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# Existence and Multiplicity of Solutions for a Steklov Eigenvalue Problem Involving The p(x)-Laplacian-like Operator

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ABSTRACT: Using the variational method, we prove the existence and multiplicity of solutions for a Steklov problem involving the $p(x)$-Laplacian-like operator, originated from a capillary phenomena. Especially, an existence criterion for infinite many pairs of solutions for the problem is obtained.

Key Words: $p(x)$-Laplacian-like, Steklov problem, mountain pass theorem, variable exponent Sobolev space.

## Contents



## 1. Introduction

This paper is devoted to the study of the existence and multiplicity of solutions for the following nonlinear eigenvalue problem for $p(x)$-Laplacian-like operator:

$$\begin{cases} -\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)+|u|^{p(x)-2}u \;=\; 0 & \text{in } \Omega,\\ \left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)\frac{\partial u}{\partial \nu} \;=\; \lambda f(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a real number, $p \in C(\overline{\Omega})$, $\nu$ is the unit outward normal to $\partial\Omega$ and $f$ is a Carathéodory function.

In recent years, differential and partial differential equations with variable exponent growth conditions have become increasingly popular. This is partly due to their frequent appearance in applications such as the modeling of electrorheological fluids, image restoration, elastic mechanics and continuum mechanics.

Capillarity can be briefly explained by considering the effects of two opposing forces: adhesion, i.e., the attractive (or repulsive) force between the molecules of the liquid and those of the container; and cohesion, i.e., the attractive force between the molecules of the liquid. The study of capillary phenomenon has gained some attention recently. This increasing interest is motivated not only by fascination in naturally occurring phenomena such as motion of drops, bubbles, and waves but also its importance in applied fields ranging from industrial and biomedical and pharmaceutical to microfluidic systems.

W. Ni and J. Serrin [16,15] initiated the study of ground states for equations of the form

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(u) \text{ in } \mathbb{R}^N, \tag{1.2}$$

with very general right hand side $f$. Radial solutions of the problem (1.2) has been studied in the context of the analysis of capillary surfaces for a function $f$ of the form $f(u) = ku$, for $k > 0$. In [3], Clément, Manasevich and Mitidieri studied the existence of positive radial solutions of problem (1.2) with $f(u) = |u|^{q-1}u$.

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Obersnel and Omari in [17] based on variational and combines critical point theory, the lower and upper solutions method and elliptic regularization, established the existence and multiplicity of positive solutions of the prescribed mean curvature problem

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda f(x,u) \ \text{ in } \Omega, \\ \qquad\qquad\qquad\quad u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda > 0$ is a parameter and $f : \mathbb{R} \to \mathbb{R}$ is a Carathéodory function whose potential satisfies a suitable oscillating behavior at zero.

For $f$ dependent of $\nabla u$ and $\lambda \in \mathbb{R}$ with Neumann boundary condition, ME. Ouaarabi and al. established in [18], established some new sufficient conditions under which the problem (1.1) possesses a weak solutions, using a topological degree for a class of demicontinuous operators of generalized $(S_+)$ type. Under Dirichlet boundary condition, M. Rodrigues [19], by using Mountain Pass lemma and Fountain theorem, studied the existence of non-trivial solutions for (1.1). Under Seklov boundary condition we cite [11,1].

Problems like (1.1), (1.2) and (1.3) play, as is well known, a role in differential geometry and in the theory of relativity.

Our purpose of this work is to extend some of the known results with $p$-Laplacian-like or with Dirichlet boundary conditions on bounded domain.

The remainder of the article is organized as follows. In section 2, we review some fundamental preliminaries about the functional framework where we will treat our problem. In Section 3, we establish the existence result (see, Theorem 3.2 and Theorem 3.3). Finally, in Section 4, we prove the full multiplicity Theorem 4.1.

## 2. Preliminaries

We start with some preliminary basic results on the theory of Lebesgue, Sobolev spaces with a variable exponent. For more details, we refer the reader to the book by Musielak [13] and the papers by Edmunds et al. [5,4], Kovácik, O., Rákosnk, [12], and Fan et al. [8,7].

