Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–11.
ISSN-0037-8712 in press
doi:10.5269/bspm.65621

Wavelet Frames in Sobolev Space Over Locally compact Abelian Group

M. M. Dixit, C. P. Pandey*, Pratima Devi

abstract: In this paper we construct wavelet frames for continuous and discrete wavelets on Sobolev space over abelian group. A necessary condition and sufficient conditions for wavelet frames in Sobolev space over Locally Compact Abelian Group are given. Moreover some important properties of continuous wavelet transform and corresponding wavelet Frames have been discussed.

Key Words: Fourier trasform, wavelet, wavelet frame, locally compact abelian groups, Sobolev space.

Contents

1. Introduction

Most of the space that we are interested in end up bring topological groups. In this section we define the terms topology and group so that we can work with them. We introduce the Haar measure, which is a translation-invariant measure.

A set S becomes a Group [4] if an operator, say $+$, can be defined such that

- $x + (y + z) = (x + y) + z, \forall$ x,y,z $\in S$
- There exist an element 0, such that $x + 0 = 0 + x = x \; \forall \; x \in S$
- For each x $\in$ S there $\exists$ an inverse element $x^{-1} = x$ such that $x + (-x) = (-x) + x = 0$.

In addition, S is a commutative group if it is also true that
$x + y = y + x \; \forall \; x, y \in$ S. Given a set S, a Topology T is a set of subsets on S that

- contains S and the empty set $\o$
- Is closed under finite intersections and infinite unions of subsets.$\times$

S ia a topological group if it has a group operation and a topology such that the maps $\alpha : G \times G \to G$ are continuous, where $\alpha(x, y) = x + y$ and $\beta(x) = x^{-1}$.

If S is locally compact, that is, every point in S is contained in a compact neighborhood, and its group operation is commutative, then we call it a Locally Compact Abelian (LCA) Group.

Given a topological space X, we define the Borel set as a set of subset of X such that:

- contains all subsets of the topology on X
- Is closed under complements, countable unions, and countable intersections of subsets
- Is the smallest set of subsets that meets these condition.

2010 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted October 30, 2022. Published December 01, 2022

A measure μ an X is a function on the Borel sets where

- $\mu(E) = \sum \mu(E_i)$ if $E \subset X$ and $E = \cup_{i\in 1} E_i$ where E_i is a countable pairwise disjoint set.
- $\mu(E)$ is finite for all $E \subset X$ where the closure of E is compact.

A measure μ is regular if for all Borel sets E, we have

$$\mu(E) = inf_{K\supset E}\ \mu(K) = sup_{k\subset E}\ \mu(K).$$

μ is invariant if $\mu(x+E) = \mu(E)\ \forall x \in x$.

Let $M(X)$ be the space of all complex-valued regular measures on X where $\| \mu \| = | \mu(S) |$ is finite.

A haar measure [15] is a measure which is non-negative,regular and invariant. In fact, Haar measure are unique up to a scalar, so we can call it the Haar measure. That is if $m-1$ and m_2 are both non-negative, regular, translation invariant measures on S, then there exists $\lambda \geq 0$ such that $m_1 = m_2$. The corresponding integral is called the Haar Integral, which is translation invariant. That is, integrals over a set E and $x+E$ are equivalent.

Given a LCA group [20] G, we define an $L^p(G)$ space to be the space of all complex valued functions f on G such that the integral $\int | f |^p\, d\mu$ exists with respect to the Haar measure.
$L^p(G)$ becomes an algebra under convolution, which is an important characteristic later on.

Definition 1.1. *A complex function [21] γ on a LCA group G is called a character of G if $| \gamma(x) | = 1$ for all $x \in G$ and if the functional equation $\gamma(x+y) = \gamma(x)\gamma(y)$ for all $(x,y) \in G$ is satisfied. The set of all continuous characters of G form a group Γ, the dual group of G. Now it is customary to write $(x,\gamma) = \gamma(x)$ satisfy the following properties:*

- $(0,\gamma) = (x,0) = 1$
- $(-x,\gamma) = (x,-\gamma) = (x,\gamma)^{-1} = \overline{(x,\gamma)}$
- $(x+y,\gamma) = (x,\gamma)(y,\gamma)$
- $(x,\gamma_1+\gamma_2) = (x,\gamma_1)(x,\gamma_2)$

Definition 1.2. *The Fourier Transform of $f \in L^1(G)$ is denoted by $\widehat{f}(\gamma)$ defined by [4]*

$$\widehat{f}(\gamma) = \int_G f(x)(-x,\gamma)dx, \tag{1.1}$$

and the Inverse Fourier Transform is defined by

$$f(x) = \int_G \widehat{f}(\gamma)(x,\gamma)d\gamma, x \in G \tag{1.2}$$

Some important properties of the fourier trasform can be proved easily:

- $\| \widehat{f} \|_{L^\infty(G)} \leq \| f \|_{L^1(G)}$
- If $f \in L^1(G)$, then $\widehat{f}$ is uniformly continuous.
- Parseval formula: If $f \in L^1(G) \cap L^2(G)$, then $\| \widehat{f} \|_{L^2(G)} = \| f \|_{L^2(G)}$
- If the convolution of f and g is defined as

$$(f*g)(x) = \int_G f(x-y)g(y)dy \tag{1.3}$$

$$\widehat{(f*g)}(\gamma) = \widehat{f}(\gamma)\widehat{g}(\gamma) \tag{1.4}$$

Definition 1.3. *For k, $0 \leq k < q$, $k = a_0 + a_1p + ... + a_{c-1}p^{c-1}$, $0 \leq a_i < p$, $i = 0, 1, 2, 3, ..., c-1$, we define*

$$v(k) = (a_0 + a_1\epsilon_1 + ... + a_{c-1}\epsilon_{c-1})p^{-1}(0 \leq k < q)$$

For $k = b_0 + b_1q + ... + a_{c-1}q^s$, $0 \leq b_i < q$, $k \geq 0$, we set

$$v(k) = v(b_0) + p^{-1}v(b_1) + ... + p^{-s}v(b_s)$$

Note that for $k, l \geq 0$, $v(k + l) \neq v(k) + v(l)$. However, it is true that for all $r, s \geq 0$, $v(rq^s) = p^{-1}v(r)$, and for $r, s \geq 0$, $0 \leq t < q^s$,
$v(rq^s + t) = v(rq^s) + v(t) = p^{-1}v(r) + v(t)$.
We denote $\chi_{v(n)}$ by χ_n $(n \geq 0)$ and use the notation $\mathbb{N}_0 = 0, 1, 2, 3, ...$ and $\mathbb{N} = 1, 2, 3, ...$ throughout this paper.

