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Picture Fuzzy Normed Linear Space

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abstract: Picture fuzzy set (PFS) is a recent advancement tool to deal with vulnerability. It is an immediate expansion of intuitionistic fuzzy set that can display vulnerability in such circumstances including more responses of these kinds: indeed, decline, no. In this manuscript the idea of Picture fuzzy normed linear space (PFNLS) is discussed for the first time. Naturally PFNLS is an hybrid concept of PFS and normed linear space. Also Convergence in PFNLS are shown. Later on Completeness property on PFNLS are explored. Finally boundedness of Cauchy sequence in PFNLS is analysed.

Key Words: Normed linear space, norm, co-norm, picture fuzzy set, picture fuzzy normed linear space.

Contents

1 Introduction **1**

2 PFNLS **2**

3 Convergence in PFNLS **3**

4 Completeness on PFNLS **4**

5 Conclusion **8**

1. Introduction

In 1965 Prof. Zadeh initiated Fuzzy set (FS) theory in [1]. Basically FS was the first step of generalizing the ancient conceptualization of classical set theory to combat with the unpredictability. However there were limitations of FS too. To overcome the limitations Atanassov [2,3,10] further generalised FS theory and launched the concept of Intuitionstic FS (IFS) theory. Here degree of neutrality was not taken into consideration. Prof. Cuong and Kreinovich added the concept of neutral membership in IFS theory and gave us a beautiful concept of new set i.e. Picture Fuzzy Set (PFS) in their paper [19]. Naturally PFS can be considered as a generalization of IFS. With the advancement of time several researchers had shown interest on PFS theory and different types of research work were done on PFS theory [9,20,22,23,24,25].

On the other hand Normed linear space is the important pillar of Functional analysis, a major branch of modern mathematics. In 1992 Prof. Felbin introduced Fuzzy normed linear space (FNLS) [4]. It was shown in [6,11] that every finite dimensional FNLS holds fuzzy norm uniquely w.r.t. fuzzy equivalence. In 1993 Prof. Felbin further established that any finite dimensional FNS is necessarily complete. In 2003 Prof. Bag and Samanta [12] decomposed the Fuzzy norm to a usual crisp norm. This paper opened the way of different approaches of study of FN spaces. As a result different articles on FNLS had been published and various developments of FNLS i.e. Intuitionistic FNLS, Neutrosophic NLS, n-FNLS [5,7,8,13,14,15,21,26,28,29] etc are done. In this article we have defined Picture Fuzzy normed linear space for the first time. Since PFS is a generalization of IFS, thus PFNLS is a much more generalised concept than the other previous concept. our manuscript is organized as following: In Section 2 the definition of PFNLS is given and a suitable example is provided for better understanding. In Section 3 Convergence in PFNLS is shown. In the next section Completeness property of PFNLS is explored. Here we have proved that that every finite dimensional PFNLS is complete. Finally Section 5 concludes our article.

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2. PFNLS

To realize PFNLS, we request our readers to read the idea of t-norm say $\odot$ and t-conorm say $\circ$ at first. The details about t-norm and t-conorm can be found in any standard article say [26]. Since PFS is a continuous development of FS and IFS. The preliminary idea of PFS, a new theory introduced by Cuong et. al. was first published in 2013 in [19]. This section introduces us the concept of PFNLS based on PFS theory. Further we have provided an example of PFNLS for smooth understanding in this section.

Definition 2.1. *Suppose U be a linear space over $\mathbb{R}$.A picture fuzzy subset*

$$A = \{((\alpha, r); P(\alpha, r), Q(\alpha, r), R(\alpha, r)) : (\alpha, r) \in U \times \mathbb{R}^+\}$$

is called a PF norm on U w.r.t. continuous t-norm $\odot$ and t-co-norm $\circ$ respectively if the following holds:

(a) $P(\alpha, r) + Q(\alpha, r) + R(\alpha, r) \leq 1 \;\; \forall (\alpha, r) \in U \times \mathbb{R}^+$.

(b) $P(\alpha, r) > 0$.

(c) $P(\alpha, r) = 1$ *iff* $\alpha = 0$.

