] is defined by

$$ (1-\Delta)^{-\eta} = \sum_{n=0}^{\infty} (\eta)_n \frac{\Delta^n}{n!}, \quad (|\Delta| < 1). $$

Considering binomial series, we have

$$ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,1-\eta;\rho) = \sum_{n=0}^{\infty} \frac{(\eta)_n}{n!}\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi+n,1;\rho), $$

which completes the proof. □

Theorem 4.5. *The integral representations of the M-Gauss hypergeometric function are obtained as:*

(a) Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$$ \begin{aligned} {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) = \frac{1}{B(\chi_2,\chi_3-\chi_2)} \int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \\ \times\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta. \end{aligned} \tag{4.1} $$

(b) Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$$ \begin{aligned} {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) = \frac{1}{B(\chi_2,\chi_3-\chi_2)} \int_0^{\infty} u^{\chi_2-1}(1+u)^{\chi_1-\chi_3}\left(1+u(1-z)\right)^{-\chi_1} \\ \times\ {}_{p}^{\alpha}M_q^{\beta}\left(-2\rho - \rho\left(u+\frac{1}{u}\right)\right) du. \end{aligned} $$

(c) Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$$
\begin{aligned}
{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) = {} & \frac{2}{B(\chi_2,\chi_3-\chi_2)} \int_0^{\frac{\pi}{2}} \sin^{2\chi_2-1}(\theta)\cos^{2\chi_3-2\chi_2-1}(\theta)\left(1-z\sin^2(\theta)\right)^{-\chi_1} \\
& \times {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\sin^2(\theta)\cos^2(\theta)}\right) d\theta.
\end{aligned}
$$

(d) Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$$
\begin{aligned}
{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) = {} & \frac{(b-a)^{1-\chi_3}}{B(\chi_2,\chi_3-\chi_2)} \int_a^b (u-a)^{\chi_2-1}(b-u)^{\chi_3-\chi_2-1}\left(1-\frac{z(u-a)}{b-a}\right)^{-\chi_1} \\
& \times {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho(b-a)^2}{(u-a)(b-u)}\right) du.
\end{aligned}
$$

Proof. Considering binomial series, we have

$$
\begin{aligned}
{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) &= \sum_{n=0}^{\infty} (\chi_1)_n \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\chi_2+n,\chi_3-\chi_2;\rho)}{B(\chi_2,\chi_3-\chi_2)} \frac{z^n}{n!} \\
&= \frac{1}{B(\chi_2,\chi_3-\chi_2)} \int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1}\, {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta.
\end{aligned}
$$

Taking $\Delta = \frac{u}{1+u}$ in (4.1), we get

$$
\begin{aligned}
{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) = {} & \frac{1}{B(\chi_2,\chi_3-\chi_2)} \int_0^{\infty} u^{\chi_2-1}(1+u)^{\chi_1-\chi_3}\left(1+u(1-z)\right)^{-\chi_1} \\
& \times {}_{p}^{\alpha}M_q^{\beta}\left(-2\rho-\rho\left(u+\frac{1}{u}\right)\right) du.
\end{aligned}
$$

Taking $\Delta = \sin^2(\theta)$ in (4.1), we obtain

$$
\begin{aligned}
{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) = {} & \frac{2}{B(\chi_2,\chi_3-\chi_2)} \int_0^{\frac{\pi}{2}} \sin^{2\chi_2-1}(\theta)\cos^{2\chi_3-2\chi_2-1}(\theta)\left(1-z\sin^2(\theta)\right)^{-\chi_1} \\
& \times {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\sin^2(\theta)\cos^2(\theta)}\right) d\theta.
\end{aligned}
$$

Taking $\Delta = \frac{u-a}{b-a}$ in (4.1), we have

$$
\begin{aligned}
{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) = {} & \frac{(b-a)^{1-\chi_3}}{B(\chi_2,\chi_3-\chi_2)} \int_a^b (u-a)^{\chi_2-1}(b-u)^{\chi_3-\chi_2-1}\left(1-\frac{z(u-a)}{b-a}\right)^{-\chi_1} \\
& \times {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho(b-a)^2}{(u-a)(b-u)}\right) du,
\end{aligned}
$$

which completes the proofs. □

Theorem 4.6. *The integral representations of the M-confluent hypergeometric function are obtained as:*

(a) Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$$
{}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) = \frac{1}{B(\chi_2,\chi_3-\chi_2)} \int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}\exp(z\Delta)\, {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right) d\Delta.
$$

(b) Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$$
{}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) = \frac{1}{B(\chi_2,\chi_3-\chi_2)} \int_0^1 u^{\chi_3-\chi_2-1}(1-u)^{\chi_2-1}\exp\left(z(1-u)\right)\, {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{u(1-u)}\right) du.
$$

(c) Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$$ {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) = \frac{1}{B(\chi_2,\chi_3-\chi_2)} \int_0^\infty u^{\chi_2-1}(1+u)^{-\chi_3} \exp\left(\frac{zu}{1+u}\right) \times {}_p^{\alpha}M_q^{\beta}\left(-2\rho-\rho\left(u+\frac{1}{u}\right)\right) du. $$

(d) Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$$ {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) = \frac{2}{B(\chi_2,\chi_3-\chi_2)} \int_0^{\frac{\pi}{2}} \sin^{2\chi_2-1}(\theta)\cos^{2\chi_3-2\chi_2-1}\exp\left(z\sin^2(\theta)\right) \times {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\sin^2(\theta)\cos^2(\theta)}\right) d\theta. $$

(e) Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. Then,

$$ {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) = \frac{(b-a)^{1-\chi_3}}{B(\chi_2,\chi_3-\chi_2)} \int_a^b (u-a)^{\chi_2-1}(b-u)^{\chi_3-\chi_2-1}\exp\left(\frac{z(u-a)}{b-a}\right) \times {}_p^{\alpha}M_q^{\beta}\left(\frac{-\rho(b-a)^2}{(u-a)(b-u)}\right) du. $$

Proof. With similar calculations, desired results are obtained. □

Note 1. *The following beta function and Pochhammer symbol equations holds true:*

$$ B(\chi_2,\chi_3-\chi_2) = \frac{\chi_3}{\chi_2}B(\chi_2+1,\chi_3-\chi_2), $$
$$ (\chi_1)_{n+1} = \chi_1(\chi_1+1)_n. $$

These equations are used in proof of two theorems given below.

