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Rate of Growth of Polynomials Not Vanishing Inside a Circle*

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ABSTRACT: For a polynomial $P(z) := \sum_{v=0}^{n} a_v z^v$ of degree n having all zeros in $|z| \geq k, k \geq 1$ Govil et al.[*ILLINOIS Journal of Math.*] proved:

$$|P'(z)| \leq n\frac{n|a_0| + k^2|a_1|}{(1+k^2)n|a_0| + 2k^2|a_1|}|P(z)|.$$

In this paper besides a refinement of above inequality, we generalize some well-known polynomial inequalities.

Key Words: Polynomials, inequalities, derivative, zeros.

Contents

1. Introduction

For each positive integer n, let $\mathcal{P}_n$ denote the linear space of all polynomials $P(z) := \sum_{v=0}^{n} a_v z^v$ of degree atmost n over the field $\mathbb{C}$ of complex numbers. If $P(z)$ is a polynomial of degree n, then concerning the estimate of $|P'(z)|$ in terms of $|P(z)|$ on $|z| = 1$, Bernstein [2] proved the following:

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|, \text{ for all } z \in \mathbb{C}. \tag{1.1}$$

By an application of maximum modulus principle we have for $R \geq 1$

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \tag{1.2}$$

Equality holds in (1.1) and (1.2) if and only if, P has all its zeros at the origin.
If we impose restrictions on the location of zeros of P, then in case of (1.1) Erdös conjectured and latter Lax [6] proved the following:

If $P \in \mathcal{P}_n$ has all zeros in $|z| \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Where as in connection with (1.2) Ankeny and Rivlin [1] proved:

If all zeros of $P \in \mathcal{P}_n$ lie in $|z| \geq 1$, then for $R \geq 1$

$$\max_{|z|=R} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)|. \tag{1.4}$$

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However, on the other hand, Turán [11] proved the following:

If $P \in \mathcal{P}_n$ has all zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.5}$$

All the above inequalities have been improved and generalized in various ways from time to time which one can find in various papers and monographs, however an interesting reader can go through [[9], [7], [10]]. In this paper we prove some results for the polynomials having all zeros in $|z| \geq k, k \geq 1$. Our results generalize as well as sharpen some well-known inequalities.

2. Basic Lemmas

First lemma is due to Frappier et al. [4].

Lemma 2.1. *If $P(z)$ is a polynomial of degree $n \geq 1$, then for $R \geq 1$*

$$\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - (R^n - R^{n-2})|P(0)|, \text{ if } n > 1 \tag{2.1}$$

and

$$\max_{|z|=R} |P(z)| \leq R \max_{|z|=1} |P(z)| - (R-1)|P(0)|, \text{ if } n = 1. \tag{2.2}$$

Lemma 2.2. *If $P(z) := \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \geq k, k \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq n \frac{n|a_0| + k^2|a_1|}{(1+k^2)n|a_0| + 2k^2|a_1|} \max_{|z|=1} |P(z)|. \tag{2.3}$$

The above lemma is due to Govil et al. [5].

Lemma 2.3. *If $P(z) := a_0 + \sum_{v=s}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \geq k, k \geq 1$, then*

$$\max_{|z|=1} |P'(z)| \leq n \frac{1 + (s/n)|a_s/a_0|k^{s+1}}{1 + k^{s+1} + (s/n)|a_s/a_0|(k^{s+1} + k^{2s})} \max_{|z|=1} |P(z)|. \tag{2.4}$$

The above lemma is due to Qazi [8].

3. Main Results

Theorem 3.1. *If $P \in \mathcal{P}_n$ be such that $P(z) = \sum_{j=0}^{n} a_j z^j$ has all zeros in $|z| \geq k, k \geq 1$, then for any $R, R \geq 1$ and $0 \leq \theta < 2\pi$, we have*

$$\begin{aligned} |P(Re^{i\theta}) - P(e^{i\theta})| \leq{}& \frac{R^n - 1}{n} \left(n \frac{n|a_0| + k^2|a_1|}{(1+k^2)n|a_0| + 2k^2|a_1|} \right) \max_{|z|=1} |P(z)| \\ &- \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right) |a_1|, \text{ for } n > 2. \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} |P(Re^{i\theta}) - P(e^{i\theta})| \leq{}& \frac{R^2 - 1}{2} \left(n \frac{n|a_0| + k^2|a_1|}{(1+k^2)n|a_0| + 2k^2|a_1|} \right) \max_{|z|=1} |P(z)| \\ &- \frac{(R-1)^2}{2} |a_1|, \text{ for } n = 2. \end{aligned}$$