(d) $P(k\alpha, r) = P(\alpha, \frac{r}{|k|})$, $k \in \mathbb{R} \setminus \{0\}$.

(e) $P(\alpha, r) \odot P(\beta, s) \leq P(\alpha + \beta, r + s)$.

(f) $P(\alpha, .)$ *is non-decreasing mapping of* $\mathbb{R}^+$ *and* $\lim_{r\to\infty} P(\alpha, r) = 1$.

(g) $Q(\alpha, r) > 0$.

(h) $Q(\alpha, r) = 0$ *if and only if* $\alpha = 0$.

(i) $Q(k\alpha, r) = Q(\alpha, \frac{r}{|k|})$, $k \in \mathbb{R} \setminus \{0\}$.

(j) $Q(\alpha, r) \circ Q(\beta, s) \geq Q(\alpha + \beta, r + s)$.

(k) $Q(\alpha, .)$ *is non-increasing function of* $\mathbb{R}^+$ *and* $\lim_{r\to\infty} Q(\alpha, r) = 0$.

(l) $R(\alpha, r) > 0$.

(m) $R(\alpha, r) = 0$ *iff* $x = 0$.

(n) $R(k\alpha, r) = R(\alpha, \frac{r}{|k|})$, $k \in \mathbb{R} \setminus \{0\}$.

(o) $R(\alpha, r) \circ R(\beta, s) \geq R(\alpha + \beta, r + s)$.

(p) $R(\alpha, .)$ *is non-increasing mapping of* $\mathbb{R}^+$ *and* $\lim_{r\to\infty} R(\alpha, r) = 0$.

Here $(U, A, \odot, \circ)$ is called a PFNLS. We will denote $(U, A, \odot, \circ)$ as (U, A) throughout this article.

Example 2.2. *Suppose $U = (\mathbb{R}, \|.\|)$ be NLS with $\|.\| = |x| \; \forall \, x \in \mathbb{R}$. Considering $a_1 \odot a_2 = \min\{a_1, a_2\}$ and $a_1 \circ a_2 = \max\{a_1, a_2\} \; \forall \, a_1, a_2 \in [0, 1]$ we take $P(\alpha, r) = \frac{r}{r+c|\alpha|}, Q(\alpha, r) = \frac{c|\alpha|}{r+c|\alpha|}, R(\alpha, r) = \frac{|\alpha|}{r}$, $c > 0$. We take $A = \{(\alpha, r); P(\alpha, r), Q(\alpha, r), R(\alpha, r)\}$. Clearly $(U, A, \odot, \circ)$ is an PFNLS.*

3. Convergence in PFNLS

Definition 3.1. *A sequence $\{s_n\}$ in a PFNLS (U, A) is said to be convergent and converges to $s \in U$ if for any $\epsilon_0 > 0,\ r > 0,\ \exists\, n_0 \in \mathbb{N}$ such $P(s_n - s, r) > 1 - \epsilon_0, Q(s_n - s, r) < \epsilon_0, R(s_n - s, r) < \epsilon_0,\ \forall n \geq n_0$ and $\epsilon_0 \in (0, 1)$.*

Theorem 3.2. *A sequence $\{s_n\}$ converges to $s \in V$ in a PFNLS (U, A) iff $\lim_{n\to\infty} P(s_n - s, r) = 1, \lim_{n\to\infty} Q(s_n - s, r) = 0, \lim_{n\to\infty} R(s_n - s, r) = 0$.*

We omit this proof as the proof is very straight forward.

Theorem 3.3. *Every convergent sequence $\{s_n\}$ in a PFNLS (U, A) has distinctive limit.*

Proof. Suppose the limit of $\{s_n\}$ is not unique i.e. $\lim_{n\to\infty} s_n = l_1$ and $\lim_{n\to\infty} s_n = l_2$. Then for $s, t \in \mathbb{R}^+$ and $s, t \to \infty$ we have,

$$\lim_{n\to\infty} P(s_n - l_1, s) = 1, \lim_{n\to\infty} Q(s_n - l_1, s) = 0, \lim_{n\to\infty} R(s_n - l_1, s) = 0,$$
$$\lim_{n\to\infty} P(s_n - l_2, t) = 1, \lim_{n\to\infty} Q(s_n - l_2, t) = 0, \lim_{n\to\infty} R(s_n - l_2, t) = 0,$$