Theorem 4.7. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. The n-order derivative of the M-Gauss hypergeometric function is obtained as:*

$$ \frac{d^n}{dz^n}\left\{{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho)\right\} = \frac{(\chi_1)_n(\chi_2)_n}{(\chi_3)_n}\left({}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1+n,\chi_2+n;\chi_3+n;z;\rho)\right). $$

Proof. Differentiating M-Gauss hypergeometric function, we have

$$ \frac{d}{dz}\left\{{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho)\right\} = \frac{d}{dz}\left\{\sum_{n=0}^{\infty}(\chi_1)_n \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\chi_2+n,\chi_3-\chi_2;\rho)}{B(\chi_2,\chi_3-\chi_2)}\frac{z^n}{n!}\right\} = \sum_{n=1}^{\infty}(\chi_1)_n \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\chi_2+n,\chi_3-\chi_2;\rho)}{B(\chi_2,\chi_3-\chi_2)}\frac{z^{n-1}}{(n-1)!}. $$

By writing $n \to n+1$, we get

$$ \frac{d}{dz}\left\{{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho)\right\} = \frac{(\chi_1)(\chi_2)}{(\chi_3)}\sum_{n=0}^{\infty}(\chi_1+1)_n \frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\chi_2+n+1,\chi_3-\chi_2;\rho)}{B(\chi_2+1,\chi_3-\chi_2)}\frac{z^n}{n!} = \frac{(\chi_1)(\chi_2)}{(\chi_3)}\left({}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1+1,\chi_2+1;\chi_3+1;z;\rho)\right). $$

By the inductive method, the more general form is obtained as:

$$ \frac{d^n}{dz^n}\left\{{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho)\right\} = \frac{(\chi_1)_n(\chi_2)_n}{(\chi_3)_n}\left({}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1+n,\chi_2+n;\chi_3+n;z;\rho)\right), $$

which completes the proof. □

Theorem 4.8. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. The n-order derivative of the M-confluent hypergeometric function is obtained as:*

$$\frac{d^n}{dz^n}\left\{{}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho)\right\} = \frac{(\chi_2)_n}{(\chi_3)_n}\left({}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2+n;\chi_3+n;z;\rho)\right).$$

Proof. By making similar calculations, the desired result is achieved. □

Theorem 4.9. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. The transformation formula of the M-Gauss hypergeometric function is obtained as:*

$${}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) = (1-z)^{-\chi_1}\left({}^{M}F_{p,q}^{(\alpha,\beta)}\left(\chi_1,\chi_3-\chi_2;\chi_3;\frac{z}{z-1};\rho\right)\right).$$

Proof. Using equation

$$\left(1-z(1-\Delta)\right)^{-\chi_1} = (1-z)^{-\chi_1}\left(1+\frac{z\Delta}{1-z}\right)^{-\chi_1}$$

and by writing $\Delta \to 1-\Delta$ in (4.1), we obtain

$$\begin{aligned}
{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) &= \frac{(1-z)^{-\chi_1}}{B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_3-\chi_2-1}(1-\Delta)^{\chi_2-1}\left(1-\frac{z\Delta}{z-1}\right)^{-\chi_1} \\
&\quad \times {}_{p}^{\alpha}M_{q}^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)d\Delta \\
&= (1-z)^{-\chi_1}\left({}^{M}F_{p,q}^{(\alpha,\beta)}\left(\chi_1,\chi_3-\chi_2;\chi_3;\frac{z}{z-1}\right)\right),
\end{aligned}$$

which completes the proof. □

Theorem 4.10. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(\rho) > 0$. The transformation formula of the M-confluent hypergeometric function is obtained as:*

$${}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) = \exp(z)\left({}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_3-\chi_2;\chi_3;-z;\rho)\right).$$

Proof. Using definition M-confluent hypergeometric function, we have

$$\begin{aligned}
{}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) &= \sum_{n=0}^{\infty}\frac{{}^{M}B_{p,q}^{(\alpha,\beta)}(\chi_2+n,\chi_3-\chi_2;\rho)}{B(\chi_2,\chi_3-\chi_2)}\frac{z^n}{n!} \\
&= \frac{1}{B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}\exp(z\Delta)\ {}_{p}^{\alpha}M_{q}^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)d\Delta.
\end{aligned}$$

By writing $\Delta \to 1-\Delta$ in above equation, we obtain

$${}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) = \exp(z)\left({}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_3-\chi_2;\chi_3;-z;\rho)\right),$$

which completes the proof. □

5. Application of Integral Transforms to Modified Special Functions

In this section, we have applied the beta, Mellin, Laplace, Sumudu, Elzaki and general integral transforms to the modified beta, Gauss hypergeometric and confluent hypergeometric functions.

5.1. Beta Transform

Theorem 5.1. *Let* $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(w) > 0$, $\Re(\omega) > 0$ *and* $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. *The beta transform of the M-beta function is obtained as:*

$$\mathfrak{B}\left\{ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho);\omega,w \right\} = B(\omega,w)\ {}^{M}B_{p+1,q+1}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,\omega;\Omega_1,\ldots,\Omega_q,\omega+w;\xi,\eta;1\right).$$

Proof. Using beta transform, we have

$$\begin{aligned}
&\mathfrak{B}\left\{ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho);\omega,w \right\} \\
&= \int_0^1 \rho^{\omega-1}(1-\rho)^{w-1}\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho)d\rho \\
&= B(\omega,w)\int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1}\ {}_{p+1}^{\ \alpha}M_{q+1}^{\beta}\left(\Lambda_1,\ldots,\Lambda_p,\omega;\Omega_1,\ldots,\Omega_q,\omega+w;\frac{-1}{\Delta(1-\Delta)}\right)d\Delta \\
&= B(\omega,w)\ {}^{M}B_{p+1,q+1}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,\omega;\Omega_1,\ldots,\Omega_q,\omega+w;\xi,\eta;1\right),
\end{aligned}$$

which completes the proof. □

Theorem 5.2. *Let* $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$, $\Re(w) > 0$, $\Re(\omega) > 0$ *and* $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. *The beta transform of the M-Gauss hypergeometric function is obtained as:*

$$\begin{aligned}
&\mathfrak{B}\left\{ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho);\omega,w \right\} \\
&= B(\omega,w)\ {}^{M}F_{p+1,q+1}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,\omega;\Omega_1,\ldots,\Omega_q,\omega+w;\chi_1,\chi_2;\chi_3;z;1\right).
\end{aligned}$$

