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Regularity and Normality Modulo Hereditary m-Spaces

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ABSTRACT: An m-structure is introduced and investigated in [14]. A hereditary class $\mathcal{H}$ is defined and investigated in [8]. In this paper, we introduce and investigate generalizations of regularity and normality in a hereditary m-space $(X, m, \mathcal{H})$ and a hereditary topological space $(X, \tau, \mathcal{H})$.

Key Words: Minimal structure, hereditary class, $\mathcal{H}$-regularity, $m\mathcal{H}$-regularity, $\mathcal{H}$-normality, $m\mathcal{H}$-normality.

Contents

1. Introduction

The notion of ideals in topological spaces was introduced by Kuratowski [10]. Janković and Hamlett [9] investigated the further properties of ideal topological spaces. Recently, many generalizations of regularity and normality in ideal topological spaces are introduced and investigated. In this paper, we define and study regularity and normality in a hereditary m-space $(X, m, \mathcal{H})$ and a heredittry topological space $(X, \tau, \mathcal{H})$. In Section 3, we introduce and investigate the notion of $m\mathcal{H}$-regularity. In the last section, the notion of $m\mathcal{H}$-normality is introduced and their properties are obtained. Recently papers [1-7] have introduced some new classes of sets via hereditary m-spaces.

2. Minimal Structures

Definition 2.1. A subfamily m of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [14] on X if $\emptyset \in m$ and $X \in m$.

By (X, m), we denote a nonempty set X with a minimal structure m on X and call it an *m-space*. Each member of m is said to be *m-open* and the complement of an m-open set is said to be *m-closed*. For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in m\}$ is denoted by $m(x)$.

Definition 2.2. *Let (X, m) be an m-space and A a subset of X. The* m-closure *$mCl(A)$ of A [11] is defined by $mCl(A) = \cap\{F \subseteq X : A \subseteq F, X \setminus F \in m\}$.*

Lemma 2.3. *[11] Let X be a nonempty set and m a minimal structure on X. For subsets A and B of X, the following properties hold:*
(1) $A \subseteq mCl(A)$ and $mCl(A) = A$ if A is m-closed,
(2) $mCl(\emptyset) = \emptyset$ and $mCl(X) = X$,
(3) If $A \subseteq B$, then $mCl(A) \subseteq mCl(B)$,
(4) $mCl(A) \cup mCl(B) \subseteq mCl(A \cup B)$,
(5) $mCl(mCl(A)) = mCl(A)$.

Definition 2.4. *A minimal structure m of a set X is said to have* property $\mathcal{B}$ *[11] if the union of any collection of elements of m is an element of m.*

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Lemma 2.5. *[14] Let (X, m) be an m-space and A a subset of X.*
(1) $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m(x)$.
(2) Let m have property $\mathcal{B}$. Then the following properties hold:
(i) A is m-closed if and only if $\mathrm{mCl}(A) = A$,
(ii) $\mathrm{mCl}(A)$ is m-closed.

Definition 2.6. A nonempty subfamily $\mathcal{H}$ of $\mathcal{P}(X)$ is called a *hereditary class* on X [8] if it satisfies the following properties: $A \in \mathcal{H}$ and $B \subset A$ implies $B \in \mathcal{H}$. A nonempty subfamily $\mathcal{I}$ of $\mathcal{P}(X)$ is called an *ideal* [9] if it is a hereditary class and it satisfies the additional condition : $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

A minimal space (X, m) with a hereditary class $\mathcal{H}$ on X is called a *hereditary minimal space* (briefly *hereditary m-space*) and is denoted by $(X, m, \mathcal{H})$.

Definition 2.7. Let $(X, m, \mathcal{H})$ be a hereditary m-space. For a subset A of X, the *minimal local function* $A^{\star}_{mH}(\mathcal{H}, m)$ of A [13] is defined as follows:

$$A^{\star}_{mH}(\mathcal{H}, m) = \{x \in X : U \cap A \notin \mathcal{H} \text{ for every } U \in m(x)\}.$$

Hereafter, $A^{\star}_{mH}(\mathcal{H}, m)$ is simply denoted by $A^{\star}_{mH}$.

Remark 2.8. Let $(X, m, \mathcal{H})$ be a hereditary m-space and A a subset of X.
(1) If $\mathcal{H} = \{\emptyset\}$ (resp. $\mathcal{P}\ (X)$), then $A^{\star}_{mH} = mCl(A)$ (resp. $A^{\star}_{mH} = \emptyset$).
(2) If $A \in \mathcal{H}$, then $A^{\star}_{mH} = \emptyset$.