Then $P(l_1 - l_2, s + t) \geq P(s_n - l_1, s) \odot P(s_n - l_2, t)$. Considering limit as $s, t, n \to \infty$ we have

$$P(l_1 - l_2, s + t) = 1$$

In a similar way we have $Q(l_1 - l_2, s + t) = 0 = R(l_1 - l_2, s + t)$. Thus $l_1 = l_2$ and we are done. □

Theorem 3.4. *If $\{s_n\}$ and $\{t_n\}$ are two convergent sequence in a PFNLS (U, A) such that $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$ respectively. Then the followings hold:*

(a) $\lim_{n\to\infty}(s_n + t_n) = (s + t)$ *in an PFNLS* $(U, A, \odot, \circ)$.

(b) $\lim_{n\to\infty}(s_n - t_n) = (s + t)$ *in an PFNLS* $(U, A, \odot, \circ)$.

(c) *if* $c \in \mathbb{R}$ *then* $\lim_{n\to\infty} cs_n = cs$ *in an PFNLS* $(U, A, \odot, \circ)$.

Proof. (a) Suppose $m_1, m_2 \in \mathbb{R}^+$. Then

$$\lim_{n\to\infty} P(s_n - s, m_1) = 1, \lim_{n\to\infty} Q(s_n - s, m_1) = 0, \lim_{n\to\infty} R(s_n - s, m_1) = 0,$$
$$\lim_{n\to\infty} P(t_n - t, m_2) = 1, \lim_{n\to\infty} Q(t_n - t, m_2) = 0, \lim_{n\to\infty} R(t_n - t, m_2) = 0,$$

Again

$$\begin{aligned}\lim_{n\to\infty} P[(s_n + t_n) - (s + t), m_1 + m_2] &\geq \lim_{n\to\infty} P(s_n - s, m_1) \odot \lim_{n\to\infty} P(t_n - t, m_2)\\ &= 1 \text{ as } m_1, m_2 \to \infty.\end{aligned}$$

Thus $\lim_{n\to\infty} P[(s_n + t_n) - (s + t), m_1 + m_2] = 1$. In a parallel way one can easily see that, $\lim_{n\to\infty} Q[(s_n + t_n) - (s + t), m_1 + m_2] = 0 = \lim_{n\to\infty} R[(s_n + t_n) - (s + t), m_1 + m_2]$.

We omit the proof of the part (b) and part (c) due to the similarity with part (a). □

4. Completeness on PFNLS

In this section we will study the completeness property on PFNLS. For this reason we have to introduce the concept of Cauchy sequence in PFNLS. After introducing the concept of Cauchy sequence in PFNLS we will study the several characteristics of complete PFNLS.

Definition 4.1. *A sequence $\{s_n\}$ in PFNLS $(U, A, \odot, \circ)$ is said to be bounded if for any $\epsilon \in (0,1)$ and $r > 0$, $P(s_n, r) > 1-\epsilon, Q(s_n, r) < \epsilon, R(s_n, r) < \epsilon \; \forall \, n \in \mathbb{N}$ hold.*

Definition 4.2. *Consider a sequence $\{s_n\}$ in PFNLS $(U, A, \odot, \circ)$. Then $\{s_n\}$ becomes a Cauchy sequence if given $\epsilon_0 \in (0,1), r > 0, \exists \; k \in \mathbb{N}$ s.t. $P(s_n - s_m, r) > 1 - \epsilon_0, Q(s_n - s_m, r) < \epsilon_0, R(s_n - s_m, r) < \epsilon_0 \; \forall \, m, n \geq k$.*

Alternatively $\{s_n\}$ is supposed to be Cauchy if $\lim_{n,m\to\infty} P(s_n - s_m, r) = 1$, $\lim_{n,m\to\infty} Q(s_n - s_m, r) = 0$, $\lim_{n,m\to\infty} R(s_n - s_m, r) = 0$ as $r \to \infty$.