Proof. Using beta transform, we get

$$\begin{aligned}
&\mathfrak{B}\{ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho);\omega,w \} \\
&= \int_0^1 \rho^{\omega-1}(1-\rho)^{w-1}\ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho)d\rho \\
&= \frac{B(\omega,w)}{B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \\
&\quad \times {}_{p+1}^{\ \alpha}M_{q+1}^{\beta}\left(\Lambda_1,\ldots,\Lambda_p,\omega;\Omega_1,\ldots,\Omega_q,\omega+w;\frac{-1}{\Delta(1-\Delta)}\right)d\Delta \\
&= B(\omega,w)\ {}^{M}F_{p+1,q+1}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,\omega;\Omega_1,\ldots,\Omega_q,\omega+w;\chi_1,\chi_2;\chi_3;z;1\right),
\end{aligned}$$

which completes the proof. □

Theorem 5.3. *Let* $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$, $\Re(w) > 0$, $\Re(\omega) > 0$ *and* $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. *The beta transform of the M-confluent hypergeometric function is obtained as:*

$$\mathfrak{B}\left\{ {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho);\omega,w \right\} = B(\omega,w)\ {}^{M}\Phi_{p+1,q+1}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,\omega;\Omega_1,\ldots,\Omega_q,\omega+w;\chi_2;\chi_3;z;1\right).$$

Proof. By making similar calculations, the desired result is achieved. □

5.2. Mellin Transform

Theorem 5.4. *Let* $\Re(\xi+s) > 0$, $\Re(\eta+s) > 0$, $\Re(\alpha) > 0$ *and* $\Re(s) > 0$. *The Mellin transform of the M-beta function is obtained as:*

$$\mathfrak{M}\left\{ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho);s \right\} = B(\xi+s,\eta+s)\ {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(s;0).$$

Proof. Using Mellin transform, we have

$$\begin{aligned}
&\mathfrak{M}\left\{{}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho);s\right\} \\
&= \int_0^\infty \rho^{s-1}\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho)d\rho \\
&= \int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1}\int_0^\infty \rho^{s-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\Lambda_1,\ldots,\Lambda_p;\Omega_1,\ldots,\Omega_q;\frac{-\rho}{\Delta(1-\Delta)}\right)d\rho d\Delta.
\end{aligned}$$

In the second integral, by substituting $k=\frac{\rho}{\Delta(1-\Delta)}$, we have

$$\begin{aligned}
\mathfrak{M}\left\{{}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho);s\right\} &= \int_0^1 \Delta^{\xi+s-1}(1-\Delta)^{\eta+s-1}d\Delta\int_0^\infty k^{s-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\Lambda_1,\ldots,\Lambda_p;\Omega_1,\ldots,\Omega_q;-k\right)dk \\
&= B(\xi+s,\eta+s)\ {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(s;0),
\end{aligned}$$

which completes the proof. □

Corollary 5.5. *If the inverse Mellin transform for $c>0$ is applied to both sides of the last equation above, we get the following formula:*

$${}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} B(\xi+s,\eta+s)\ {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(s;0)\rho^{-s}ds.$$

Theorem 5.6. *Let $\Re(\chi_3)>\Re(\chi_2)>0$, $\Re(\alpha)>0$ and $\Re(s)>0$. The Mellin transform of the M-Gauss hypergeometric function is obtained as:*

$$\mathfrak{M}\left\{{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho);s\right\} = \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(s)B(\chi_2+s,\chi_3+s-\chi_2)}{B(\chi_2,\chi_3-\chi_2)}\,{}_2F_1(\chi_1,\chi_2+s;\chi_3+2s;z).$$

Proof. Using Mellin transform, we obtain

$$\begin{aligned}
&\mathfrak{M}\{{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho);s\} \\
&= \int_0^\infty \rho^{s-1}\ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho)d\rho \\
&= \frac{1}{B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1}\int_0^\infty \rho^{s-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)d\rho d\Delta.
\end{aligned}$$

In second integral, by substituting $k=\frac{\rho}{\Delta(1-\Delta)}$, we have

$$\int_0^\infty \rho^{s-1}\ {}_{p}^{\alpha}M_q^{\beta}\left(\frac{-\rho}{\Delta(1-\Delta)}\right)d\rho = \Delta^s(1-\Delta)^s\ {}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(s;0).$$

Thus, we get

$$\mathfrak{M}\left\{{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho);s\right\} = \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(s;0)B(\chi_2+s,\chi_3+s-\chi_2)}{B(\chi_2,\chi_3-\chi_2)}\ {}_2F_1(\chi_1,\chi_2+s;\chi_3+2s;z),$$

which completes the proof. □

Corollary 5.7. *If the inverse Mellin transform for $c>0$ is applied to both sides of the last equation above, we get the following formula:*

$$\begin{aligned}
{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} &\frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(s;0)B(\chi_2+s,\chi_3+s-\chi_2)}{B(\chi_2,\chi_3-\chi_2)} \\
&\times {}_2F_1(\chi_1,\chi_2+s;\chi_3+2s;z)\rho^{-s}ds.
\end{aligned}$$

Theorem 5.8. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ and $\Re(s) > 0$. The Mellin transform of the M-confluent hypergeometric function is obtained as:*

$$\mathfrak{M}\left\{ {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho);s\right\} = \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(s;0)B(\chi_2+s,\chi_3+s-\chi_2)}{B(\chi_2,\chi_3-\chi_2)}\Phi(\chi_2+s;\chi_3+2s;z).$$

Proof. By making similar calculations, the desired result is achieved. □

Corollary 5.9. *If the inverse Mellin transform for $c > 0$ is applied to both sides of the last equation above, we get the following formula:*

$${}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{{}^{M}\Gamma_{p,q}^{(\alpha,\beta)}(s;0)B(\chi_2+s,\chi_3+s-\chi_2)}{B(\chi_2,\chi_3-\chi_2)}\Phi(\chi_2+s;\chi_3+2s;z)\rho^{-s}ds.$$