Distributions over LCA Group:

We denote $\mathscr{S}(G)$ the space of all finite linear combinations of characteristics functions of ball of G. The Fourier transform is homeomorphism of $\mathscr{S}(G)$ onto $\mathscr{S}(G)$. The distribution space of $\mathscr{S}(G)$ is denoted by $\mathscr{S}'(G)$.

The Fourier transform of $g \in \mathscr{S}(G)$ is denoted by $\widehat{f}(\omega)$ and defined by

$$\widehat{f}(\omega) = \int_G f(x)(-x, \omega)dx, \omega \in G$$

and the Inverse Fourier Transform defined by

$$f(x) = \int_G \widehat{f}(\omega)(x, \omega)dw, x \in G$$

The Fourier Transform and inverse Fourier Transforms of a distributions $f \in \mathscr{S}'(G)$ is defined by

$$\langle \widehat{f}, \varphi\rangle = \langle f, \widehat{\varphi}\rangle, \langle f^v, \varphi\rangle = \langle f, \varphi^v\rangle, for\ all\ \varphi \in \mathscr{S}(G)$$

Definition 1.4. ***Sobolev space over LCA groups***

Let $s \in G$, Sobolev space over LCA group [8] denote by $H^s_\gamma(G)$, defined by the space of all $f \in \mathscr{S}(G)$ such that

$$\int_{G^\Lambda} (1 + \gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 d\xi\ is\ finite$$

where $f \in L^2(G), \xi \in \Gamma$. We denote Γ by the set

$$\Gamma = \{\gamma : G^\Lambda \to [0, \infty) : \exists_{c_\gamma} \forall \alpha, \beta \in G^\Lambda \gamma(\alpha\beta) \leq c_\gamma[\gamma(\alpha) + \gamma(\beta)]\}$$

Moreover, for $f \in H^s_\gamma(G)$; its norm $\| f \|_{H^s_\gamma(G)}$ is defined as follows:

$$\| f \|^2_{H^s_\gamma(G)} = \int_{G^\Lambda} (1 + \gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 d\xi$$

2. A Necessary condition of wavelet frame for $H^s_\gamma(G)$

Let $\psi \in H^s_\gamma(G)$, $\psi_{j,k}(\xi) = q^{j/2}\psi(\mathfrak{p}^{-j}\xi - v(k))$ $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$, where q *and* $\mathfrak{p}$ are integers. The function system $\psi_{j,k}(\xi)_{(j,k)\in\mathbb{Z}\times\mathbb{N}_0}$ a wavelet frame for $H^s_\gamma(G)$, if there are two constants C, $D \geqslant 0$ such that

$$C \| f \|^2_{H^s_\gamma(G)} \leq \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |< f, \psi_{j,k} >|^2 \leq D \| f \|^2_{H^s_\gamma(G)} \tag{2.1}$$

satisfies for all $f \in H^s_\gamma(G)$.

Theorem 2.1. *If* $\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}_0$ *is a wavelet frame for* $H^s_\gamma(G)$ *with bounds* C *and* D *then* $C \leq (1+\gamma(\xi)^2)^s \sum_{j\in\mathbb{Z}} \mid \widehat{\psi}(\mathfrak{p}^j\omega) \mid^2 \leq D \quad a.e.w. \in G.$

Proof: For $f \in \mathscr{S}(G)$ and $\psi \in H^s_\gamma(G)$, we have

$$\sum_{k=0}^{\infty} |< f, \psi_{j,k} >_{H^s_\gamma(G)}|^2$$

$$= \sum_{k=0}^{\infty} \int_G \mid (1+\gamma(\xi)^2)^s \widehat{f}(\xi) q^{j/2} \overline{\psi(\mathfrak{p}^j\xi - v(k)^\Lambda} d\xi \mid^2$$

$$= \sum_{k=0}^{\infty} \int_G \mid (1+\gamma(\xi)^2)^s \widehat{f}(\xi) q^{-j/2} \overline{\widehat{\psi}(\mathfrak{p}^j\xi)}(x, \mathfrak{p}^j\xi) d\xi \mid^2$$

$$= \sum_{k=0}^{\infty} q^{-j} \int_G (1+\gamma(\xi)^2)^s \widehat{f}(\xi) \overline{\widehat{\psi}(\mathfrak{p}^j\xi)}(x, \mathfrak{p}^j\xi) d\xi$$

$$\times \{ \int_G (1+\gamma(\xi)^2)^s) \overline{\widehat{f}(\xi)} \psi(\mathfrak{p}^j\xi) \overline{(x, \mathfrak{p}^j\xi)} d\xi \}$$

$$= \sum_{k=0}^{\infty} q^{j} \int_G (1+\gamma(\mathfrak{p}^{-j}\xi)^2)^s \widehat{f}(\mathfrak{p}^{-j}\xi) \overline{\widehat{\psi}(\xi)}(x, \xi) d\xi$$

$$\times \{ \int_G (1+\gamma(\mathfrak{p}^{-j}\xi)^2)^s \overline{\widehat{f}(\mathfrak{p}^{-j}\xi)} \widehat{\psi}(\xi) \overline{(x, \xi)} d\xi$$

$$= \sum_{k=0}^{\infty} q^{j} \int_G \{ \sum_{l=0}^{\infty} \int_G (1+\gamma(\mathfrak{p}^{-j}(\xi + v(l))^2)^s \ \widehat{f}(\mathfrak{p}^{-j}(\xi + v(l))))$$

$$\times (x, (\xi + v(l))) \overline{\widehat{\psi}(\xi + v(l))} d\xi \} \times \{ (1+\gamma(\mathfrak{p}^{-j}\xi)^2)^s \overline{\widehat{f}(\mathfrak{p}^{-j}\xi)} \widehat{\psi}(\xi) \times (x, \xi) \} d\xi$$