3. $m\mathcal{H}$-regular spaces

Definition 3.1. *A hereditary m-space $(X, m, \mathcal{H})$ is said to be* $m\mathcal{H}$-regular *if for each m-closed set F and a point $p \notin F$, there exist disjoint m-open sets U and V such that $p \in U$ and $F - V \in \mathcal{H}$.*

Definition 3.2. *A hereditary topological space $(X, \tau, \mathcal{H})$ is said to be* $\mathcal{H}$-regular *if for each closed set F and a point $p \notin F$, there exist disjoint open sets U and V such that $p \in U$ and $F - V \in \mathcal{H}$.*

Definition 3.3. *[12] A minimal space (X, m) is said to be* m-regular *if for each m-closed set F and each $x \notin F$, there exist disjoint m-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Clearly, for the hereditary class $\mathcal{H} = \{\emptyset\}$, m-regularity and $m\mathcal{H}$-regularity coincide.

Example 3.4. *Let $X = \{a, b, c, d\}$, $m = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. Let $F = \{a, b, d\}$ be m-closed set and $c \notin F$, since there no any disjoint m-open set U, V such that $a \in U$ and $F \subseteq V$, then $(X, m, \mathcal{H})$ is not m-regular. But $(X, m, \mathcal{H})$ is $m\mathcal{H}$-regular, because if we conceder F be any m-closed set and $x \notin F$ as the following table*

$F = \{a, b, d\}$	$c \notin F$	$U = \{c\}$ *and* $V = \{a, b\}$, $U \cap V = \emptyset$ *and* $F - V \in \mathcal{H}$
$F = \{c, d\}$	$a \notin F$	$U = \{a, b\}$ *and* $V = \{c\}$, $U \cap V = \emptyset$ *and* $F - V \in \mathcal{H}$
$F = \{c, d\}$	$b \notin F$	$U = \{a, b\}$ *and* $V = \{c\}$, $U \cap V = \emptyset$ *and* $F - V \in \mathcal{H}$
$F = \{d\}$	$a \notin F$	$U = \{a, b\}$ *and* $V = \{c\}$, $U \cap V = \emptyset$ *and* $F - V \in \mathcal{H}$
$F = \{d\}$	$b \notin F$	$U = \{a, b\}$ *and* $V = \{c\}$, $U \cap V = \emptyset$ *and* $F - V \in \mathcal{H}$
$F = \{d\}$	$c \notin F$	$U = \{c\}$ *and* $V = \{a, b\}$, $U \cap V = \emptyset$ *and* $F - V \in \mathcal{H}$

Theorem 3.5. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and m have property $\mathcal{B}$. Then the following are equivalent.*

1. *X is $m\mathcal{H}$-regular.*

2. *For each $x \in X$ and any m-open set U containing x, there is an m-open set V containing x such that $mCl(V) - U \in \mathcal{H}$.*

3. *For each $x \in X$ and any m-closed set A not containing x, there is an m-open set V containing x such that $mCl(V) \cap A \in \mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and U be an m-open set containing x. Then, there exist disjoint m-open sets V and W such that $x \in V$ and $(X - U) - W \in \mathcal{H}$. If $(X - U) - W = H \in \mathcal{H}$, then $(X - U) \subseteq W \cup H$. Now $V \cap W = \emptyset$ implies that $V \subseteq X - W$ and so $mCl(V) \subseteq X - W$. Now $mCl(V) - U \subseteq (X - W) \cap (W \cup H) = (X - W) \cap H \subseteq H \in \mathcal{H}$.

(2) $\Rightarrow$ (3): Let $x \in X$ and A be an m-closed set in X such that $x \notin A$. Then, there exists an m-open set V containing x such that $mCl(V) - (X - A) \in \mathcal{H}$ which implies that $mCl(V) \cap A \in \mathcal{H}$.