5.3. Laplace Transform

Theorem 5.10. *Let $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(s) > 0$ and $\Lambda_1,\ldots,\Lambda_p,\Omega_1,\ldots,\Omega_q \neq 0,-1,-2,\ldots$. The Laplace transform of the M-beta function is obtained as:*

$$\mathfrak{L}\left\{ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho);s\right\} = \frac{1}{s}\left({}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\xi,\eta;\frac{1}{s}\right)\right).$$

Proof. Using Laplace transform, we have

$$\begin{aligned}\mathfrak{L}\left\{ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho);s\right\} &= \int_0^\infty \exp(-s\rho)\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho)d\rho \\ &= \frac{1}{s}\int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1}{}_{p+1}^{\ \alpha}M_q^{\beta}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\frac{-\frac{1}{s}}{\Delta(1-\Delta)}\right)d\Delta \\ &= \frac{1}{s}\left({}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\xi,\eta;\frac{1}{s}\right)\right),\end{aligned}$$

which completes the proof. □

Corollary 5.11. *If the inverse Laplace transform for $c > 0$ is applied to both sides of the last equation above, we get the following formula:*

$${}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp(s\rho)}{s}\ {}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\xi,\eta;\frac{1}{s}\right)ds.$$

Theorem 5.12. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$, $\Re(s) > 0$ and $\Lambda_1,\ldots,\Lambda_p,\Omega_1,\ldots,\Omega_q \neq 0,-1,-2,\ldots$. The Laplace transform of the M-Gauss hypergeometric function is obtained as:*

$$\mathfrak{L}\left\{ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho);s\right\} = \frac{1}{s}\left({}^{M}F_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\chi_1,\chi_2;\chi_3;z;\frac{1}{s}\right)\right).$$

Proof. Using Laplace transform, we get

$$\begin{aligned}\mathfrak{L}\{ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho);s\} &= \int_0^\infty \exp(-s\rho)\ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho)d\rho \\ &= \frac{1}{s}\frac{1}{B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \\ &\quad \times {}_{p+1}^{\ \alpha}M_q^{\beta}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\frac{-\frac{1}{s}}{\Delta(1-\Delta)}\right)d\Delta \\ &= \frac{1}{s}\left({}^{M}F_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\chi_1,\chi_2;\chi_3;z;\frac{1}{s}\right)\right),\end{aligned}$$

which completes the proof. □

Corollary 5.13. *If the inverse Laplace transform for $c>0$ is applied to both sides of the last equation above, we get the following formula:*

$$ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(s\rho)}{s}\ {}^{M}F_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\chi_1,\chi_2;\chi_3;z;\frac{1}{s}\right)ds. $$

Theorem 5.14. *Let $\Re(\chi_3)>\Re(\chi_2)>0$, $\Re(\alpha)>0$, $\Re(s)>0$ and $\Lambda_1,\ldots,\Lambda_p,\Omega_1,\ldots,\Omega_q\neq 0,-1,-2,\ldots$. The Laplace transform of the M-confluent hypergeometric function is obtained as:*

$$ \mathfrak{L}\left\{{}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho);s\right\}=\frac{1}{s}\left({}^{M}\Phi_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\chi_2;\chi_3;z;\frac{1}{s}\right)\right). $$

Proof. By making similar calculations, the desired result is achieved. □

Corollary 5.15. *If the inverse Laplace transform for $c>0$ is applied to both sides of the last equation above, we get the following formula:*

$$ {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(s\rho)}{s}\ {}^{M}\Phi_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\chi_2;\chi_3;z;\frac{1}{s}\right)ds. $$

5.4. Sumudu Transform

Theorem 5.16. *Let $\Re(\xi)>0$, $\Re(\eta)>0$, $\Re(\alpha)>0$, $\Re(s)>0$ and $\Lambda_1,\ldots,\Lambda_p,\Omega_1,\ldots,\Omega_q\neq 0,-1,-2,\ldots$. The Sumudu transform of the M-beta function is obtained as:*

$$ S\left\{{}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho);s\right\}={}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\xi,\eta;s\right). $$

Proof. Using Sumudu transform, we have

$$ \begin{aligned} S\left\{{}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho);s\right\}&=\frac{1}{s}\int_0^\infty\exp\left(\frac{-\rho}{s}\right)\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho)d\rho\\ &=\int_0^1\Delta^{\xi-1}(1-\Delta)^{\eta-1}{}_{p+1}^{\ \alpha}M_q^{\beta}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\frac{-s}{\Delta(1-\Delta)}\right)d\Delta\\ &={}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\xi,\eta;s\right), \end{aligned} $$

which completes the proof. □

Theorem 5.17. *Let $\Re(\chi_3)>\Re(\chi_2)>0$, $\Re(\alpha)>0$, $\Re(s)>0$ and $\Lambda_1,\ldots,\Lambda_p,\Omega_1,\ldots,\Omega_q\neq 0,-1,-2,\ldots$. The Sumudu transform of the M-Gauss hypergeometric function is obtained as:*

$$ S\left\{{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho);s\right\}={}^{M}F_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\chi_1,\chi_2;\chi_3;z;s\right). $$

Proof. Using Sumudu transform, we get

$$ \begin{aligned} S\{{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho);s\}&=\frac{1}{s}\int_0^\infty\exp\left(\frac{-\rho}{s}\right)\ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho)d\rho\\ &=\frac{1}{B(\chi_2,\chi_3-\chi_2)}\int_0^1\Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1}\\ &\quad\times{}_{p+1}^{\ \alpha}M_q^{\beta}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\frac{-s}{\Delta(1-\Delta)}\right)d\Delta\\ &={}^{M}F_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\chi_1,\chi_2;\chi_3;z;s\right), \end{aligned} $$

which completes the proof. □

Theorem 5.18. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$, $\Re(s) > 0$ and $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. The Sumudu transform of the M-confluent hypergeometric function is obtained as:*

$$S\left\{{}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho);s\right\} = {}^{M}\Phi_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\chi_2;\chi_3;z;s\right).$$

Proof. By making similar calculations, the desired result is achieved. □

5.5. Elzaki Transform

Theorem 5.19. *Let $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Re(s) > 0$ and $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. The Elzaki transform of the M-beta function is obtained as:*

$$E\left\{{}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho);s\right\} = s^2\left({}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\xi,\eta;s\right)\right).$$