Theorem 4.3. *In PFNLS $(U, A, \odot, \circ)$ a sequence which is convergent must be a Cauchy sequence.*

Proof. Consider $\{s_n\}$ in $(U, A, \odot, \circ)$ s.t. $\lim_{n\to\infty} s_n = s$. Then for $l_1 \in \mathbb{R}^+, \exists k \in \mathbb{N}$ s.t. for $m, n \geq k$

$$\lim_{n,m\to\infty} P(s_n - s_m, l_1) \geq \lim_{n\to\infty} P(s_n - s, \frac{l_1}{2}) \odot \lim_{m\to\infty} P(s_m - s, \frac{l_1}{2}) \geq 1$$

Thus $\lim_{n,m\to\infty} P(s_n - s_m, l_1) = 1$. □

Similarly we can show that $\lim_{n,m\to\infty} Q(s_n - s_m, l_1) = 0$, $\lim_{n,m\to\infty} R(s_n - s_m, l_1) = 0$. Hence $\{s_n\}$ is Cauchy in $(U, A, \odot, \circ)$. However our next example exemplifies that the converse of Theorem 4.3 is not true.

Example 4.4. *Recall the PFNLS $(U, A, \odot, \circ)$ (as in Example 2.2) where $U = (\mathbb{R}, \|.\|), \|x\| = |x| \; \forall \; x \in \mathbb{R}$, $a_1 \odot a_2 = \min\{a_1, a_2\}$, $a_1 \circ a_2 = \max\{a_1, a_2\}$ for all $a_1, a_2 \in [0,1]$. We consider $P(x,t) = \frac{t}{t+k|x|}, Q(x,t) = \frac{k|x|}{t+k|x|}, R(x,t) = \frac{|x|}{t}$ $t > 0$. Further we suppose $\{s_n\}$ is Cauchy. Then $\lim_{n,m\to\infty} \frac{t}{t+|s_n - s_m|} = 1$, $\lim_{n,m\to\infty} \frac{|s_n - s_m|}{t+|s_n-s_m|} = 0$, $\lim_{n,m\to\infty} \frac{|s_n-s_m|}{t} = 0$. Hence $\lim_{n,m\to\infty} P(s_n - s_m, t) = 1$, $\lim_{n,m\to\infty} Q(s_n - s_m, t) = 0$, $\lim_{n,m\to\infty} R(s_n - s_m, t) = 0$. We consider $U_1 = \{\frac{1}{k} | k \in \mathbb{N}\}$. It is clear that $(U_1, A, \odot, \circ)$ is a PFNLS and $\{s_n\}$ is also a Cauchy sequence in $(U_1, A, \odot, \circ)$. However $\{s_n\}$ is not convergent.*

Theorem 4.5. *Consider two Cauchy sequence $\{s_n\}, \{t_n\}$ of vectors and a Cauchy sequence $\{\mu_n\}$ of scalars in a PFNLS $(U, A, \odot, \circ)$. Then $\{s_n + t_n\}, \{\mu_n s_n\}$ are also Cauchy in $(U, A, \odot, \circ)$.*

Proof. As the proof is very straight-forward so we have omitted it. □

Theorem 4.6. *Suppose $(U, A, \odot, \circ)$ be a PFNLS. If $\exists$ a convergent sub-sequence of any Cauchy sequence in $(U, A, \odot, \circ)$, then $(U, A, \odot, \circ)$ be a complete PFNLS.*

Proof. Consider a Cauchy sequence $\{s_n\}$ in $(U, A, \odot, \circ)$. Suppose $\{s_{n_r}\}$ be a convergent sub-sequence of $\{s_n\}$ converging to an element s. Being a Cauchy sequence $\{s_n\}$ we have for $t > 0$ and $t \to \infty$,

$$\lim_{n,r\to\infty} P(s_n - s_{n_r}, \frac{t}{2}) = 1,$$

$$\lim_{n,r\to\infty} Q(s_n - s_{n_r}, \frac{t}{2}) = 0,$$

$$\lim_{n,r\to\infty} R(s_n - s_{n_r}, \frac{t}{2}) = 0,$$

Also we have,

$$\lim_{n,r\to\infty} P(s_{n_r}-s,\frac{t}{2})=1,$$
$$\lim_{n,r\to\infty} Q(s_{n_r}-s,\frac{t}{2})=0,$$
$$\lim_{n,r\to\infty} R(s_{n_r}-s,\frac{t}{2})=0,$$