Since $f \in \mathscr{S}(G)$ so the $\sum_{l=0}^{\infty}$ contains only finite non-zero terms and $(x, v(l)) = 1$ for all $k, l \in \mathbb{N}_0$, then we get

$$\sum_{k=0}^{\infty} |< f, \psi_{j,k} >_{H^s_\gamma(G)}|^2 = \sum_{k=0}^{\infty} q^j \int_G (\int_G \{ \sum_{l=0}^{\infty} (1+\gamma(\mathfrak{p}^{-j}(\xi + v(l)))^2)^s \ \widehat{f}(\mathfrak{p}^{-j}(\xi + v(l)))(x, \xi) \times \overline{\widehat{\psi}(\xi + v(l))} \} d\xi)$$
$$\times \{ (1+\gamma(\mathfrak{p}^{-j}\xi)^2)^s \overline{\widehat{f}(\mathfrak{p}^{-j}\xi)} \widehat{\psi}(\xi)(x, \xi) \} d\xi.$$

By the convergence theorem of Fourier Series on $\mathfrak{D}$, we get

$$\sum_{k=0}^{\infty} |< f, \psi_{j,k} >_{H^s_\gamma(G)}|^2 = \int_G (1+\gamma(\xi)^2)^s \widehat{f}(\xi) \overline{\widehat{\psi}}(\mathfrak{p}^j\xi) \{ \sum_{k=0}^{\infty} (1+\gamma(\xi + \mathfrak{p}^{-j}v(k))^2)^s \overline{\widehat{f}}(\xi + \mathfrak{p}^{-j}v(k))$$
$$\times \widehat{\psi}(\mathfrak{p}^j\xi + v(k)) \} d\xi. \tag{2.2}$$

Let A_j is the set of regular point of $(1+\gamma(\xi)^2)^s \mid \widehat{\psi}(\mathfrak{P}^j\xi) \mid^2$, so for all $\xi \in A_j$.

$$q^l \int_{\xi - \xi_0 \in \mathfrak{p}^l} (1+\gamma(\xi)^2)^s \mid \widehat{\psi}(\mathfrak{p}^j\xi) \mid^2 d\xi \to (1+\gamma(\xi_0)) \mid \widehat{\psi}(\mathfrak{p}^j\xi_0) \mid^2, as \ l \to +\infty.$$

If $A = \bigcup_{j\in\mathbb{Z}} A_j^c$, then $\mid A \mid = 0$.
Suppose that $\xi_0 \in \mathbb{F} - A$. So for each fixed positive integer M, set

$$\widehat{f}(\xi) = \frac{q^{\frac{l}{2}} \varphi_l(\xi - \xi_0)}{\widehat{v}^{\frac{s}{2}}(\xi)} \ for \ all \ \ l \geq M,$$

where ψ_l is the characteristic function of $\xi_0+\mathfrak{P}^l$.
Then for $l\in\mathbb{N}$ and $j\geq -M$, $\widehat{f}(\xi)\overline{\widehat{f}}(\xi+\mathfrak{p}^{-j}v(l))=0$. Since ξ and $(\xi+\mathfrak{p}^{-j}v(l))$ can not be in $\xi_0+\mathfrak{P}^m$ simultaneously. Now, we have

$$\sum_{j\geq -M}\sum_{k=0}^{\infty}|<f,\psi_{j,k}>|^2=\sum_{j\geq -M}\int_{\xi+\mathfrak{P}^l}q^l\widehat{v}^s(\xi)\mid\widehat{\psi}(\mathfrak{p}^j\xi)\mid^2 d\xi\leq D. \tag{2.3}$$

Let $l\to+\infty$ and $M\to+\infty$, we have

$$\sum_{j\in\mathbb{Z}}\widehat{v}^s(\xi_0)\mid\widehat{\psi}(\mathfrak{p}^j\xi)\mid^2\leq D. \tag{2.4}$$

To prove the left hand inequality,

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|<f,\psi_{j,k}>|^2=T_1+T_2, \tag{2.5}$$

where

$$T_1=\sum_{j\geq -M}\sum_{k\in\mathbb{N}_0}|<f,\psi_{j,k}>|^2\ and\ T_2=\sum_{j\leq -M}\sum_{k\in\mathbb{N}_0}|<f,\psi_{j,k}>|^2\ .$$