Let $p \in C(\overline{\Omega})$ be a variable exponent. Throughout this paper, we denote

$$p^- := \inf_{x\in\overline{\Omega}} p(x), p^+ := \sup_{x\in\overline{\Omega}} p(x)$$

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N; \\ +\infty, & \text{if } p(x) \geq N, \end{cases}$$

and

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : 1 < p^- < p^+ < \infty\}.$$

For $p, \alpha \in C_+(\overline{\Omega})$ we define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is a measurable and } \int_\Omega |u(x)|^{p(x)} dx < +\infty\},$$

with the norm

$$|u|_{p(x)} = \inf\{\tau > 0; \int_\Omega \left|\frac{u(x)}{\tau}\right|^{p(x)} dx \leq 1\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\| = |\nabla u|_{p(x)} + |u|_{p(x)}.$$

The properties of $W^{1,p(x)}(\Omega)$ and the properties concerning the embedding results are given in the following propositions.

**Proposition 2.1.** *[6,7,8]*

*(1) Both $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are separable, reflexive Banach spaces;*

*(2) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$,then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$is compact and continuous;*

*(3) Assume that the boundary of $\Omega$ possesses the cone property and $s \in C_+(\overline{\Omega})$ and $s(x) < p^\partial(x)$ for $x \in \overline{\Omega}$, then there is a compact and continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\partial\Omega)$.*

The mapping

$$\rho(u) := \int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |u|^{p(x)}dx, \forall u \in W^{1,p(x)}(\Omega),$$

plays an important role in manipulating the generalized Lebesgue-Sobolev spaces.

**Proposition 2.2.** *For $u, u_k \in W^{1,p(x)}(\Omega), k = 1, 2, 3, ...$ we have*

. $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;

. $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+}$;

. $\|u_k\| \to 0$ *if and only if* $\rho(u_k) \to 0$;

. $\|u_k\| \to \infty$ *if and only if* $\rho(u_k) \to \infty$.

Next, we recall the following version of the Mountain Pass Theorem [10], that will be used in the proof of Theorem 3.3.

**Theorem 2.3.** *[10] Let $X$ endowed with the norm $\|.\|_X$, be a Banach space. Assume that $\phi \in C^1(X;\mathbb{R})$ satisfies the Palais-Smale (PS) condition. Also, assume that $\phi$ has a mountain pass geometry, that is,*

**(i)** *there exists two constants $\eta > 0$ and $\rho \in \mathbb{R}$ such that $\phi(u) \geq \rho$ if $\|u\|_X = \eta$;*

**(ii)** *$\phi(0) < \rho$ and there exists $e \in X$ such that $\|e\|_X > \eta$ and $\phi(e) < \rho$.*

*Then $\phi$ has a critical point $u_0 \in X$ such that $u_0 \neq 0$ and $u_0 \neq e$ with critical value*

$$\phi(u_0) = \inf_{\gamma \in P} \sup_{u \in \gamma} \phi(u) \geq \rho > 0,$$

*where $P$ denotes the class of the paths $\gamma \in C([0,1];X)$ joining $0$ to $e$.*

Finally, we remind the Weierstrass type theorem that will be used in the proof of Theorem 3.2.

**Theorem 2.4.** *[2] Assume that $X$ is a reflexive Banach space and the function $\Phi : X \to \mathbb{R}$ is coercive and (sequentially) weakly lower semicontinuous on $X$. Then, $\Phi$ is bounded from below on $X$ and attains its infimum on $X$.*

Let $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x,0) = 0$ for all $x \in \partial\Omega$ and let

$$F(x,t) := \int_0^t f(x,s)ds, \ (x,t) \in \partial\Omega \times \mathbb{R}, \tag{2.1}$$

and the family of functions

$$\mathcal{F} = \{G_\alpha : G_\alpha(x,t) = f(x,t)t - \alpha F(x,t), \alpha \in [p^-, p^+]\}.$$

Noticing that when $p(x) = p$ is a constant, $\mathcal{F} = \{f(x,t)t - pF(x,t),\}$ consists of only one element.