Proof. Using Elzaki transform, we have

$$\begin{aligned}
E\left\{{}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho);s\right\} &= s\int_0^\infty \exp\left(\frac{-\rho}{s}\right)\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho)d\rho \\
&= s^2\int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1}\ {}_{p+1}^{\ \alpha}M_q^{\beta}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\frac{-s}{\Delta(1-\Delta)}\right)d\Delta \\
&= s^2\left({}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\xi,\eta;s\right)\right),
\end{aligned}$$

which completes the proof. □

Theorem 5.20. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$, $\Re(s) > 0$ and $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. The Elzaki transform of the M-Gauss hypergeometric function is obtained as:*

$$E\left\{{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho);s\right\} = s^2\left({}^{M}F_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\chi_1,\chi_2;\chi_3;z;s\right)\right).$$

Proof. Using Elzaki transform, we get

$$\begin{aligned}
E\{{}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho);s\} &= s\int_0^\infty \exp\left(\frac{-\rho}{s}\right)\ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho)d\rho \\
&= \frac{s^2}{B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \\
&\quad \times {}_{p+1}^{\ \alpha}M_q^{\beta}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\frac{-s}{\Delta(1-\Delta)}\right)d\Delta \\
&= s^2\left({}^{M}F_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\chi_1,\chi_2;\chi_3;z;s\right)\right),
\end{aligned}$$

which completes the proof. □

Theorem 5.21. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$, $\Re(s) > 0$ and $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. The Elzaki transform of the M-confluent hypergeometric function is obtained as:*

$$E\left\{{}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho);s\right\} = s^2\left({}^{M}\Phi_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1,\ldots,\Lambda_p,1;\Omega_1,\ldots,\Omega_q;\chi_2;\chi_3;z;s\right)\right).$$

Proof. By making similar calculations, the desired result is achieved. □

5.6. General Integral Transform

Theorem 5.22. *Let $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$, $\Lambda_1, \ldots, \Lambda_p$, $\Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$ and $p(s)$, $q(s)$ are positive real functions. The general integral transform of the M-beta function is obtained as:*

$$T\left\{ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho); s\right\} = \frac{p(s)}{q(s)} \left({}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \xi, \eta; \frac{1}{q(s)}\right)\right).$$

Proof. Using general integral transform, we have

$$\begin{aligned} T\left\{ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho); s\right\} &= p(s)\int_0^\infty \exp\big(-q(s)\rho\big)\ {}^{M}B_{p,q}^{(\alpha,\beta)}(\xi,\eta;\rho)d\rho \\ &= \frac{p(s)}{q(s)}\int_0^1 \Delta^{\xi-1}(1-\Delta)^{\eta-1}\ {}_{p+1}^{\ \alpha}M_q^{\beta}\left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \frac{-\frac{1}{q(s)}}{\Delta(1-\Delta)}\right)d\Delta \\ &= \frac{p(s)}{q(s)}\left({}^{M}B_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \xi, \eta; \frac{1}{q(s)}\right)\right), \end{aligned}$$

which completes the proof. □

Theorem 5.23. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$, $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$ and $p(s)$, $q(s)$ are positive real functions. The general integral transform of the M-Gauss hypergeometric function is obtained as:*

$$T\left\{ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho); s\right\} = \frac{p(s)}{q(s)} \left({}^{M}F_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \chi_1, \chi_2; \chi_3; z; \frac{1}{q(s)}\right)\right).$$

Proof. Using general integral transform, we get

$$\begin{aligned} T\{ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho); s\} &= p(s)\int_0^\infty \exp\big(-q(s)\rho\big)\ {}^{M}F_{p,q}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z;\rho)d\rho \\ &= \frac{p(s)}{q(s)}\frac{1}{B(\chi_2,\chi_3-\chi_2)}\int_0^1 \Delta^{\chi_2-1}(1-\Delta)^{\chi_3-\chi_2-1}(1-z\Delta)^{-\chi_1} \\ &\quad \times {}_{p+1}^{\ \alpha}M_q^{\beta}\left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \frac{-\frac{1}{q(s)}}{\Delta(1-\Delta)}\right)d\Delta \\ &= \frac{p(s)}{q(s)}\left({}^{M}F_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \chi_1, \chi_2; \chi_3; z; \frac{1}{q(s)}\right)\right), \end{aligned}$$

which completes the proof. □

Theorem 5.24. *Let $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$, $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$ and $p(s)$, $q(s)$ are positive real functions. The general integral transform of the M-confluent hypergeometric function is obtained as:*

$$T\left\{ {}^{M}\Phi_{p,q}^{(\alpha,\beta)}(\chi_2;\chi_3;z;\rho); s\right\} = \frac{p(s)}{q(s)} \left({}^{M}\Phi_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \chi_2; \chi_3; z; \frac{1}{q(s)}\right)\right).$$

Proof. By making similar calculations, the desired result is achieved. □

6. Solution of Fractional Differential Equations Involving Modified Special Functions

In this section, we have obtained the solution of fractional differential equations involving the modified special functions.

Example 6.1. *Let* $1 < \Re(\varepsilon) \leq 2$, $\Re(\xi) > 0$, $\Re(\eta) > 0$, $\Re(\alpha) > 0$ *and* $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. *Assume that, the fractional differential equation*

$$ {}^cD_\rho^\varepsilon \{f(\rho)\} = {}^MB_{p,q}^{(\alpha,\beta)} (\Lambda_1, \ldots, \Lambda_p; \Omega_1, \ldots, \Omega_q; \xi, \eta; \varepsilon\rho) $$

and initial conditions

$$ f(0) = f'(0) = 0 $$

are given. Considering equation (2.1) and applying Laplace transform to both sides of the fractional differential equation, we have

$$ \mathfrak{L}\left\{ {}^cD_\rho^\varepsilon \{f(\rho)\}; s\right\} = \mathfrak{L}\left\{ {}^MB_{p,q}^{(\alpha,\beta)} (\Lambda_1, \ldots, \Lambda_p; \Omega_1, \ldots, \Omega_q; \xi, \eta; \varepsilon\rho); s\right\}, $$

and then

$$ s^\varepsilon F(s) - s^{\varepsilon-1} f(0) - s^{\varepsilon-2} f'(0) = \frac{1}{s}\left({}^MB_{p+1,q}^{(\alpha,\beta)} \left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \xi, \eta; \frac{\varepsilon}{s}\right)\right). $$