By condition of frame, $T_1\geq C-T_2$. Since we have already show that $T_1=\sum_{j>-M}(1+\gamma(\xi_0))\mid\widehat{\psi}(\mathfrak{p}^{-j}\xi_0)\mid^2$. So, we only need to show that $T_2\to 0$ as $M\to\infty$. Now, using the fact $\mathscr{S}'(G)$ is dence in $H^s_\gamma(G)$ in (2.2) and Schwaz's inequality, we have

$$\begin{aligned}T_2\leq\sum_{j\geq -M}\sum_{k=0}^{\infty}&\{\int_G(1+\gamma(\xi)^2)^s\mid\widehat{g}(\xi)\mid^2\mid\widehat{\psi}(\mathfrak{p}^j\xi)\mid^2 d\xi\}^{\frac{1}{2}}\\ &\times\{\int_G(1+\gamma(\xi+\mathfrak{p}^{-j}v(k))^2)^s\mid\widehat{g}(\xi+\mathfrak{p}^{-j}v(k))\mid^2\mid\widehat{\psi}(\mathfrak{p}^j\xi+v(k))\mid^2 d\xi\}^{\frac{1}{2}}\ .\end{aligned}$$

where $\widehat{f}=((1+\gamma(\xi)^2)^{-\frac{s}{2}}\widehat{g})$ and $\widehat{g}\in\mathscr{S}(G)$.
Since $\widehat{g}\in\mathscr{S}(G)$, so there exists a characteristic function $\varphi_{r(\xi-\xi_0)}$ of the set $\xi_0+\mathfrak{P}^r$ where r is some integers. Now $\widehat{g}$ can be written as $\widehat{g}(\xi)=q^{\frac{r}{2}}\varphi_r(\xi-\xi_0)$.If $\xi+\mathfrak{p}^{-j}v(k)\in\xi_0+\mathfrak{P}^r$, then $\mid\mathfrak{p}^{-j}v(k)\mid\leq q^{-r}$, hence $\mid v(k)\mid\leq q^{-r-j}$. Then summation index k is bounded by q^{-r-j}. So using this, we get

$$T_2\leq q^{-r}\int_{\mathfrak{p}^{-j}\xi_0+\mathfrak{P}^{-j+r}}(1+\gamma(\mathfrak{p}^{-j}\xi)^2)^s\mid\widehat{\psi}(\xi)\mid^2 d\xi.$$

Suppose that $\xi_0\neq 0$. For any $\in>0$, choose $J<0$ enough small satisfies the following two inequalities:

- $q^J<\mid\xi_0\mid=q^\rho$ such that $J+\rho<0$.
- $\int_{\mathfrak{P}^{-J-\rho}}(1+\gamma(\mathfrak{p}^{-J}\xi)^2)^s\mid\widehat{\psi}(\xi)\mid^2 d\xi<\in$.

We have

$$\mathfrak{p}^{-j}\xi_0+\mathfrak{P}^{-j+r}\subset\mathfrak{P}^{-J-\rho}\ for\ all\ j\leq J. \tag{2.6}$$

Since $\mid\mathfrak{p}^{-j}\xi_0\mid=q^jq^\rho\leq q^Jq^\rho$ and $\mathfrak{P}^{-j+r}\subset\mathfrak{P}^{-J-\rho}$.
Hence, $T_2\to 0$ as $j\to-\infty$. Therefore there exists j such that

$$T_2<\in$$

Hence we obtain required result. □

3. Sufficient conditions of wavelet frame for $H^s_\gamma(G)$

To find the sufficient conditions of wavelet frame for $H^s_\gamma(G)$.
We need the following Lemma.

Lemma 3.1. *Let f be in $\mathscr{S}(G)$ and $\psi \in H^s_\gamma(G)$. If* $sup\{(1+\gamma(\xi)^2)^s \sum_{j\in\mathbb{Z}} \mid \widehat{\psi}(\mathfrak{p}^j\xi) \mid^2 : \xi \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\} < +\infty$, *then*

$$\sum_{j\in\mathbb{Z}}\sum_{k=0}^{\infty} |< f, \psi_{j,k} >_{H^s_\gamma(G)}|^2 = \int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 (1+\gamma(\xi)^2)^s \sum_{j\in\mathbb{Z}} \mid \widehat{\psi}(\mathfrak{p}^j\xi) \mid^2 d\xi + T_2, \tag{3.1}$$

where,

$$T_2 = \sum_{j\in\mathbb{Z}} \int_G (1+\gamma(\xi)^2)^s \widehat{f}(\xi)\overline{\widehat{\psi}}(\mathfrak{p}^j\xi)[\sum_{l=1}^{\infty}(1+\gamma(\xi+\mathfrak{p}^{-j}v(l))^2)^s \overline{\widehat{f}}(\xi+\mathfrak{p}^{-j}v(l)) \times \widehat{\psi}(\mathfrak{p}^j\xi+v(l))]d\xi \tag{3.2}$$

Then iterated series in (3.2) *is absolutely convergent.*

Proof: Let $f \in \mathscr{S}(G)$ so the $\sum_{l=0}^{\infty}$ in (3.2) contains only finite non-zero terms. Hence,

$$\sum_{j\in\mathbb{Z}} \int_G (1+\gamma(\xi)^2)^s \overline{\widehat{\psi}}(\mathfrak{p}^j\xi)\widehat{f}(\xi)[\sum_{l=0}^{\infty}(1+\gamma(\xi+\mathfrak{p}^{-j}v(l))^2)^s \overline{\widehat{f}}(\xi+\mathfrak{p}^{-j}v(l))\widehat{\psi}(\mathfrak{p}^j\xi+v(l))]d\xi$$

$$= \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} \int_G (1+\gamma(\xi)^2)^s \widehat{f}(\xi)\overline{\widehat{\psi}}(\mathfrak{p}^j\xi)(1+\gamma(\xi+\mathfrak{p}^{-j}v(l))^2)^s \overline{\widehat{f}}(\xi)+\mathfrak{p}^{-j}v(l))\widehat{\psi}(\mathfrak{p}^j\xi+v(l))d\xi \tag{3.3}$$