We limit ourselves to the subcritical case, i.e. we assume that

$H_1(f)$ There exists $c_1 > 0$ and $p(x) < \alpha(x) < p^\partial(x)$ where $\alpha \in C_+(\partial\Omega)$ satisfying

$$|f(x,t)| \leq c_1(1 + |t|^{\alpha(x)-1}) \text{ for all } (x,t) \in \partial\Omega \times \mathbb{R}.$$

$H_2(f)$ $f(x,t)t \geq 0$, and $f(x,t)$ is superlinear at infinity, that is

$$\lim_{|t|\to\infty} \frac{f(x,t)t}{|t|^{p^+}} = +\infty, \text{ uniformly with respect to a.e. } x \in \partial\Omega.$$

$H_3(f)$ $f(x,t) = o(|t|^{p^+-1})$, $t \to 0$ for $x \in \partial\Omega$ uniformly.

$H_4(f)$ The function is odd with respect to its second variable, that is

$$f(x,-t) = -f(x,t), \quad \text{for a.e. } x \in \partial\Omega, t \in \mathbb{R}.$$

$H_5(f)$ $\exists M > 0$, $\theta > p^+$ such that $f$ satisfies the Ambrosetti-Rabinowitz condition

$$0 < \theta F(x,t) \leq tf(x,t), \text{ for all } |t| \geq M, \text{ and a.e. } x \in \partial\Omega.$$

The condition $H_1(f)$ implies that the functional $\Phi : W^{1,p(x)}(\Omega) \to \mathbb{R}$,

$$\Phi(u) := \int_\Omega \left( \frac{|\nabla u|^{p(x)}}{p(x)} + \frac{1}{p(x)}\left(\sqrt{1 + |\nabla u|^{2p(x)}}\right) + \frac{|u|^{p(x)}}{p(x)} \right) dx - \lambda \int_{\partial\Omega} F(x,u)d\sigma$$

is well defined and of class $C^1$. It is well known that the critical points of $\Phi$ are weak solutions of (1.1). The condition $H_2(f)$ characterizes the problem (1.1) as superlinear at infinity.

Consider the following functional

$$\varphi(u) := \int_\Omega \left( \frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} + \frac{1}{p(x)}\left(\sqrt{1 + |\nabla u|^{2p(x)}}\right)\right) dx, u \in X := W^{1,p(x)}(\Omega).$$

Let $\varphi \in C^1(X, \mathbb{R})$ and the derivative operator of $\varphi$ in weak sense $\varphi : X \to X^*$ is such that

$$(\varphi'(u), v) = \int_\Omega \left( |\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv + \frac{|\nabla u|^{2p(x)-2}\nabla u \nabla v}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) dx, \ \forall u, v \in X. \tag{2.2}$$

**Proposition 2.5.** *[19] The functional $\varphi : X \to \mathbb{R}$ is convex. The mapping $\varphi' : X \to X^*$ is a strictly monotone, bounded homeomorphism, and is of type $(S_+)$, namely, $u_n \rightharpoonup u$ in $X$ and $\limsup_{n\to\infty} a(\varphi'(u_n) - \varphi'(u), u_n - u)) \leq 0$ implies $u_n \to u \in X$.*

## 3. Existence results

In this section, we establish the existence of solution to problem (1.1).

**Definition 3.1.** *We recall that $u \in W^{1,p(x)}(\Omega)$ is a weak solution of the problem (1.1), if*

$$\int_\Omega \left( |\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}} \right) \cdot \nabla v dx + \int_\Omega |u|^{p(x)-2}uv dx = \lambda \int_{\partial\Omega} f(x,u)v d\sigma$$

*for all* $v \in W^{1,p(x)}(\Omega)$.