Then

$$P(s_n-s,t)=P(s_n-s_{n_r}+s_{n_r}-s,t)\geq P(s_n-s_{n_r})\odot P(s_{n_r}-s,t)$$

which implies that $\lim_{n\to\infty} P(s_n-s,t)=1$. In a parallel way we can see that

$$\lim_{n\to\infty} Q(s_n-s,t)=0,\ \lim_{n,r\to\infty} R(s_n-s,t)=0$$

Hence we are done. □

Theorem 4.7. *Suppose $(U,A,\odot,\circ)$ be a PFNLS and*

(q) $\nu\circ\nu=\nu,\nu\odot\nu=\nu\ \forall\ \nu\in[0,1]$.

(u) $\forall\ t>0\ P(x,t)>0,\ Q(x,t)>0,\ R(x,t)>0$ *implies that* $x=\underline{0}$.

We define $\|x\|_\gamma^1=\bigwedge\{t:P(x,t)\geq\gamma\},\ \|x\|_\gamma^2=\bigvee\{t:Q(x,t)\leq\gamma\},\ \|x\|_\gamma^3=\bigvee\{t:R(x,t)\leq\gamma\}$ *where* $\gamma\in(0,1)$. *Then* $\{|x\|_\gamma^1\},\ \{|x\|_\gamma^2\},\ \{|x\|_\gamma^3\}$ *are ascending family of norms on* U. *Above three norms are combined as* γ*-norms i.e.* $|x\|_\gamma$ *on* $(U,A,\odot,\circ)$.

Proof. We know that for $x\in U,\ P(x,t)=0$ for $t<0$ implies $\bigwedge\{t:P(x,t)\geq\gamma\}\geq 0,\ \gamma\in(0,1)\implies |x\|_\gamma^1\geq 0$.

Now if $|x\|_\gamma^1=0\implies\bigwedge\{t:P(x,t)\geq\gamma\}=0$ implies that $x=\underline{0}$. Conversely $x=\underline{0}\implies P(x,t)=1\ \forall t>0\implies\forall\gamma\in(0,1)\bigwedge\{t:P(x,t)\geq\gamma\}=0\implies |x\|_\gamma^1=0$.

Again if $c=0$ we are done. If $c\neq 0$, then

$$|cx\|_\gamma^1=\bigwedge\{r:P(cx,r)\geq\gamma\}=\bigwedge\{r:P(x,\frac{r}{|c|})\gamma=\bigwedge|c|\{t:P(x,t)\geq\gamma\}$$

.

Finally,

$$\begin{aligned}
&|x\|_\gamma^1+|y\|_\gamma^1\\
&=\quad\bigwedge\{t:P(x,t)\geq\gamma\}+\bigwedge\{r:P(y,r)\geq\gamma\}\\
&=\quad\bigwedge\{t+r:P(x,t)\geq\gamma,P(y,r)\geq\gamma\}\\
&=\quad\bigwedge\{t+r:P(x,t)\odot P(y,r)\geq\gamma\odot\gamma\}\\
&\geq\quad\bigwedge\{t+r:P(x+y,t+r)\geq\gamma\}\\
&=\quad\|x+y\|_\gamma^1.
\end{aligned}$$

.

Suppose $\gamma_1<\gamma_2$ and $\gamma_1,\gamma_2\in(0,1)$.Then $\{t:P(x,t)\geq\gamma_2\}\subset\{t:P(x,t)\geq\gamma_1\}$ which clearly implies that $\|x\|_{\gamma_2}^1\geq\|x\|_{\gamma_1}^1$.Thus $\{\|x\|_\gamma^1:\gamma\in(0,1)\}$ is an ascending on U.

Similarly we can validate that $\{|x\|_\gamma^2\geq 0,\ \gamma\in(0,1)\}$.