We claim that,

$$\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} |< f, \psi_{j,k} >_{H^s_\gamma(G)}|^2 = \sum_{j\in\mathbb{Z}} \int_G \mid \widehat{f}(\xi) \mid^2 (1+\gamma(\xi)^2)^s \mid \overline{\widehat{\psi}}(\mathfrak{p}^j\xi) \mid^2 (1+\gamma(\xi)^2)^s d\xi + T_2 \tag{3.4}$$

holds for all $f \in \mathscr{S}(G)$,We have

$$\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} |< f, \psi_{j,k} >_{H^s_\gamma(G)}|^2 = \int_G (1+\gamma(\xi)^2)^2 s(\xi) \mid \widehat{f}(\xi) \mid^2 \mid \overline{\widehat{\psi}}(\mathfrak{p}^j\xi) \mid^2 d\xi + T_2, \tag{3.5}$$

where

$$T_2 = \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} \int_G (1+\gamma(\xi)^2)^s \widehat{f}(\xi)\overline{\widehat{\psi}}(\mathfrak{p}^j\xi)(1+\gamma(\xi+\mathfrak{p}^{-j}v(l))^2)^s \overline{\widehat{f}}(\xi+\mathfrak{p}^{-j}v(l)) \times \widehat{\psi}(\mathfrak{p}^j\xi+v(l))d\xi. \tag{3.6}$$

By using the condition $sup\{(1+\gamma(\xi)^2)^s \sum_{j\in\mathbb{Z}} \mid \overline{\widehat{\psi}}(\mathfrak{p}^j\xi) \mid^2 : \xi \in \mathfrak{P}^{-1} \backslash\mathfrak{D}\} < +\infty$
and Levi's Lemma for integral, we get

$$= \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} |< f, \psi_{j,k} >_{H^s_\gamma(G)}|^2 = \int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 (1+\gamma(\xi)^2)^s \sum_{j\in\mathbb{Z}} \mid \overline{\widehat{\psi}}(\mathfrak{p}^j\xi) \mid^2 d\xi + T_2. \tag{3.7}$$

Now, we show that series (3.6) is absolutely convergent.

$$\mid T_2 \mid \leq \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} \int_G (1+\gamma(\xi)^2)^s \widehat{f}(\xi)\overline{\widehat{\psi}}(\mathfrak{p}^j\xi)(1+\gamma(\xi+\mathfrak{p}^{-j}v(l))^2)^s \overline{\widehat{f}}(\xi+\mathfrak{p}^{-j}v(l))^2)^s \widehat{\psi}(\mathfrak{p}^j\xi+v(l)) \mid d\xi$$

$$\leq \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} \int_G (1+\gamma(\xi)^2)^{\frac{s}{2}} \mid \widehat{f}(\xi) \mid (1+\gamma(\xi+\mathfrak{p}^{-j}v(l))^2)^{\frac{s}{2}} \mid \overline{\widehat{f}}(\xi+\mathfrak{p}^{-j}v(l)) \mid \frac{1}{2}[(1+\gamma(\xi)^2)^s \mid (\widehat{\psi}^j\xi) \mid^2$$

$$+ (1+\gamma(\xi+\mathfrak{p}^{-j}v(l))^2)^s] \mid \widehat{\psi}(\psi^j\xi+v(l)) \mid^2]d\xi.$$

$$| T_2 |\leq| \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_G (1+\gamma(\mathfrak{p}^{-j}\xi)^2)^s \mid \widehat{f}(\mathfrak{p}^{-j}\xi) \mid (1+\gamma(\mathfrak{p}^{-j}\xi+\mathfrak{p}^{-j}v(l))^2)^s \, |\widehat{f}(\mathfrak{p}^{-j}\xi+\mathfrak{p}^{-j}v(l)) \mid \times (1+\gamma(\mathfrak{p}^{-j}\xi)^2)^s \mid \widehat{\psi}(\xi) \mid^2 d\xi. \tag{3.8}$$

Since '$g \in \mathscr{S}(G)$, there exist a constant $J > 0$ such that for all $|j| > J$

$$\widehat{f}(\mathfrak{p}^{-j}\xi)\widehat{f}(\mathfrak{p}^{-j}\xi+\mathfrak{p}^{-j}v(l)) = 0 \tag{3.9}$$

On the other hand, for each $\mid j \mid > J$, there exist a constant L such that for all $l \geq L$.

$$\widehat{f}(\mathfrak{p}^{-j}\xi+\mathfrak{p}^{-j}v(l)) = 0. \tag{3.10}$$

Therefore only finite number of terms of the iterated series in (3.8) are non-zero.

$$| T_2 |\leq C||(1+\gamma(.)^2)^s\widehat{f}(.)||^2_\infty ||\psi||_{H^2_\gamma(G)}. \tag{3.11}$$

Hence the T_2 is absolutely convergent. The proof is complete. □

Now using above lemma, we establish sufficient condition of frame for $H^s_\gamma(G)$.
Let

$$\triangle_1 = ess\ sup\{(1+\gamma(\xi)^2)^s \sum_{j\in\mathbb{Z}} \mid \widehat{\psi}(\mathfrak{p}^j\xi) \mid^2 : \xi \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\}, \tag{3.12}$$

and

$$\triangle_2 = ess\ inf\{(1+\gamma(\xi)^2)^s \sum_{j\in\mathbb{Z}} \mid \widehat{\psi}(\mathfrak{p}^j\xi) \mid^2 : \xi \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\}. \tag{3.13}$$

We set

$$\beta_\psi(v(l)) = Sup\{\sum_{j\in\mathbb{Z}} \mid h_\psi(v(l),\mathfrak{p}^j\xi) \mid : \xi \in \mathfrak{P}^{-1}\backslash\mathfrak{D}\}, \tag{3.14}$$

where

$$h_\psi(v(l),\xi) = \sum_{l\in\mathbb{N}_0}(1+\gamma(\xi)^2)^s\widehat{\psi}(\mathfrak{p}^{-j}\xi)\overline{\widehat{\psi}(\mathfrak{p}^{-j}\xi+v(l))}. \tag{3.15}$$

Suppose that $Q = \{1,2,3,4,...q-1\}$ and $q\mathbb{N}_0 = \{qk : k = 0,1,2,3,...\}$.