Our main results of this section are the following

**Theorem 3.2.** *If $\lambda > 0$, and $f$ satisfies the condition*

$$|f(x,t)| \leq c_1(1 + |t|^{\beta-1}), \text{ where } 1 \leq \beta < p^-, \tag{3.1}$$

*then the problem (1.1) has a weak solution.*

**Theorem 3.3.** *Assume that $f$ satisfies $H_1(f)$-$H_5(f)$, and let $p, \alpha \in C_+(\overline{\Omega})$ such that $p^+ < \alpha^- < \alpha^+ < p^\partial(x)$, for $x \in \partial\Omega$. Then for any $\lambda > 0$ the problem (1.1) has a nontrivial solution.*

*Proof of Theorem 3.2.* From (3.1) we have $|F(x,t)| \leq C(1+|t|^\beta)$ and

$$\begin{aligned}
\Phi(u) &= \int_\Omega \left( \frac{|\nabla u|^{p(x)}}{p(x)} + \frac{1}{p(x)} \left( \sqrt{1+|\nabla u|^{2p(x)}} \right) + \frac{|u|^{p(x)}}{p(x)} \right) dx - \lambda \int_{\partial\Omega} F(x,u) d\sigma \\
&\geq \frac{\|u\|^{p^-} + \sqrt{1+\|u\|^{2p^-}}}{p^+} - \lambda C \int_{\partial\Omega} |u|^\beta d\sigma - \lambda C_1 \\
&\geq \frac{2\|u\|^{p^-}}{p^+} - \lambda C_2 \|u\|^\beta - \lambda C_1 \to \infty, \text{ as } \|u\| \to \infty.
\end{aligned}$$

As $\Phi$ is weakly lower semicontinuous, $\Phi$ has a minimum point $u \in W^{1,p(x)}(\Omega)$ and $u$ is a weak solution of the problem (1.1) which completes the proof. $\square$

**Definition 3.4.** *We say $\Phi$ satisfies the Palais-Smale condition in $X$, if any sequence $\{u_n\} \subset X$ such that $\{\Phi(u_n)\}$ is bounded and $\|\Phi'(u_n)\| \to 0$ as $n \to \infty$, has a convergent subsequence.*

**Lemma 3.5.** *If $f$ satisfies the Ambrosetti-Rabinowitz condition $H_5(f)$, then $\Phi$ satisfies the Palais-Smale condition.*

*Proof.* Let $c > 0$ and $\{u_n\} \subset X$, $\{\Phi(u_n)\}$ is bounded and $\|\Phi'(u_n)\| \to 0$. We first show that $\{u_n\}$ is bounded. To do so, we argue by contradiction and we assume that, up to a subsequence, $\|u_n\| \to \infty$. Then, using $H_5(f)$, for sufficiently large $n$ we have

$$\begin{aligned}
c + 1 + \|u_n\| &\geq \Phi(u_n) - \frac{1}{\theta} \langle \Phi'(u_n), u_n \rangle \\
&= \int_\Omega \left( \frac{1}{p(x)} - \frac{1}{\theta} \right) \left( |\nabla u_n|^{p(x)} + |u_n|^{p(x)} \right) dx + \int_\Omega \frac{1}{\sqrt{1+|\nabla u|^{2p(x)}}} dx \\
&\quad - \lambda \int_{\partial\Omega} \left( F(x,u_n) - \frac{1}{\theta} f(x,u_n) \right) d\sigma \\
&\geq \left( \frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|^{p^-} - \lambda \int_{\{x \in \partial\Omega; |u_n| \geq M\}} \left( F(x,u_n) - \frac{1}{\theta} f(x,u_n) \right) d\sigma \\
&\quad - \lambda |\partial\Omega| \sup\{F(x,t) - f(x,t)t; x \in \partial\Omega, |t| < M\}.
\end{aligned}$$

Using Proposition 2.1 and $H_5(f)$, we deduce that, for sufficiently large $n$,

$$c + 1 + \|u_n\| \geq \left( \frac{1}{p^+} - \frac{1}{\theta} \right) \|u_n\|^{p^-} - \lambda |\partial\Omega| \sup\{F(x,t) - f(x,t)t; x \in \partial\Omega, |t| < M\}.$$