By using initial conditions, we get

$$ F(s) = \frac{1}{s^{\varepsilon+1}}\left({}^MB_{p+1,q}^{(\alpha,\beta)} \left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \xi, \eta; \frac{\varepsilon}{s}\right)\right). $$

Applying inverse Laplace transform to both sides of the last equation and by making the necessary calculations, we obtain the solution function as:

$$ f(\rho) = \frac{\rho^\varepsilon}{\Gamma(1+\varepsilon)}\left({}^MB_{p+1,q+1}^{(\alpha,\beta)} (\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q, 1+\varepsilon; \xi, \eta; \varepsilon\rho)\right). $$

Example 6.2. *Let* $1 < \Re(\varepsilon) \leq 2$, $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ *and* $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. *Assume that, the fractional differential equation*

$$ {}^cD_\rho^\varepsilon \{f(\rho)\} = {}^MF_{p,q}^{(\alpha,\beta)} (\Lambda_1, \ldots, \Lambda_p; \Omega_1, \ldots, \Omega_q; \chi_1, \chi_2; \chi_3; z; \varepsilon\rho) $$

and initial conditions

$$ f(0) = f'(0) = 0 $$

are given. Considering equation (2.1) and applying Laplace transform to both sides of the fractional differential equation, we have

$$ \mathfrak{L}\left\{ {}^cD_\rho^\varepsilon \{f(\rho)\}; s\right\} = \mathfrak{L}\left\{ {}^MF_{p,q}^{(\alpha,\beta)} (\Lambda_1, \ldots, \Lambda_p; \Omega_1, \ldots, \Omega_q; \chi_1, \chi_2; \chi_3; z; \varepsilon\rho); s\right\}, $$

and then

$$ s^\varepsilon F(s) - s^{\varepsilon-1} f(0) - s^{\varepsilon-2} f'(0) = \frac{1}{s}\left({}^MF_{p+1,q}^{(\alpha,\beta)} \left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \chi_1, \chi_2; \chi_3; z; \frac{\varepsilon}{s}\right)\right). $$

By using initial conditions, we get

$$ F(s) = \frac{1}{s^{\varepsilon+1}}\left({}^MF_{p+1,q}^{(\alpha,\beta)} \left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \chi_1, \chi_2; \chi_3; z; \frac{\varepsilon}{s}\right)\right). $$

Applying inverse Laplace transform to both sides of the last equation and by making the necessary calculations, we obtain the solution function as:

$$ f(\rho) = \frac{\rho^\varepsilon}{\Gamma(1+\varepsilon)}\left({}^MF_{p+1,q+1}^{(\alpha,\beta)} (\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q, 1+\varepsilon; \chi_1, \chi_2; \chi_3; z; \varepsilon\rho)\right). $$

Example 6.3. *Let* $1 < \Re(\varepsilon) \leq 2$, $\Re(\chi_3) > \Re(\chi_2) > 0$, $\Re(\alpha) > 0$ *and* $\Lambda_1, \ldots, \Lambda_p, \Omega_1, \ldots, \Omega_q \neq 0, -1, -2, \ldots$. *Assume that, the fractional differential equation*

$$^{c}D_{\rho}^{\varepsilon}\left\{f(\rho)\right\} = {}^{M}\Phi_{p,q}^{(\alpha,\beta)}\left(\Lambda_1, \ldots, \Lambda_p; \Omega_1, \ldots, \Omega_q; \chi_2; \chi_3; z; \varepsilon\rho\right)$$

and initial conditions

$$f(0) = f'(0) = 0$$

are given. Considering equation (2.1) and applying Laplace transform to both sides of the fractional differential equation, we have

$$\mathfrak{L}\left\{^{c}D_{\rho}^{\varepsilon}\left\{f(\rho)\right\}; s\right\} = \mathfrak{L}\left\{^{M}\Phi_{p,q}^{(\alpha,\beta)}\left(\Lambda_1, \ldots, \Lambda_p; \Omega_1, \ldots, \Omega_q; \chi_2; \chi_3; z; \varepsilon\rho\right); s\right\},$$

and then

$$s^{\varepsilon}F(s) - s^{\varepsilon-1}f(0) - s^{\varepsilon-2}f'(0) = \frac{1}{s}\left({}^{M}\Phi_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \chi_2; \chi_3; z; \frac{\varepsilon}{s}\right)\right).$$

By using initial conditions, we get

$$F(s) = \frac{1}{s^{\varepsilon+1}}\left({}^{M}\Phi_{p+1,q}^{(\alpha,\beta)}\left(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q; \chi_2; \chi_3; z; \frac{\varepsilon}{s}\right)\right).$$

Applying inverse Laplace transform to both sides of the last equation and by making the necessary calculations, we obtain the solution function as:

$$f(\rho) = \frac{\rho^{\varepsilon}}{\Gamma(1+\varepsilon)}\left({}^{M}\Phi_{p+1,q+1}^{(\alpha,\beta)}(\Lambda_1, \ldots, \Lambda_p, 1; \Omega_1, \ldots, \Omega_q, 1+\varepsilon; \chi_2; \chi_3; z; \varepsilon\rho)\right).$$

We presented the approximate graphs of the solution function $f(\rho)$ of Example 6.1 below.

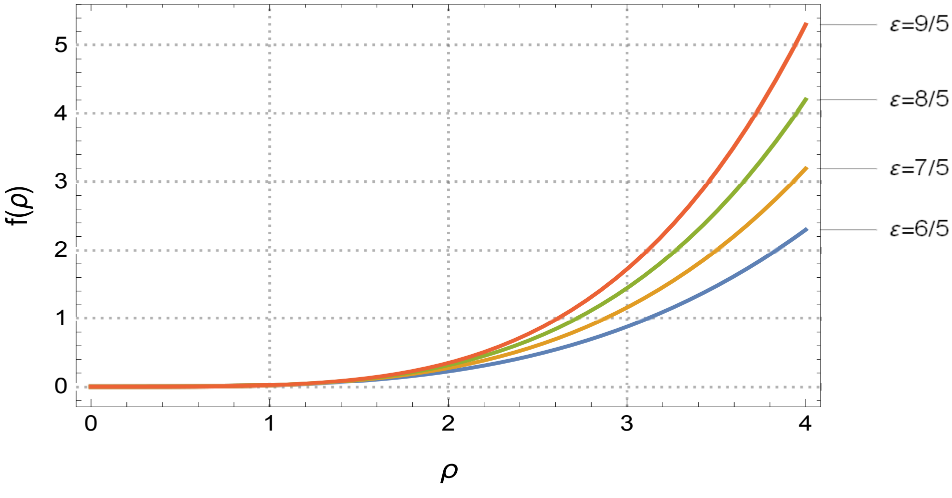

Figure 1: The behavior of the solution function $f(\rho)$ of Example 6.1 for different values of ε, where $p = q = \Lambda_1 = \Omega_1 = \alpha = \beta = 1$, $\xi = \eta = 4$, generalized M-series index $n = 0, 1, 2$ and $0 < \rho < 4$.