Proof. Since all zeros of $P(z)$ lie in $|z| \geq k, k \geq 1$, therefore using Lemma 2.2, we have for the polynomial $P(z)$ of degree $n \geq 2$

$$\max_{|z|=1} |P'(z)| \leq n \frac{n|a_0| + k^2|a_1|}{(1+k^2)n|a_0| + 2k^2|a_1|} \max_{|z|=1} |P(z)|. \tag{3.2}$$

We first consider the case when $n > 2$, hence we have for every $\theta, 0 \leq \theta < 2\pi$

$$\begin{aligned} |P(Re^{i\theta}) - P(e^{i\theta})| &= \left| \int_1^R P'(re^{i\theta}) e^{i\theta} dr \right| \\ &\leq \int_1^R |P'(re^{i\theta})| dr. \end{aligned} \tag{3.3}$$

Since $P(z)$ is of degree $n > 2$, therefore the polynomial $P'(z)$ is of degree $(n-1) \geq 2$. Hence using inequality (2.1) of Lemma 2.1 to $P'(z)$, we get for $n \geq 3$

$$\begin{aligned} |P(Re^{i\theta}) - P(e^{i\theta})| &\leq \int_1^R \left(r^{n-1} \max_{|z|=1} |P'(z)| - (r^{n-1} - r^{n-3})|P'(0)| \right) dr \\ &= \frac{R^n - 1}{n} \max_{|z|=1} |P'(z)| - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right) |P'(0)|. \end{aligned} \tag{3.4}$$

Using (3.2) in (3.4), it follows that for $n > 2$

$$\begin{aligned} |P(Re^{i\theta}) - P(e^{i\theta})| \leq \frac{R^n - 1}{n} &\left(n \frac{n|a_0| + k^2|a_1|}{(1+k^2)n|a_0| + 2k^2|a_1|} \right) \max_{|z|=1} |P(z)| \\ &- \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right) |a_1|. \end{aligned}$$

The proof for the case $n = 2$ can be obtained on similar lines, by using (2.2) of Lemma 2.1 instead of inequality (2.1). □

This completes the proof of Theorem 3.1.

If we divide both sides of (3.1) by $R - 1$ and letting $R \to 1$, we get a result due to Govil et al. [5]. Also on simplification (3.1) reduces to

$$\begin{aligned} |P(Re^{i\theta})| \leq &\frac{(R^n + k^2)n|a_0| + (R^n + 1)k^2|a_1|}{(1+k^2)n|a_0| + 2k^2|a_1|} \max_{|z|=1} |P(z)| \\ &- \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2} \right) |a_1|. \end{aligned} \tag{3.5}$$

On dividing both sides of (3.5) by R^n and letting $R \to \infty$, we get

$$|a_n| + \frac{|a_1|}{n} \leq \frac{n|a_0| + k^2|a_1|}{(1+k^2)n|a_0| + 2k^2|a_1|} \max_{|z|=1} |P(z)|. \tag{3.6}$$

For $k = 1$, (3.6) reduces to

$$|a_n| + \frac{|a_1|}{n} \leq \frac{1}{2} \max_{|z|=1} |P(z)|.$$

This inequality can also be obtained by applying an inequality of Visser [12] to $P'(z)$ and then combining it with inequality (1.3).

We next prove the following generalization of Theorem 3.1.

Theorem 3.2. *If $P(z) := a_0 + \sum_{v=s}^{n} a_v z^v$ be such that all the zeros of $P(z)$ lie in $|z| \geq k, k \geq 1$, then*

$$|P(Re^{i\theta}) - P(e^{i\theta})| \leq \frac{R^n - 1}{n}\left(n\frac{1 + (s/n)|a_s/a_0|k^{s+1}}{1 + k^{s+1} + (s/n)|a_s/a_0|(k^{s+1} + k^{2s})}\max_{|z|=1}|P(z)|\right) - \left(\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right)|a_1|, \text{ for } n > 2. \tag{3.7}$$

and

$$|P(Re^{i\theta}) - P(e^{i\theta})| \leq \frac{R^2 - 1}{2}\left(n\frac{1 + (s/n)|a_s/a_0|k^{s+1}}{1 + k^{s+1} + (s/n)|a_s/a_0|(k^{s+1} + k^{2s})}\max_{|z|=1}|P(z)|\right) - \frac{(R-1)^2}{2}|a_1|, \text{ for } n = 2.$$

For $s = 1$, (3.7) reduces to (3.1).

Proof. The proof of Theorem 3.2 follows on similar lines as that of Theorem 3.1 by using Lemma 2.3 instead of Lemma 2.2. □

Remark 3.3. *On dividing both sides of (3.7) by $R - 1$ and letting $R \to 1$, it reduces to a result due to Qazi [8].*

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