Dividing by $\|u_n\|^{p^-}$ and Letting $n$ go to infinity in the above inequality, then we obtain a contradiction. Therefore, $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. For a subsequence of $\{u_n\}$, $u_n \rightharpoonup u$ weakly in $W^{1,p(x)}(\Omega)$, strongly in $L^{p(x)}(\Omega)$. Therefore $\langle \Phi'(u_n), (u_n - u) \rangle \to 0$, $\int_\Omega |u_n|^{p(x)-2}(u_n)(u_n - u)dx \to 0$ and by $H_1(f)$ we have $\int_{\partial\Omega} f(x,u_n)(u_n - u)d\sigma \to 0$. Consequently by Proposition 2.5, $u_n \to u$ strongly in $W^{1,p(x)}(\Omega)$, and so $\Phi$ satisfies the Palais-Smale (PS) condition. $\square$

*Proof of Theorem 3.3.* By Lemma 3.5, $\Phi$ satisfies Palais-Smale (PS) condition in $W^{1,p(x)}(\Omega)$. To apply Theorem 2.3, we will show that $\Phi$ possesses the mountain pass geometry.

Since $p^+ < \alpha^- \leq \alpha(x) < p^\partial(x)$, for all $x \in \partial\Omega$, we have from Proposition 2.1 that $W^{1,p(x)}(\Omega) \hookrightarrow L^{p^+}(\partial\Omega)$ and $W^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\partial\Omega)$ with a continuous and compact embeddings. So, there exist $c_i > 0, i = 1,2$ such that

$$|u|_{p^+,\partial\Omega} \leq c_1 \|u\| \text{ and } |u|_{\alpha(x),\partial\Omega} \leq c_2 \|u\| \ \forall u \in W^{1,p(x)}(\Omega). \tag{3.2}$$

Let $\varepsilon$ be small enough such $\varepsilon\lambda c_1^{p^+} \leq \frac{1}{2p^+}$. By the assumptions $H_1(f)$ and $H_3(f)$, we have

$$F(x,t) \leq \varepsilon |t|^{p^+} + C(\varepsilon)|t|^{\alpha(x)} \ \forall (x,t) \in \partial\Omega \times \mathbb{R}.$$

Therefore, in view of (3.2), for $\|u\|$ sufficiently small we get

$$\begin{aligned}
\Phi(u) &\geq \int_\Omega \big(\frac{2}{p^+}|\nabla u|^{p^+} + \frac{1}{p^+}|u|^{p^+}\big)dx - \lambda\varepsilon \int_{\partial\Omega} |u|^{p^+} d\sigma - \lambda C(\varepsilon) \int_{\partial\Omega} |u|^{\alpha(x)} d\sigma \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \lambda\varepsilon c_1^{p^+}\|u\|^{p^+} - \lambda C(\varepsilon) c_2^{\alpha^-}\|u\|^{\alpha^-} \\
&\geq \|u\|^{p^+}\left(\frac{1}{2p^+} - C(\varepsilon)c_2^{\alpha^-}\|u\|^{\alpha^- - p^+}\right).
\end{aligned}$$

As $\alpha^- > p^+$, by the standard argument, there exists $r > 0$ such that $\Phi(u) \geq \rho > 0$ for every $u \in W^{1,p(x)}(\Omega)$ and $\|u\| = r$.

From $H_2(f)$ it follows that

$$F(x,t) \geq C|t|^\theta \ x \in \partial\Omega, |t| \geq M, \theta > p^+.$$

For $w \in W^{1,p(x)}(\Omega)\backslash\{0\}$ and $t > 1$, we have

$$\begin{aligned}
\Phi(tw) &= \int_\Omega \frac{1}{p(x)}|\nabla tw|^{p(x)}dx + \int_\Omega \sqrt{1 + |\nabla tw|^{2p(x)}} + \int_\Omega \frac{1}{p(x)}|tw|^{p(x)}dx \\
&\quad - \lambda \int_{\partial\Omega} F(x,tw)d\sigma \\
&\leq t^{p^+} \int_\Omega \frac{1}{p(x)}\left(|\nabla w|^{p(x)} + |w|^{p(x)} + \sqrt{1 + |\nabla w|^{2p(x)}}\right) \\
&\quad - \lambda C t^\theta \int_{\partial\Omega} |w|^\theta d\sigma - \lambda C_1,
\end{aligned}$$

which implies $\Phi(tw) \to -\infty (t \to +\infty)$. Since $\Phi(0) = 0$, $\Phi$ satisfies the condition of Mountain Pass Theorem 2.3. So $\Phi$ has at least one nontrivial critical point, i.e., the problem (1.1) has a nontrivial weak solution. □

## 4. Infinitely many solutions

In this section, we prove under some condition on the function $f$ that the problem (1.1) possesses infinitely many nontrivial weak solutions. The proof is based on Bartsch's fountain theorem.