(3) $\Rightarrow$ (1): Let A be an m-closed set in X such that $x \notin A$. Then, there is an m-open set V containing x such that $mCl(V) \cap A \in \mathcal{H}$. Since $mCl(V) \cap A \in \mathcal{H}$, then $A - (X - mCl(V)) \in \mathcal{H}$. V and $(X - mCl(V))$ are the required disjoint m-open sets such that $x \in V$ and $A - (X - mCl(V)) \in \mathcal{H}$. Hence X is $m\mathcal{H}$-regular. □

In [17] Zahid defined a space $(X, \tau, \mathcal{I})$ to be paracompact modulo $\mathcal{I}$, or simply $\mathcal{I}$-paracompact, iff every open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ such that $X - \mathcal{V} \in I$.

A space $(X, \tau, \mathcal{H})$ is said to be *paracompact modulo* $\mathcal{H}$, or simply $\mathcal{H}$-*paracompact* if every open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ such that $X - \mathcal{V} \in \mathcal{H}$.

Theorem 3.6. *If a hereditary topological space $(X, \tau, \mathcal{H})$ is $\mathcal{H}$-paracompact and Hausdorff, then $(X, \tau, \mathcal{H})$ is $\mathcal{H}$-regular.*

Proof. Let F be a closed subset of X and let $p \in X$ such that $p \notin F$. For each $x \in F$ there exist disjoint open sets U_x containing p and V_x containing x, particular, $p \notin Cl(V_x)$. Now $\mathcal{U} = \{V_x : x \in F\} \cup \{X - F\}$ is an open cover of X and hence there exists a precise locally finite open refinement $\mathcal{V} = \{V'_x : x \in F\} \cup \{W\}$ such that $V'_x \subseteq V_x$ for each $x \in F$, $W \subseteq X - F$, and $X - \cup\mathcal{V} = H \in \mathcal{H}$. Let $U = X - \cup\{C1(V'_x) : x \in F\}$ and $V = \cup\{V'_x : x \in F\}$. Then U and V are disjoint open sets, $p \in U$, and $F - V \subseteq H$, then $F - V \in \mathcal{H}$ and $(X, \tau, \mathcal{H})$ is $\mathcal{H}$-regular. □

A hereditary topological space $(X, \tau, \mathcal{H})$ is said to be *compact modulo* $\mathcal{H}$, or simply $\mathcal{H}$-*compact* if every open cover $\mathcal{U}$ of X contains a finite subcollection $\mathcal{F}$ such that $X \backslash \cup \mathcal{F} \in \mathcal{H}$.

It is well-known that a Hausdorff space (X, τ) is said to be H-*closed* if for every open cover $\{V_\alpha : \alpha \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $\cup\{Cl(V_\alpha) : \alpha \in \Delta_0\} = X$.

Theorem 3.7. *If a hereditary topological space $(X, \tau, \mathcal{H})$ is Hausdorff, $\mathcal{H}$-compact and $\mathcal{H} \cap \tau = \{\emptyset\}$, then it is H-closed and $\mathcal{H}$-regular.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any open cover of X. Then, there exists a finite subset Δ_0 of Δ such that $X - \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Then $X - \cup\{Cl(U_\alpha) : \alpha \in \Delta_0\} \subseteq X - \cup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Therefore, $X - \cup\{Cl(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H} \cap \tau = \{\emptyset\}$ and hence $\cup\{Cl(U_\alpha) : \alpha \in \Delta_0\} = X$. Hence (X, τ) is H-closed. Since $\mathcal{H}$-compact spaces are $\mathcal{H}$-paracompact, by Theorem 3.2 $(X, \tau, \mathcal{H})$ is $\mathcal{H}$-regular.

□

Theorem 3.8. *If a hereditary topological space $(X, \tau, \mathcal{H})$ is Hausdorff, countably $\mathcal{H}$-compact and first countable, then $(X, \tau, \mathcal{H})$ is $\mathcal{H}$-regular.*

Proof. Assume $(X, \tau, \mathcal{H})$ is a countably $\mathcal{H}$-compact, first countable and Hausdorff space. Let $F \subseteq X$ be closed and $p \notin F$. Let $\mathcal{U} = \{U_n : n = 1, 2, 3, ...\}$ be a countable nbd base at p. Since X is Hausdorlf, we have that $\{p\} = \cap_{n=1}^{\infty} Cl(U_n)$ and hence $\{X - F\} \cup \{X - Cl(U_n) : n = 1, 2, 3, ...\}$ is a countable open cover of X . Let $\mathcal{F} = \{X - F\} \cup \{X - Cl(U_m) : m = n_1, n_2, n_3, ..., n_k\}$ be a finite subcollection such that $X - \cup\mathcal{F} = H \in \mathcal{H}$. Now let $U = \cup_{i=1}^{k} U_{n_i}$ and $V = \cup_{i=1}^{k}[X - Cl(U_{n_i})]$. Then $p \in U$, $F - V \subseteq H \in \mathcal{H}$ and also, U and V are disjoint open sets. Hence $(X, \tau, \mathcal{H})$ is $\mathcal{H}$-regular. □