Theorem 3.2. *Suppose $\psi \in H^s_\gamma(G)$ such that*

$$\rho_1(\psi) = \triangle_2 - \sum_{m\in q\mathbb{N}_0+Q}[\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}} > 0,$$
$$\rho_2(\psi) = \triangle_1 + \sum_{m\in q\mathbb{N}_0+Q}[\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}} < +\infty.$$

Then $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ is wavelet frame for $H^s_\gamma(G)$ with bounds $\rho_1(\psi)$ and $\rho_2(\psi)$.

Proposition 3.3. *For a given $l \in \mathbb{N}$, there exists $k \in \mathbb{N}$ and unique $m \in q\mathbb{N}_0 + Q$ such that $l = q^k m$. Thus we have $\{v(l)\}_{l\in\mathbb{N}} = \{\mathfrak{p}^{-k}v(m)\}_{(k,m)\in\mathbb{N}_0\times\{q\mathbb{N}_0+Q\}}$. Since the last series in equation (3.2) is absolutely convergent. Therfore equation (3.2) become*

$$T_2 = \sum_{j\in\mathbb{Z}}\int_G (1+\gamma(\xi)^2)^s\widehat{f}(\xi)\overline{\widehat{\psi}}(\mathfrak{p}^j\xi)[\sum_{l\in\mathbb{N}}(1+\gamma(\xi+\mathfrak{p}^{-j}v(l))^2)^s\overline{\widehat{f}}(\xi+\mathfrak{p}^{-j}v(l)) \times \widehat{\psi}(\mathfrak{p}^j\xi+v(l))]d\xi$$

$$= \sum_{j\in\mathbb{Z}}\int_G (1+\gamma(\xi)^2)^s\widehat{f}(\xi)[\sum_{k\in\mathbb{N}_0}\sum_{m\in q\mathbb{N}_0+Q}\overline{\widehat{\psi}}(\mathfrak{p}^j\xi)(1+\gamma(\xi+\mathfrak{p}^{-j-k}v(m))^2)^s\overline{\widehat{f}}(\xi+\mathfrak{p}^{-j-k}v(m)) \times \overline{\widehat{\psi}}(\mathfrak{p}^j\xi+\mathfrak{p}^{-\mathfrak{k}}v(m))]d\xi$$

$$\int_G (1+\gamma(\xi)^2)^s \widehat{f}(\xi)[\sum_{k\in\mathbb{N}_0}\sum_{m\in q\mathbb{N}_0+Q}\sum_{j\in\mathbb{Z}} \overline{\widehat{\psi}}(\mathfrak{p}^{j-k}\xi)(1+\gamma(\xi+\mathfrak{p}^{-j}v(m))^2)^s \overline{\widehat{f}}(\xi+\mathfrak{p}^{-j}v(m)) \times \overline{\widehat{\psi}}(\mathfrak{p}^{j-k}\xi+\mathfrak{p}^{-k}v(m))]d\xi$$

$$= \int_G (1+\gamma(\xi)^2)^{\frac{s}{2}} \widehat{f}(\xi)[\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+Q}(1+\gamma(\xi+\mathfrak{p}^{-j}v(m))^2)^{\frac{s}{2}} \overline{\widehat{f}}(\xi+\mathfrak{p}^{-j}v(m)) \sum_{k\in\mathbb{N}_0}(1+\gamma(\xi+\mathfrak{p}^{-j}v(m))^2)^{\frac{s}{2}}\overline{\widehat{\psi}}(\mathfrak{p}^{j-k}\xi)(1+\gamma(\xi)^2)^{\frac{s}{2}}\widehat{\psi}(\mathfrak{p}^{-k}(\mathfrak{p}^j\xi+v(m)))]d\xi$$

$$= \int_G (1+\gamma(\xi)^2)^{\frac{s}{2}} \widehat{f}(\xi)[\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+Q}(1+\gamma(\xi+\mathfrak{p}^{-j}v(m))^2)^{\frac{s}{2}} h_\psi(v(m),\mathfrak{p}^j\xi) \times \overline{\widehat{f}}(\xi+\mathfrak{p}^{-j}v(m))]d\xi$$

$$\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+Q}\int_G (1+\gamma(\xi)^2)^{\frac{s}{2}} \widehat{f}(\xi)(1+\gamma(\xi+\mathfrak{p}^{-j}v(m))^2)^{\frac{s}{2}} \overline{\widehat{f}}(\xi+\mathfrak{p}^{-j}v(m)) \times h_\psi(v(m),\mathfrak{p}^j\xi)d\xi.$$

We derive further that

$$|T_2| \le \int_G (1+\gamma(\xi)^2)\frac{s}{2} \mid \widehat{f}(\xi) \mid^2 [\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+Q}(1+\gamma(\xi+\mathfrak{p}^{-j}v(m))^2)^{\frac{s}{2}} \mid \widehat{f}(\xi+\mathfrak{p}^{-j}v(m)) \mid \times \mid h_\psi(v(m),\mathfrak{p}^j\xi) \mid]d\xi$$

$$\le \sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+Q}[\int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 \mid h_\psi(v(m),\mathfrak{p}^j\xi) \mid]^{\frac{1}{2}} \times [\int_G (1+\gamma(\xi+\mathfrak{p}^{-j}v(m))^2)^{\frac{s}{2}} \mid \widehat{f}(\xi+\mathfrak{p}^{-j}v(m)) \mid^2 \mid h_\psi(v(m),\mathfrak{p}^j\xi) \mid d\xi]^{\frac{1}{2}}$$

$$\le \sum_{m\in q\mathbb{N}_0+Q}[\sum_{j\in\mathbb{Z}}\int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 \mid h_\psi(v(m),\mathfrak{p}^j\xi) \mid d\xi]^{\frac{1}{2}} \times [\sum_{j\in\mathbb{Z}}\int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 \mid h_\psi(-v(m),\mathfrak{p}^j\xi) \mid d\xi]^{\frac{1}{2}}$$

$$\le \sum_{m\in q\mathbb{N}_0+Q}[\int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 \beta_\psi(v(m))d\xi]^{\frac{1}{2}}[\int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 \mid \beta_\psi(-v(m))d\xi]^{\frac{1}{2}}$$

$$= \int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 d\xi \sum_{m\in q\mathbb{N}_0+Q}[\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}}.$$