The main result of this section is the following.

**Theorem 4.1.** *Assume that $f$ satisfies $H_1(f), H_2(f)$ and $H_4(f)$. If $0 < \lambda < p^+\alpha^+$ and $\alpha^- > p^+$, then $\Phi$ has a sequence of critical points $\{u_n\}$ such that $\Phi(u_n) \to +\infty$ and the problem (1.1) has infinite many (pairs) of solutions.*

Since $W^{1,p(x)}(\Omega)$ is reflexive and separable (and its dual), then $X$ and $X^*$ are too. Let $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{span\{e_j | j = 1,2,3...\}}, \ X^* = \overline{span\{e_j^* | j = 1,2,3...\}},$$

and

$$\langle e_j, e_j^* \rangle = \begin{cases} 1, \ i = j \\ 0, \ i \neq j, \end{cases}$$

where $\langle .,. \rangle$ denotes the duality product between$X$ and $X^*$. For convenience, we write

$$X_j = span\{e_j\}, \ Y_k = \oplus_{j=1}^k X_j, \ Z_k = \oplus_{j=k}^\infty X_j.$$

Let us recall the version of the Fountain theorem 4.2 which will be used in the sequel.

**Theorem 4.2.** *(Fountain theorem, [20]). Under assumption*
*$(A_1)$, let $\varphi \in C^1(X, \mathbb{R})$ be an invariant functional. If, for every $k \in \mathbb{N}$, there exists $\rho_k > \gamma_k > 0$ such that*
*$(A_2)$ $a_k := \max_{u \in Y_k, \|u\|=\rho_k} \varphi(u) \leq 0$;*
*$(A_3)$ $b_k := \inf_{u \in Y_k, \|u\|=\gamma_k} \varphi(u) \to \infty, k \to \infty$;*
*$(A_4)$ $\varphi$ satisfies the Palais-Smale (PS)c condition for every $c > 0$,*
*then $\varphi$ has an unbounded sequence of critical values.*

In order to prove Theorem 4.1, we need the following lemma.

**Lemma 4.3.** *If $\alpha \in C_+(\overline{\Omega})$, $\alpha(x) < p^\partial(x)$ for any $x \in \overline{\Omega}$, denote*

$$\beta_k = \sup\{|u|_{\alpha(x),\partial\Omega} \mid \|u\| = 1, u \in Z_k\},$$

*then* $\lim_{k\to\infty} \beta_k = 0$.

*Proof.* Obviously, $0 < \beta_{k+1} \leq \beta_k$, so $\beta_k \to \beta \geq 0$. Let $u_k \in Z_k$ satisfy

$$\|u\| = 1, 0 \leq \beta_k - |u_k|_{\alpha(x),\partial\Omega} < \frac{1}{k}.$$

Then there exists a subsequence of $\{u_k\}$ (which we still denote by $u_k$) such that $u_k \rightharpoonup u$, and

$$\langle e_j^*, u\rangle = \lim_{k\to\infty} \langle e_j^*, u_k\rangle = 0, j = 1, 2, ...$$

which implies that $u = 0$, and so $u_k \rightharpoonup 0$. Since the imbedding from $W^{1,p(x)}(\Omega)$ to $L^{\alpha(x)}(\partial\Omega)$ is compact, then $u_k \to 0$ in $L^{\alpha(x)}(\partial\Omega)$. Hence we get $\beta_k \to 0$. □