4. $m\mathcal{H}$-normal spaces

Definition 4.1. *A hereditary m-space $(X, m, \mathcal{H})$ is said to be* $m\mathcal{H}$-normal *if for every pair of disjoint m-closed sets A and B of X, there exist disjoint m-open sets U and V such that $A - U \in \mathcal{H}$ and $B - V \in \mathcal{H}$.*

Definition 4.2. *A hereditary topological space $(X, \tau, \mathcal{H})$ is said to be* $\mathcal{H}$-normal *if for every pair of disjoint closed sets A and B of X, there exist disjoint open sets U and V such that $A - U \in \mathcal{H}$ and $B - V \in \mathcal{H}$.*

Clearly, if $\mathcal{H} = \{\emptyset\}$, then normality and $\mathcal{H}$-normality coincide.

Definition 4.3. *A minimal space (X, m) is said to be* m-normal *if for every pair of disjoint m-closed sets A and B of X, there exist disjoint m-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.*

Clearly, if $\mathcal{H} = \{\emptyset\}$, then m-normality and $m\mathcal{H}$-normality coincide. Since $\emptyset \in \mathcal{H}$, it is clear that every m-normal space is an $m\mathcal{H}$-normal space for every hereditary class $\mathcal{H}$ but not conversely.

Example 4.4. *Consider the Modified Fort space ([16], Example 27) in which $X = \mathbb{N} \cup \{x_1\} \cup \{x_2\}$, where $\mathbb{N}$ is the set of all natural numbers, with the m-space defined as follows: Any subset of $\mathbb{N}$ is m-open and any set containing x_1 or x_2 is m-open if it contains all but a finite number of points of $\mathbb{N}$. This space is not m-normal. Let $\mathcal{H}_f$ be the hereditary class of all finite subsets of X. We prove that $(X, m, \mathcal{H}_f)$ is $m\mathcal{H}$-normal. Let A and B be two disjoint m-closed sets in X.*

Case (i). Let A and B are subsets of $\mathbb{N}$, then A and B are m-open. If $G = A$ and $H = B$. Then, since $\emptyset \in \mathcal{H}_f$, $A - G \in \mathcal{H}_f$ and $B - H \in \mathcal{H}_f$.

Case (ii). Suppose $x_1 \in A$ and $x_2 \notin A$. Let $G = A - \{x_1\}$ and $H = (X - A) - \{x_2\}$. Then G and H are disjoint. Since $G \subseteq \mathbb{N}$, G is m-open and $A - G = \{x_1\} \in \mathcal{H}_f$. Since $H \subseteq \mathbb{N}$, H is m-open and $B - H \subseteq B \cap A \subseteq A$. Since $x_2 \notin A$, A is finite and so $A \in \mathcal{H}_f$ which implies that $B - H \in \mathcal{H}_f$. Thus, there exist disjoint m-open sets G and H such that $A - G \in \mathcal{H}_f$ and $B - H \in \mathcal{H}_f$.

Case (iii). Suppose $x_1, x_2 \in A$. Let $G = A - \{x_1, x_2\}$ and $H = B$. Then G and H are disjoint. Since $G \subseteq \mathbb{N}$, G is m-open and $A - G = \{x_1, x_2\} \in \mathcal{H}_f$. $x_1, x_2 \notin B$ implies that $B \subseteq \mathbb{N}$ and so B is m-open. Thus there exist disjoint m-open sets G and H such that $A - G \in \mathcal{H}_f$ and $B - H \in \mathcal{H}_f$.

Thus, in all the three cases, there exist disjoint m-open sets G and H such that $A - G \in \mathcal{H}_f$ and $B - H \in \mathcal{H}_f$. Hence $(X, m, \mathcal{H}_f)$ is $m\mathcal{H}$-normal.

The following theorem characterizes $m\mathcal{H}$-normal spaces.