Now it follows from equation (3.1) *in Lemma* 3.1 *that*

$$\int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 \{\sum_{j\in\mathbb{Z}}(1+\gamma(\xi)^2)^s \mid \widehat{\psi}(\mathfrak{p}^j\xi) \mid^2 - \sum_{m\in q\mathbb{N}_0+Q}[\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}}\}d\xi \le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|< f,\psi_{j,k} >_{H^s_\gamma(G)}|^2, \tag{3.16}$$

and

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}|< f,\psi_{j,k} >_{H^s_\gamma(G)}|^2 \le \int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 \{\sum_{j\in\mathbb{Z}}(1+\gamma(\xi)^2)^s \mid \widehat{\psi}(\mathfrak{p}^j\xi) \mid^2 + \sum_{m\in q\mathbb{N}_0+Q}[\beta_\psi(v(m))\beta_\psi(-v(m))]^{\frac{1}{2}}\}d\xi. \tag{3.17}$$

Taking infimum and suprimum in above two inequality respectively, we get

$$\rho_2(\psi)||f||_{H^s_\gamma} \leq \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} |< g,\psi_{j,k} >|^2 \leq \rho_1(\psi)||g||_{H^s_\gamma(G)}. \tag{3.18}$$

The proof of the theorem 3.1 *is complete.*

Theorem 3.4. *Suppose* $\psi \in H^s_\gamma(G)$ *such that*

$$\begin{aligned}\triangle_3(\psi) = ess\ inf_{\xi\in\mathfrak{P}^{-1}\backslash\mathfrak{D}}\{(1+\gamma(\xi)^2)^s\sum_{j\in\mathbb{Z}} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\mid^2 \\ -(1+\gamma(\xi)^2)^s\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\overline{\widehat{\psi}(\mathfrak{p}^j\xi+v(l))}\} > 0,\end{aligned} \tag{3.19}$$

$$\triangle_4(\psi) = ess\ sup_{\xi\in\mathfrak{P}^{-1}\backslash\mathfrak{D}}\{(1+\gamma(\xi)^2)^s\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\overline{\widehat{\psi}(\mathfrak{p}^j\xi+v(k))}\mid\} < +\infty. \tag{3.20}$$

Then $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ *is a wavelet frame for* $H^s_\gamma(G)$ *with bounds* $\triangle_3(\psi)$ *and* $\triangle_4(\psi)$.

Proof: We use Lemma 3.1 to calculate T_2 in (3.2) for $f \in \mathscr{S}(G)$ with another way. We first deduce that

$$\begin{aligned}\mid T_2\mid =\mid \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_G (1+\gamma(\xi)^2)^s\overline{\widehat{f}(\xi)}\widehat{\psi}(\mathfrak{p}^j\xi)(1+\gamma(\xi+\mathfrak{p}^{-j}v(l))^2)^s\widehat{f}(\xi+\mathfrak{p}^{-j}v(l)) \\ \times\overline{\widehat{\psi}(\mathfrak{p}^j\xi+v(l))}d\xi\mid\end{aligned}$$

$$\begin{aligned}&\leq \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\{\int_G \mid \widehat{f}(\xi)\mid^2 (1+\gamma(\xi)^2)^{2s} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\overline{\widehat{\psi}(\mathfrak{p}^j\xi+v(l))}\mid d\xi\}^{\frac{1}{2}} \\ &\qquad\times\{\int_G \mid \widehat{f}(\xi+\mathfrak{p}^{-j}v(l))\mid^2 (1+\gamma(\xi+\mathfrak{p}^{-j}v(l))^2)^{2s} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\overline{\widehat{\psi}(\mathfrak{p}^j\xi+v(l))}\mid d\xi\}^{\frac{1}{2}} \\ &= \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\{\int_G \mid \widehat{f}(\xi)\mid^2 (1+\gamma(\xi)^2)^{2s} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\overline{\widehat{\psi}(\mathfrak{p}^j\xi+v(l))}\mid d\xi\}^{\frac{1}{2}} \\ &\qquad\times\{\int_G \mid \widehat{f}(\xi)\mid^2 (1+\gamma(\xi)^2)^{2s} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\overline{\widehat{\psi}(\mathfrak{p}^j\xi-v(l))}\mid d\xi\}^{\frac{1}{2}}.\end{aligned}$$

Since $\{v(k) : k \in \mathbb{N}\} = \{-v(k) : k \in \mathbb{N}_0\}$, we have

$$\mid T_2\mid \leq \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}}\int_G \mid \widehat{f}(\xi)\mid^2 (1+\gamma(\xi)^2)^{2s} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\overline{\widehat{\psi}(\mathfrak{p}^j\xi-v(l))}\mid d\xi. \tag{3.21}$$

By Levi Lemma we obtain,

$$\mid T_2\mid \leq \int_G \mid \widehat{f}(\xi)\mid^2 (1+\gamma(\xi)^2)^{2s}\{\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\overline{\widehat{\psi}(\mathfrak{p}^j\xi-v(l))}\mid\}d\xi. \tag{3.22}$$