Now if $|x\|_\gamma^2 = 0 \implies \bigvee\{t : Q(x,t) \le \gamma = 0 \implies x = \underline{0}$. Conversely

$$x = \underline{0} \implies Q(x,t) = 0\, \forall t > 0 \implies \forall \gamma \in (0,1), \bigvee\{t : Q(x,t) \le \gamma\} = 0 \implies |x\|_\gamma^2 = 0$$

.

Also if $c = 0$ we are again done.If $c \neq 0$, then $|cx\|_\gamma^2 = \bigvee\{r : Q(cx, r) \le \gamma\} = \bigvee\{r : Q(x, \frac{r}{|c|})\gamma = \bigvee |c|\{t : P(x,t) \le \gamma\}$.

Lastly

$$\begin{aligned}
&|x\|_\gamma^2 + |y\|_\gamma^2 \\
&= \bigvee\{t : Q(x,t) \le \gamma\} + \bigvee\{r : Q(y,r) \le \gamma\} \\
&= \bigvee\{t + r : Q(x,t) \le \gamma, Q(y,r) \le \gamma\} \\
&= \bigvee\{t + r : Q(x,t) \odot Q(y,r) \le \gamma \odot \gamma\} \\
&\le \bigvee\{t + r : Q(x+y, t+r) \le \gamma \\
&= \|x+y\|_\gamma^2.
\end{aligned}$$

Let $\gamma_1 < \gamma_2$ and $\gamma_1, \gamma_2 \in (0,1)$. Clearly $\{t : Q(x,t) \le \gamma_1\} \subset \{t : Q(x,t) \le \gamma_2\}$. Thus $\|x\|_{\gamma_2}^2 \le \|x\|_{\gamma_1}^2$.Hence $\{\|x\|_\gamma^2 : \gamma \in (0,1)\}$ is an ascending norm on U. In a parallel method one can demonstrate that $\{\|x\|_\gamma^3 : \gamma \in (0,1)\}$ is also an ascending norm on U. □

Lemma 4.8. *Suppose $(U, A, \odot, \circ)$ be a PFNLS which satisfies the Theorem 4.7. Suppose $\{u_1, u_2, \dots, u_n\}$ be a set of l.i. vectors of U. Then for each $\gamma \in (0,1)$, $\exists\, K > 0$ s.t. for any scalars $\gamma_1, \gamma_2, \dots, \gamma_n$,*

$$\|\gamma_1 u_1 + \gamma_2 u_2 + \dots \gamma_n u_n\|_\gamma \ge K \sum_{i=1}^{n} |\gamma_i|,$$

where K is a constant.

Proof. Here $(U, A, \odot, \circ)$ is a PFNLS and $\|x\|_\gamma$, where $x \in U, \gamma \in (0,1)$ is an induced norm on U. Hence $(U, \|x\|_\gamma)$ is a NLS. Therefore for each $\gamma \in (0,1), \exists\, K > 0$ s.t. for any scalars $\gamma_1, \gamma_2, \dots, \gamma_n$,

$$\|\gamma_1 u_1 + \gamma_2 u_2 + \dots \gamma_n u_n\|_\gamma^1 \ge K \sum_{i=1}^{n} |\gamma_i|,$$

where K is a constant holds in $(U, \|x\|_\gamma)$. □

Theorem 4.9. *Every finite dimensional PFNLS which satisfies Theorem 4.7 is complete.*

Proof. Suppose $(U, A, \odot, \circ)$ is a r-dimensional PFNLS. Consider a Cauchy Sequence $\{s_n\}$ in $(U, A, \odot, \circ)$. Then $\exists$ a basis of U say $\{u_1, u_2, \dots, u_r\}$ and scalars $c_1^n, c_2^n, \dots, c_r^n$ such that $s_n = c_1^n u_1 + \dots + c_r^n u_r$ holds. Again we know that

$$\begin{aligned}
&\lim_{n\to\infty} P(s_{n+p} - s_n, t) = 1\ \forall\, t > 0, p = 1, 2, \dots \\
&\implies \lim_{n\to\infty} P(\sum_{i=1}^{n+p} c_i^{n+p} u_i - \sum_{i=1}^{n} c_i^n u_i, t) = 1 \\
&\implies \lim_{n\to\infty} P((\sum_{i=1}^{n+p} c_i^{n+p} - \sum_{i=1}^{n} c_i^n) u_i, t) = 1.
\end{aligned}$$