Theorem 4.5. *Let $(X, m, \mathcal{H})$ be a hereditary m-space and m have property $\mathcal{B}$. Then the following are equivalent.*

1. *$(X, m, \mathcal{H})$ is $m\mathcal{H}$-normal.*
2. *For every m-closed set F and m-open set G containing F, there exists an m-open set V such that $F - V \in \mathcal{H}$ and $mCl(V) - G \in \mathcal{H}$.*
3. *For each pair of disjoint m-closed sets A and B, there exists an m-open set U such that $A - U \in \mathcal{H}$ and $mCl(U) \cap B \in \mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2): Let F be m-closed and G be an m-open set such that $F \subseteq G$. Then $X - G$ is an m-closed set such that $(X - G) \cap F = \emptyset$. By hypothesis, there exist disjoint m-open sets U and V such that $(X - G) - U \in \mathcal{H}$ and $F - V \in \mathcal{H}$. Now $U \cap V = \emptyset$ implies that $mCl(V) \subseteq X - U$ and so $(X - G) \cap mCl(V) \subseteq (X - G) \cap (X - U)$ which implies that $mCl(V) - G \subseteq (X - G) - U \in \mathcal{H}$. Therefore, $mCl(V) - G \in \mathcal{H}$.

(2) $\Rightarrow$ (3): Let A and B be disjoint m-closed subsets of X. Then there exists an m-open set U such that $A - U \in \mathcal{H}$ and $mCl(U) - (X - B) \in \mathcal{H}$ which implies that $A - U \in \mathcal{H}$ and $mCl(U) \cap B \in \mathcal{H}$.

(3) $\Rightarrow$ (1): Let A and B be disjoint m-closed subsets in X. Then there exists an m-open set U such that $A - U \in \mathcal{H}$ and $mCl(U) \cap B \in \mathcal{H}$. Now $mCl(U) \cap B \in \mathcal{H}$ implies that $B - (X - mCl(U)) \in \mathcal{H}$. Put $V = X - mCl(U)$, then V is an m-open set such that $B - V \in \mathcal{H}$ and $U \cap V = U \cap (X - mCl(U)) = \emptyset$. Hence $(X, m, \mathcal{H})$ is $m\mathcal{H}$-normal. $\square$

If we put $\mathcal{H} = \{\emptyset\}$, then by Theorem 4.5 we obtain characterizations of m-normal spaces.

Corollary 4.6. *Let (X, m) be an m-space, where m has property $\mathcal{B}$. Then the following are equivalent:*

1. *(X, m) is m-normal;*
2. *For every m-closed set F and m-open set G containing F, there exists an m-open set V such that $F \subset V$ and $mCl(V) \subset G$;*
3. *For each pair of disjoint m-closed sets A and B, there exists an m-open set U such that $A \subset U$ and $mCl(U) \cap B = \emptyset$.*

Theorem 4.7. *If a hereditary m-space $(X, m, \mathcal{H})$ is $m\mathcal{H}$-normal, where m has property $\mathcal{B}$, and $Y \subseteq X$ is m-closed, then $(Y, m_Y, \mathcal{H}_Y)$ is $m_Y\mathcal{H}_Y$-normal.*

Proof. Let A and B be disjoint m_Y-closed subsets of Y. Since Y is m-closed, A and B are disjoint m-closed subsets of X. By hypothesis, there exist disjoint m-open sets U and V in X such that $A - U \in \mathcal{H}$ and $B - V \in \mathcal{H}$. If $A - U = H_1 \in \mathcal{H}$ and $B - V = H_2 \in \mathcal{H}$, then $A \subseteq U \cup H_1$ and $B \subseteq V \cup H_2$. Since $A \subseteq Y$, $A \subseteq Y \cap (U \cup H_1)$ and so $A \subseteq (Y \cap U) \cup (Y \cap H_1)$. Therefore, $A - (Y \cap U) \subseteq (Y \cap H_1) \in \mathcal{H}_Y$. Similarly, $B - (Y \cap V) \subseteq (Y \cap H_2) \in \mathcal{H}_Y$. Put $U_1 = Y \cap U$ and $V_1 = Y \cap V$, then U_1 and V_1 are disjoint m_Y-open sets such that $A - U_1 \in \mathcal{H}_Y$ and $B - V_1 \in \mathcal{H}_Y$. Hence $(Y, m_Y, \mathcal{H}_Y)$ is $m_Y\mathcal{H}_Y$-normal. □

In [15] Theorem 2.6, it was established that in an ideal topological space every $\mathcal{I}$-paracompact, Hausdorff space is $\mathcal{I}$-normal.