7. Conclusion

In this paper, M-gamma, M-beta, M-Gauss hypergeometric, and M-confluent hypergeometric functions are given. Also, some of their properties of these modified functions are presented and then beta, Mellin, Laplace, Sumudu, Elzaki and general integral transformations are applied to these modified functions. Then, solutions of fractional differential equations involving modified special functions are obtained and the approximate graphs of the solution function $f(\rho)$ of Example 6.1 for specific values are presented in Figure 1.

The relations of the special functions defined here with the other special functions in the literature are listed below:

Abubakar [1],

$$
\begin{aligned}
{}^{M}B_{p,q}^{(1,1)}(\xi,\eta;\rho) &= \frac{\Gamma(\Omega_1)\ldots\Gamma(\Omega_q)}{\Gamma(\Lambda_1)\ldots\Gamma(\Lambda_p)}\,{}^{\Psi}B_{\rho}^{1,1}\left[\begin{matrix}(\Lambda_i,1)_{1,p}\\(\Omega_j,1)_{1,q}\end{matrix}\,\middle|\,\xi,\eta\right],\\
{}^{M}F_{p,q}^{(1,1)}(\chi_1,\chi_2;\chi_3;z;\rho) &= \frac{\Gamma(\Omega_1)\ldots\Gamma(\Omega_q)}{\Gamma(\Lambda_1)\ldots\Gamma(\Lambda_p)}\,{}^{\Psi}F_{\rho}^{1,1}\left[\begin{matrix}(\Lambda_i,1)_{1,p}\\(\Omega_j,1)_{1,q}\end{matrix}\,\middle|\,\chi_1,\chi_2;\chi_3;z\right],\\
{}^{M}\Phi_{p,q}^{(1,1)}(\chi_2;\chi_3;z;\rho) &= \frac{\Gamma(\Omega_1)\ldots\Gamma(\Omega_q)}{\Gamma(\Lambda_1)\ldots\Gamma(\Lambda_p)}\,{}^{\Psi}\Phi_{\rho}^{1,1}\left[\begin{matrix}(\Lambda_i,1)_{1,p}\\(\Omega_j,1)_{1,q}\end{matrix}\,\middle|\,\chi_2;\chi_3;z\right].
\end{aligned}
$$

Classic functions [4,21],

$$
\begin{aligned}
{}^{M}\Gamma_{1,1}^{(1,1)}(1;1;\xi;0) &= \Gamma(\xi),\\
{}^{M}B_{1,1}^{(1,1)}(1;1;\xi,\eta;0) &= B(\xi,\eta),\\
{}^{M}F_{1,1}^{(1,1)}(1;1;\chi_1,\chi_2;\chi_3;z;0) &= {}_2F_1(\chi_1,\chi_2;\chi_3;z),\\
{}^{M}\Phi_{1,1}^{(1,1)}(1;1;\chi_2;\chi_3;z;0) &= \Phi(\chi_2;\chi_3;z).
\end{aligned}
$$

Ata and Kıymaz [5],

$$
\begin{aligned}
{}^{M}\Gamma_{p,q}^{(1,1)}(\xi;\rho) &= \frac{\Gamma(\Omega_1)\ldots\Gamma(\Omega_q)}{\Gamma(\Lambda_1)\ldots\Gamma(\Lambda_p)}\,{}^{\Psi}\Gamma_{\rho}\left[\begin{matrix}(\Lambda_i,1)_{1,p}\\(\Omega_j,1)_{1,q}\end{matrix}\,\middle|\,\xi\right],\\
{}^{M}B_{p,q}^{(1,1)}(\xi,\eta;\rho) &= \frac{\Gamma(\Omega_1)\ldots\Gamma(\Omega_q)}{\Gamma(\Lambda_1)\ldots\Gamma(\Lambda_p)}\,{}^{\Psi}B_{\rho}\left[\begin{matrix}(\Lambda_i,1)_{1,p}\\(\Omega_j,1)_{1,q}\end{matrix}\,\middle|\,\xi,\eta\right],\\
{}^{M}F_{p,q}^{(1,1)}(\chi_1,\chi_2;\chi_3;z;\rho) &= \frac{\Gamma(\Omega_1)\ldots\Gamma(\Omega_q)}{\Gamma(\Lambda_1)\ldots\Gamma(\Lambda_p)}\,{}^{\Psi}F_{\rho}\left[\begin{matrix}(\Lambda_i,1)_{1,p}\\(\Omega_j,1)_{1,q}\end{matrix}\,\middle|\,\chi_1,\chi_2;\chi_3;z\right],\\
{}^{M}\Phi_{p,q}^{(1,1)}(\chi_2;\chi_3;z;\rho) &= \frac{\Gamma(\Omega_1)\ldots\Gamma(\Omega_q)}{\Gamma(\Lambda_1)\ldots\Gamma(\Lambda_p)}\,{}^{\Psi}\Phi_{\rho}\left[\begin{matrix}(\Lambda_i,1)_{1,p}\\(\Omega_j,1)_{1,q}\end{matrix}\,\middle|\,\chi_2;\chi_3;z\right].
\end{aligned}
$$

Ata [6,7],

$$
\begin{aligned}
{}^{M}\Gamma_{1,2}^{(\alpha,\beta)}(1;1,1;\xi;\rho) &= {}^{\Psi}\Gamma_{\rho}^{(\alpha,\beta)}(\xi),\\
{}^{M}B_{1,2}^{(\alpha,\beta)}(1;1,1;\xi,\eta;\rho) &= {}^{\Psi}B_{\rho}^{(\alpha,\beta)}(\xi,\eta),\\
{}^{M}F_{1,2}^{(\alpha,\beta)}(1;1,1;\chi_1,\chi_2;\chi_3;z;\rho) &= {}^{\Psi}F_{\rho}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z),\\
{}^{M}\Phi_{1,2}^{(\alpha,\beta)}(1;1,1;\chi_2;\chi_3;z;\rho) &= {}^{\Psi}\Phi_{\rho}^{(\alpha,\beta)}(\chi_2;\chi_3;z).
\end{aligned}
$$