Now for $\gamma \in (0,1), \exists$ an integer $N(t,\gamma)$ such that

$$P((\sum_{i=1}^{n+p} c_i^{n+p} - \sum_{i=1}^{n} c_i^n)u_i, t) > \gamma \ \forall n > N$$
$$\implies |((\sum_{i=1}^{n+p} c_i^{n+p} - \sum_{i=1}^{n} c_i^n)u_i| \leq t \ \forall n > N.$$

Since $t > 0$ is arbitrary, hence $((\sum_{i=1}^{n+p} c_i^{n+p} - \sum_{i=1}^{n} c_i^n)u_i| \to 0$ as $n \to \infty$ which obviously implies that $\{s_n\}$ is a Cauchy sequence which converges in $s = \sum c_i u_i$ for some $s \in U$. Now

$$\begin{aligned} P(s_n - s, t) &= P(\sum_{i=1}^{n} c_i^n u_i - \sum_{i=1}^{n} c_i u_i, t) \\ &= P(\sum_{i=1}^{n} (c_i^n - c_i) u_i, t) \\ \geq P(u_1, \frac{t}{n|c_1^n - c_1|}) \circ \ldots \circ P(u_n, \frac{t}{n|c_n^n - c_1|}) \\ \implies \lim_{n\to\infty} P(s_n - s, t) \geq 1 \circ 1 \ldots \circ 1 = 1 \ \forall\, t > 0 \\ \lim_{n\to\infty} P(s_n - s, t) = 1 \ \forall\, t > 0. \end{aligned}$$

Again we have $\forall\, t > 0$,

$$\begin{aligned} Q(s_n - s, t) &= Q(\sum_{i=1}^{n} c_i^n u_i - \sum_{i=1}^{n} c_i u_i, t) \\ &= Q(\sum_{i=1}^{n} (c_i^n - c_i) u_i, t) \\ \leq Q(u_1, \frac{t}{n|c_1^n - c_1|}) \odot \ldots \odot Q(u_n, \frac{t}{n|c_n^n - c_1|}) \\ \implies \lim_{n\to\infty} Q(s_n - s, t) \leq 1 \odot 1 \ldots \odot 1 = 1 \ \forall\, t > 0 \\ \lim_{n\to\infty} Q(s_n - s, t) = 0 \ \forall\, t > 0. \end{aligned}$$

Similarly one can demonstrate that $\lim_{n\to\infty} R(s_n - s, t) = 0 \ \forall\, t > 0$. Hence $\{s_n\}$ in $(U, A, \circ, \odot)$ converges to $s \in U$.Thus $(U, A, \circ, \odot)$ is complete. □

Definition 4.10. *Suppose $(U, A, \circ, \odot)$ be an PFNLS. A subset $W \subseteq U$ is said to be*

(i) *closed if for any sequence $\{s_n\}$ in W converges to $s \in W$. Mathematically*

$$\lim_{n\to\infty} P(s_n - s, t) = 1, \lim_{n\to\infty} Q(s_n - s, t) = 0, \lim_{n\to\infty} R(s_n - s, t) = 0 \implies s \in W.$$

(ii) *bounded iff $\exists\, t > 0,\ 0 < r < 1$ s.t.*

$$P(s,t) > 1 - r, Q(s,t) < r \text{ and } R(s,t) < r \ \forall s \in W.$$

(iii) *closure of W_1 if for any $s \in W$, $\exists$ a sequence $\{s_n\}$ in W_1 s.t.*

$$\lim_{n\to\infty} P(s_n - s, t) = 1, \lim_{n\to\infty} Q(s_n - s, t) = 0, \lim_{n\to\infty} R(s_n - s, t) = 0 \ \forall\, t \in \mathbb{R}^+.$$

In that case $W_1 = \overline{W}$.