Theorem 4.8. *If a hereditary topological space $(X, \tau, \mathcal{H})$ is a Lindelof and $\mathcal{H}$-regular space, then $(X, \tau, \mathcal{H})$ is $\mathcal{H}$-normal.*

Proof. Let A and B be two disjoint closed subsets of X. Since $(X, \tau, \mathcal{H})$ is $\mathcal{H}$-regular, for each $a \in A$, there exist open sets U_a and V such that $a \in U_a$, $U_a \cap V = \emptyset$ and $B - V \in \mathcal{H}$. Hence $Cl(U_a) \cap V = \emptyset$ and $Cl(U_a) \cap B \subseteq B \cap (X - V) = B - V \in \mathcal{H}$. Therefore, $Cl(U_a) \cap B \in \mathcal{H}$. Since the collection $\{U_a \cap A : a \in A\}$ is a cover of A by open subsets of A and A is a Lindelof subspace of X, $A = \cup\{U_i \cap A : i \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of all natural numbers, which implies that $A \subseteq \cup\{U_i : i \in \mathbb{N}\}$. Hereafter, a_i is simply denoted by i. Also $Cl(U_i) \cap B \in \mathcal{H}$ for every $i \in \mathbb{N}$. Similarly, we can find a countable collection $\{V_i : i \in \mathbb{N}\}$ of open sets such that $B \subseteq \cup\{V_i : i \in \mathbb{N}\}$ and $Cl(V_i) \cap A = H_i \in \mathcal{H}$ for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $G_n = U_n - \cup\{Cl(V_i) : i = 1, 2, ..., n\}$ and $H_n = V_n - \cup\{Cl(U_i) : i = 1, 2, ..., n\}$. Let $G = \cup\{G_n : n \in \mathbb{N}\}$ and $H = \cup\{H_n : n \in \mathbb{N}\}$. Since G_n and H_n are open for each $n \in \mathbb{N}$, G and H are open subsets of X. Clearly, $G \cap H = \emptyset$. Now we prove that $A - G \in \mathcal{H}$. Let $x \in A$. Then $x \in U_m$ for some $m \in \mathbb{N}$. Also, $Cl(V_n) \cap A = H_n \in \mathcal{H}$ for every $n \in \mathbb{N}$ implies that $A \subseteq H_n \cup (X - Cl(V_n))$ for every $n \in \mathbb{N}$. Therefore, $x \in A$ implies that $x \in H_n \cup (X - Cl(V_n))$ for every n and so $x \in H_n$ or $x \notin Cl(V_n)$ for every n. Hence $x \in U_m - \cup\{Cl(V_j) : j = 1, 2, ..., m\}$ or $x \in \cap\{H_j : j \in \mathbb{N}\} = K \in \mathcal{H}$. Since $x \in G_m$, $x \in G$ and so $x \in G \cup K$. Hence $A \subseteq G \cup K$ which implies that $A - G \subseteq K \in \mathcal{H}$. Similarly, we can prove that $B - H \in \mathcal{H}$. Hence $(X, \tau, \mathcal{H})$ is $\mathcal{H}$-normal. □

Theorem 4.9. *If a hereditary m-space $(X, m, \mathcal{H})$, where m has property $\mathcal{B}$, is $m\mathcal{H}$-normal and $f : (X, m, \mathcal{H}) \to (Y, n, f(\mathcal{H}))$ is an mn-homeomorphism, then $(Y, n, f(\mathcal{H}))$ is $nf(\mathcal{H})$-normal space.*

Proof. Let A and B be disjoint n-closed subsets of Y. Since f is mn-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint m-closed subsets of X. Since $(X, m, \mathcal{H})$ is $\mathcal{H}$-normal, there exist disjoint m-open sets U and V in X such that $f^{-1}(A) - U \in \mathcal{H}$ and $f^{-1}(B) - V \in \mathcal{H}$. Since $f^{-1}(A) - U \in \mathcal{H}$, then $f(f^{-1}(A) - U) \in f(\mathcal{H})$ and $A - f(U) \in f(\mathcal{H})$. Similarly, $B - f(V) \in f(\mathcal{H})$. Since $f(U)$ and $f(V)$ are disjoint n-open sets in Y, it follows that $(Y, n, f(\mathcal{H}))$ is $nf(\mathcal{H})$-normal space. □

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