*Proof of Theorem 4.1.* According to $H(f_0), H(f_4)$, $\Phi$ is an even functional and satisfies (PS) condition. We will prove that if $k$ is large enough, then there exist $\rho_k > \gamma_k > 0$ such that

$$(A_1)\ a_k := \max\{\Phi(u) | u \in Y_k, \|u\| = \rho_k\} \leq 0$$

$$(A_2)\ b_k := \inf\{\Phi(u) | u \in Z_k, \|u\| = \gamma_k\} \to \infty, k \to \infty.$$

In what follows, we will use the mean value theorem in the following form: for every $\alpha \in C_+(\partial\Omega)$ and $u \in L^{\alpha(x)}(\partial\Omega)$, there is $\zeta \in \partial\Omega$ such that

$$\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma = |u|^{\alpha(\zeta)}_{\alpha,\partial\Omega}. \tag{4.1}$$

Indeed, it is well known that there is $\zeta \in \partial\Omega$ such that

$$1 = \int_{\partial\Omega} \left(|u|/|u|_{\alpha,\partial\Omega}\right)^{\alpha(x)} d\sigma = \int_{\partial\Omega} |u|^{\alpha(x)} dx / |u|^{\alpha(\zeta)}_{\alpha,\partial\Omega}.$$

Then, (4.1) holds. The assertion of this theorem can be obtained from Fountain Theorem 4.2.
$(A_2)$ For any $u \in Z_k, \|u\| = \gamma_k \geq 1$ ($\gamma_k$ will be specified below). Using $H_5(f)$ and (4.1) we deduce

$$\begin{aligned}
\Phi(u) &= \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1 + |\nabla u|^{2p(x)}}\right) + \int_\Omega \frac{1}{p(x)}|u|^{p(x)} dx - \lambda \int_{\partial\Omega} F(x, u) d\sigma \\
&\geq \int_\Omega \frac{1}{p^+}\left(2|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx - \lambda c \int_{\partial\Omega} |u|^{\alpha(x)} d\sigma - \lambda c_1 \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - \lambda c |u|^{\alpha(\zeta)}_{\alpha,\partial\Omega} - \lambda c_1, \text{ where } \zeta \in \partial\Omega, \\
&\geq \begin{cases} \frac{1}{p^+}\|u\|^{p^-} - \lambda c - \lambda c_2 & |u|_{(\alpha,\partial\Omega)} \leq 1; \\ \frac{1}{p^+}\|u\|^{p^-} - \lambda c \beta_k \|u\|^{\alpha^+} - \lambda c_2 & |u|_{(\alpha,\partial\Omega)} > 1 \end{cases} \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - \lambda c \beta_k^{\alpha^+} \|u\|^{\alpha^+} - \lambda c_3 \\
&= \gamma_k^{p^-}\left(\frac{1}{p^+} - \lambda c \beta_k^{\alpha^+} \gamma_k^{\alpha^+ - p^-}\right) - \lambda c_3.
\end{aligned}$$

We fix $\gamma_k$ as follows

$$\gamma_k := \left(c\alpha^+ \beta_k^{\alpha^+}\right)^{\frac{1}{p^- - \alpha^+}},$$

then

$$\Phi(u) \geq \gamma_k^{p^-}\left(\frac{1}{p^+} - \frac{\lambda}{\alpha^+}\right).$$

Because $p^+ < \alpha^+$, $\beta_k \to 0$ (Lemma 4.3) and $0 < \lambda < p^+\alpha^+$, it follows $\gamma_k \to +\infty$, as $k \to +\infty$. Consequently $\Phi(u) \to +\infty$ as $\|u\| \to +\infty$ with $u \in Z_k$. The assertion $(A_2)$ is valid.

$(A_1)$ From $H_2(f)$, we have

$$F(x,t) \geq c_1|t|^\theta - c_2.$$

By $\theta > p^+$ and $dimY_k = k$, it is easy to see that

$$\int_{\partial\Omega} F(x,u)d\sigma \to -\infty, \text{ as } \|u\| \to \infty,$$

for $u \in Y_k$. The assertion $(A_1)$ holds. Applying the fountain theorem, we achieve the proof of Theorem 4.1. □

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