Theorem 4.11. *Suppose $(U, A, \circ, \odot)$ be an PFNLS. Every Cauchy sequence in $(U, A, \circ, \odot)$ is bounded.*

Proof. Suppose $\{s_n\}$ be a Cauchy sequence in PFNLS $(U, A, \circ, \odot)$. We consider a fixed l_0 such that $l_0 \in (0, 1)$. Then $\lim_{n\to\infty} P(s_n - s_{n+p}, t) = 1 > l_0 \ \forall \ t > 0, p = 1, 2$. For another $\tilde{t} > 0 \ \exists \ \tilde{n}$ such that $P(s_n - s_{n+p}, \tilde{t}) > l_0, p = 1, 2, \ldots$. Since $\lim_{n\to\infty} P(s, t) = 1$ we have $\forall \ s_i, t_i > 0$ such that $P(s_i, t) > l_0 \ \forall t > t_i, i = 1, 2, \ldots$. Suppose $t_0 = \tilde{t} + \max\{t_i\}, \ i = 1, \ldots, \tilde{n}$.Then

$$\begin{aligned} P(s_n, t_0) &\geq P(s_n, \tilde{t} + t_{\tilde{n}}) \\ &= P(s_n - s_{\tilde{n}} + s_{\tilde{n}}, \tilde{t} + t_{\tilde{n}}) \\ &> l_0 \circ l_0 = l_0 \ \forall \ n > \tilde{n}. \end{aligned}$$

So $\forall \ n = 1, \ldots, \tilde{n}, \ P(s_n, t_0) \geq P(s_n, t_n) > l_0$. Hence $P(s_n, t_0) > l_0 \ \forall \ n = 1, 2, \ldots$. Again we have $Q(s_n - s_{n+p}, t) = 0 < (1 - l_0) \ \forall \ t > 0, \ p = 1, 2, \ldots$. Now again for all $\tilde{t} > 0 \ \exists \ \tilde{n}$ we have $Q(s_n - s_{n+p}, \tilde{t}) < 1 - l_0 \ \forall \ n \geq \tilde{n} \ p = 1, 2, \ldots$. Since $\lim_{t\to\infty} Q(s, t) = 0$, we have we have $\forall \ s_i, \tilde{t_i} > 0$ such that $Q(s_i, t) < 1 - l_0 \ \forall t > \tilde{t_i}, i = 1, 2, \ldots$. Suppose $\tilde{t_0} = \tilde{t} + \max\{\tilde{t_i}\}, \ i = 1, \ldots, \tilde{n}$.Then

$$\begin{aligned} Q(s_n, \tilde{t_0}) &\leq Q(s_n, \tilde{t} + \tilde{t_n}) \\ &= Q(s_n - s_{\tilde{n}} + s_{\tilde{n}}, \tilde{t} + \tilde{tn}) \\ &< (1 - l_0) \odot (1 - l_0) = 1 - l_0 \ \forall \ n > \tilde{n}. \end{aligned}$$

So $\forall \, n = 1, \ldots, \tilde{n}, \ \underline{Q}(s_n, \tilde{t_0}) \geq Q(\underline{s}_n, t_n) < 1 - l_0$. Hence $Q(s_n, \tilde{t}) < 1 - l_0 \ \forall \ n = 1, 2, \ldots$. Now for the same procedure and for $\overline{\tilde{t_0}} = \tilde{t} + \max\{\overline{\tilde{t_i}}\}, \ i = 1, \ldots, \tilde{n}$ we can again show that $R(s_n, \tilde{t}) < 1 - l_0 \ \forall \ n = 1, 2, \ldots$. Let $\overline{t} = \max\{t_0, \tilde{t_0}\}, \overline{\tilde{t_0}}$. Finally we have

$$P(s_n, \overline{t}) > l_0, Q(s_n, \overline{t}) < 1 - l_0, R(s_n, \overline{t}) < 1 - l_0 \ \forall \ n = 1, 2, \ldots.$$

Thus $\{s_n\}$ is bounded in $(U, A, \circ, \odot)$ and we are done. □

5. Conclusion

In this manuscript we have discussed the concept of PFNLS for the first time. The concept of PF norm is also discussed. Later convergence of a sequence as well as Cauchy sequence in PFNLS have been examined. All these concepts are illustrated with examples. Finally associated properties and characteristics of PFNLS have been shown here. In future we will study the concept of PF n-NLS and PF Metric space.

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