Chaudhry et al. [9,10,11],

$$
\begin{aligned}
{}^{M}\Gamma_{1,1}^{(1,1)}(1;1;\xi;\rho) &= \Gamma_{\rho}(\xi),\\
{}^{M}B_{1,1}^{(1,1)}(1;1;\xi,\eta;\rho) &= B(\xi,\eta;\rho),\\
{}^{M}F_{1,1}^{(1,1)}(1;1;\chi_1,\chi_2;\chi_3;z;\rho) &= F_{\rho}(\chi_1,\chi_2;\chi_3;z),\\
{}^{M}\Phi_{1,1}^{(1,1)}(1;1;\chi_2;\chi_3;z;\rho) &= \Phi_{\rho}(\chi_2;\chi_3;z).
\end{aligned}
$$

Kulip et al. [22],

$$ {}^{M}\Gamma_{1,2}^{(\alpha,\beta)}(\gamma;\delta,1;\xi;\rho) = {}^{W}\Gamma_{\rho}^{(\alpha,\beta;\gamma,\delta)}(\xi), $$
$$ {}^{M}B_{1,2}^{(\alpha,\beta)}(\gamma;\delta,1;\xi,\eta;\rho) = {}^{W}B_{\rho}^{(\alpha,\beta;\gamma,\delta)}(\xi,\eta). $$

Lee et al. [23],

$$ {}^{M}B_{1,1}^{(1,1)}(1;1;\xi,\eta;\rho) = B(\xi,\eta;\rho;1), $$
$$ {}^{M}F_{1,1}^{(1,1)}(1;1;\chi_1,\chi_2;\chi_3;z;\rho) = F_{\rho}(\chi_1,\chi_2;\chi_3;z;1), $$
$$ {}^{M}\Phi_{1,1}^{(1,1)}(1;1;\chi_2;\chi_3;z;\rho) = \Phi_{\rho}(\chi_2;\chi_3;z;1). $$

Özergin et al. [26],

$$ {}^{M}\Gamma_{1,1}^{(1,1)}(\alpha;\beta;\xi;\rho) = \Gamma_{\rho}^{(\alpha,\beta)}(\xi), $$
$$ {}^{M}B_{1,1}^{(1,1)}(\alpha;\beta;\xi,\eta;\rho) = B_{\rho}^{(\alpha,\beta)}(\xi,\eta), $$
$$ {}^{M}F_{1,1}^{(1,1)}(\alpha;\beta;\chi_1,\chi_2;\chi_3;z;\rho) = F_{\rho}^{(\alpha,\beta)}(\chi_1,\chi_2;\chi_3;z), $$
$$ {}^{M}\Phi_{1,1}^{(1,1)}(\alpha;\beta;\chi_2;\chi_3;z;\rho) = \Phi_{\rho}^{(\alpha,\beta)}(\chi_2;\chi_3;z). $$

Parmar [27],

$$ {}^{M}\Gamma_{1,1}^{(1,1)}(\alpha;\beta;\xi;\rho) = \Gamma_{\rho}^{(\alpha,\beta;1)}(\xi), $$
$$ {}^{M}B_{1,1}^{(1,1)}(\alpha;\beta;\xi,\eta;\rho) = B_{\rho}^{(\alpha,\beta;1)}(\xi,\eta), $$
$$ {}^{M}F_{1,1}^{(1,1)}(\alpha;\beta;\chi_1,\chi_2;\chi_3;z;\rho) = F_{\rho}^{(\alpha,\beta;1)}(\chi_1,\chi_2;\chi_3;z), $$
$$ {}^{M}\Phi_{1,1}^{(1,1)}(\alpha;\beta;\chi_2;\chi_3;z;\rho) = \Phi_{\rho}^{(\alpha,\beta;1)}(\chi_2;\chi_3;z). $$

Rahman et al. [29],

$$ {}^{M}B_{1,1}^{(\alpha,1)}(1;1;\xi,\eta;\rho) = B_{\rho}^{\alpha;1}(\xi,\eta), $$
$$ {}^{M}F_{1,1}^{(\alpha,1)}(1;1;\chi_1,\chi_2;\chi_3;z;\rho) = F_{\rho}^{\alpha;1}(\chi_1,\chi_2;\chi_3;z), $$
$$ {}^{M}\Phi_{1,1}^{(\alpha,1)}(1;1;\chi_2;\chi_3;z;\rho) = \Phi_{\rho}^{\alpha;1}(\chi_2;\chi_3;z). $$

Sadab et al. [30],

$$ {}^{M}B_{1,1}^{(\alpha,1)}(1;1;\xi,\eta;\rho) = B_{\alpha}^{\rho}(\xi,\eta), $$
$$ {}^{M}F_{1,1}^{(\alpha,1)}(1;1;\chi_1,\chi_2;\chi_3;z;\rho) = F_{\rho,\alpha}(\chi_1,\chi_2;\chi_3;z), $$
$$ {}^{M}\Phi_{1,1}^{(\alpha,1)}(1;1;\chi_2;\chi_3;z;\rho) = \Phi_{\rho,\alpha}(\chi_2;\chi_3;z). $$

Srivastava et al. [33],

$$ {}^{M}B_{1,1}^{(1,1)}(\alpha;\beta;\xi,\eta;\rho) = B_{\rho}^{(\alpha,\beta;1,1)}(\xi,\eta), $$
$$ {}^{M}F_{1,1}^{(1,1)}(\alpha;\beta;\chi_1,\chi_2;\chi_3;z;\rho) = F_{\rho}^{(\alpha,\beta;1,1)}(\chi_1,\chi_2;\chi_3;z). $$

In future studies, similar generalizations can be defined for special functions such as univariate (Bessel, Mittag-Leffler, Wright, etc.) and multivariate (Lauricella, Srivastava, Horn, etc.) which used in mathematical physics. Various properties of these functions can be examined and their integral transforms (Laplace, Mellin, Fourier, etc.) can be calculated. Thus, potential applications can be found in various fields of physics and engineering.

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