Using equation 3.1, we get

$$\begin{aligned}&\int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi)\mid^2 \{(1+\gamma(\xi)^2)^s\sum_{j\in\mathbb{Z}} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\mid^2 -(1+\gamma(\xi)^2)^s\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\overline{\widehat{\psi}(\mathfrak{p}^j\xi+v(l))}\mid\}d\xi \\ &\leq \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}_0} |< f,\psi_{j,k} >|^2\end{aligned} \tag{3.23}$$

and

$$\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}_0} |< f,\psi_{j,k} >|^2 \leq \int_G (1+\gamma(\xi)^2)^s \mid \widehat{f}(\xi) \mid^2 \{(1+\gamma(\xi)^2)^s \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}} \mid \widehat{\psi}(\mathfrak{p}^j\xi)\overline{\widehat{\psi}(\mathfrak{p}^j\xi+v(l))} \mid\} d\xi \quad (3.24)$$

Taking infimum in equation (3.23) and supremum in equation (3.24), we obtain that

$$\triangle_3(\psi)||f||^2_{H^s_\gamma(G)} \leq \sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{N}_0} |< f,\psi_{j,k} >|^2 \leq \triangle_4(\psi)||f||^2_{H^s_\gamma(G)} \quad (3.25)$$

hold for all $g \in \mathscr{S}(G)$. The proof of theorem 3.2 is complete. □

Acknowledgments

The authors are thankful to the referee for giving valuable suggestions and comments to improvements in this paper.

References

1. C.K.Chui, An introduction to Wavelet Analysis and its Apllications, *Academic Press, Boston, MA*, 1, (1992).
2. C.P.Pandey and R.S.Pathak, Laguerre Wavelet Transforms,*Integral Transform and Special Funcions* (2009) pp 505-518.
3. M.H.Taibleson, Fourier Analysis on Local Field, *Princeton univ. Press*, 1975.
4. M.M. Dixit, C.P. Pandey, Deepanjan Das, Generalized continuous wavelet transform on locally compact abelian group,*Adv. Inequal. Appl.*,(2019), Article ID 10.
5. R.S.Pathak and Ashish Pathak, On Convolution for Wavelets Transform, *International Journal of Wavelets, multiresolution and information processing*, 6(5) (2008) pp 739-747.
6. M. M. Dixit, C. P. Pandey , A.k.Pandey, and D.Das, The Generalized Wavelet Transform on Sobolev type Spaces, *Advances in Mathematics: Scientific Journal* 9 (2020), no.12,pp 11003–11010.
7. C. P. Pandey, P Phukan, Continuous and Discrete Wavelet Transforms associated with Hermite Transform, *International Journal of Analysis and Applications* Vol 18, Number 4 (2020),pp 531-549.
8. Przemyslaw Gorka,Tomasz Kotrzewa and Enrique G.Reyes, Sobolev Space on Locally Compact Abelian Group:Compact Embeddings and local space,*Journal of Function Spaces*, Article ID 404738(2014) 6 pages.
9. Yang, Q., Taylor, K.F.: Multiresolution analysis and Haar-likewavelet bases on locally compact groups. *J. Appl. Funct. Anal.* 7(4), 413–439 (2012).
10. Raj Kumar, Satyapriya, Firdous A. Shah, Explicit construction of wavelet frames on locally compact Abelian groups, *Analysis and Mathematical Physics* (2022) 12:83.
11. Shah, F.A.,Wahid, A.:Wavelet packets on locally compact Abelian groups. *Analele St. Univ. Ovidius Constanta*. 18(2), 223–240 (2010).
12. Ashish Pathak, Dileep Kumar, Guru P Singh, The Necessary and Sufficient Conditions for Wavelet Frames in Sobolev Space Over Local Fields, *Bol. Soc. Paran. Mat.* (3s.) v. 39 3 (2021): 81–92.
13. Gorka, Przemys law; Kostrzewa, Tomasz Sobolev spaces on metrizable groups. *Ann. Acad. Sci. Fenn. Math.* 40 (2015), no. 2, 837–849.
14. A.Ghaani Farashahi, Generalized wavelet transforms over finite fields. *Linear Multilinear Algebra* 68 (2020),
15. A. Ghaani Farashahi, Galois wavelet transforms over finite fields. *Rocky Mountain J. Math.* 49 (2019), no. 1, 79–99. no. 8, 1585–1604.
16. A. Ghaani Farashahi, Structure of finite wavelet frames over prime fields. *Bull. Iranian Math. Soc.* 43 (2017), no. 1, 109–120.
17. A. Ghaani Farashahi, Abstract harmonic analysis of wave packet transforms over locally compact abelian groups, *Banach Journal of Mathematical Analysis*, 11(1) (2017), 50-71.
18. A. Ghaani Farashahi, Multivariate wave-packet transforms, Zeitschrift f¨ur Analysis und ihre Anwendungen *Journal of Analysis and its Applications*, Volume 36, Issue 4, 2017, pp. 481-500.
19. A. Ghaani Farashahi, Wave packet transforms over finite cyclic groups, *Linear Algebra and its Applications*, 489 (2016), 75-92.
20. A. Ghaani Farashahi, Wave packet transform over finite fields, *Electronic Journal of Linear Algebra*, 30(2015),507-529.
21. A. Ghaani Farashahi, Cyclic wave packet transform on finite Abelian groups of prime order. *Int. J. Wavelets Multiresolut. Inf. Process.* 12 (2014), no. 6, 1450041, 14 pp.

22. Arefijamaal, A. A.; Kamyabi-Gol, R. A. On the square integrability of quasi regular representation on semidirect product groups. *J. Geom. Anal. 19* (2009), no. 3, 541–552.

23. Arefijamaal, A. A.; Kamyabi-Gol, R. A. On construction of coherent states associated with semidirect products. *Int. J. Wavelets Multiresolut. Inf. Process.* 6 (2008), no. 5, 749–759.

M. M. Dixit, C. P. Pandey, Pratima Devi,
Department of Mathematics,
North Eastern Regional Institute of Science and Technology,
India.
**Corresponding author: C.P.Pandey*
E-mail address: mmdixit79@gmail.com
E-mail address: cpp.nerist@gmail.com
E-mail address: pd4229872@gmail.com