Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–8.
ISSN-0037-8712 in press
doi:10.5269/bspm.66247

On Submanifolds of a Sasakian Manifold

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abstract: In this paper, we study submanifolds of a Sasakian manifolds provide with a torqued vector field and also accepting a Ricci-Yamabe soliton of both Sasakian manifold and Sasakian spaceform. Further, we obtain some important results which categorize the submanifolds admitting a Ricci-Yamabe soliton of Sasakian spaceform.

Key Words: Ricci-Yamabe soliton, Sasakian Manifold, Einstein manifold, torqued vector field.

Contents

1. Introduction

The concepts of Ricci flow and Yamabe flow are introduced in 1988 by Hamilton [13]. Ricci soliton and Yamabe soliton appear as the limit of the solutions of the Ricci flow and Yamabe flow respectively. Ricci flow and Yamabe flow have been deliberated by many geometers (See [10,12,14]). The Ricci-Yamabe flow is studied by Crasmareanu and Guler [9]. Some related developments can be found in [4,5,15-24]. This flow for the metrics on the Riemannian manifolds is defined as

$$\frac{\partial}{\partial \mathfrak{t}}g(\mathfrak{t}) = -2p\ Ric(\mathfrak{t}) + q\, r(\mathfrak{t})g(\mathfrak{t}),\ go = g(0) \tag{1.1}$$

A soliton to the Ricci-Yamabe flow is called Ricci-Yamabe soliton (RYS) and it is precised on Riemannain manifold (g, V, λ, p, q) satisfying

$$\mathfrak{L}_V g + 2p\ S + (2\lambda - qr)g = 0, \tag{1.2}$$

where S is the Ricci tensor, r is the scalar curvature, $\mathfrak{L}_V$ is the Lie derivative along the vector field and p, q are the scalars. (M, g) is called RYS expanding if $\lambda > 0$; RYS shrinking if $\lambda < 0$; and RYS steady if $\lambda = 0$.

The equation (1.2) as RYS of type (p, q) is said to be p-Ricci soliton and q-Yamabe soliton when $q = 0$ and $p = 0$ respectively. Riemannian manifold which admit torqued vector fields were first defined by Chen [8]. By this definition, a non-zero vector field τ on a Riemannian manifold $(\bar{M}, g)$ is called torqued vector field, which is given by

$$\bar{\nabla}_{U_1} = fU_1 + \pi(U_1)\tau, \quad \pi(\tau) = 0, \tag{1.3}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$, for any $U_1 \in \gamma(T\bar{M})$. The torqued function f and 1-form π has torqued form of τ.

2010 *Mathematics Subject Classification:* 53D10, 53D15.
Submitted December 11, 2022. Published July 21, 2023

2. Preliminaries

Let $\bar{M}$ be an odd dimensional almost contact metric manifold with an almost contact metric structure (φ, ξ, η, g). Such that φ is a tensor field, ξ is a vector field, η is a 1-form tensor field on $\bar{M}$ and the Riemannain metric g please the following equations (See [6,7,11]):

$$\left.\begin{aligned} \varphi^2 U_1 = -U_1 + \eta(U_1)\xi, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \\ g(\varphi U_1, \varphi U_2) = g(U_1, U_2) - \eta(U_1)\eta(U_2),\ g(U_1, \varphi U_2) = -g(\varphi U_1, U_2) \end{aligned}\right\} \tag{2.1}$$

for any $U_1, U_2 \in \gamma(T\bar{M})$.

An almost contact metric manifold $(\bar{M}, \varphi, \xi, \eta, g)$ is said to be Sasakian manifold if it satisfy the following conditions

$$(\bar{\nabla}_{U_1\varphi})U_2 = g(U_1, U_2)\xi - \eta(U_2)U_1, \quad (\bar{\nabla}_{U_1})\xi = -\varphi U_1 \tag{2.2}$$

Further, a Sasakian manifold $\bar{M}$ with constant ϕ-sectional curvature c is a Sasakian space form and it is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of a Sasakian space form (See [1,2,12]) is given by

$$\begin{aligned} \bar{R}(U_1, U_2)U_3 = \frac{c+3}{4}[g(U_2, U_3)U_1 - g(U_1, U_3)U_2] + \frac{c-1}{4}[g(U_1, \phi U_3)\phi U_2 \\ - g(U_2, \phi U_3)\phi U_1 + 2g(U_1, \phi U_2)\phi U_3 + \eta(U_1)\eta(U_3)U_2 \\ - \eta(U_2)\eta(U_3)U_1 + g(U_1, U_3)\eta(U_2)\xi - g(U_2, U_3)\eta(U_1)\xi] \end{aligned} \tag{2.3}$$

Let M be a submanifold of dimension m of a manifold $\bar{M}$ $(m < n)$ with metric g. Also ∇ and $\nabla^{\perp}$ be the incited connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M individually. The Weingarten and Gauss equations are

$$\bar{\nabla}_{U_1} U_2 = \nabla_{U_1} U_2 + h(U_1, U_2), \tag{2.4}$$

$$\bar{\nabla}_{U_1} V = -A_V U_1 + \nabla^{\perp}_{U_1} V, \tag{2.5}$$

for all $U_1, U_2 \in \gamma(TM)$ and $V \in \gamma(T^{\perp}M)$, where h and A_V are second fundamental form and the shape operator respectively for the immersion of M into $\bar{M}$. From (2.2) and (2.4), which follows that

$$\nabla_{U_1}\xi = -\varphi U_1, \tag{2.6}$$

$$h(U_1, \xi) = 0, \tag{2.7}$$

where ∇ is the Levi-Civita connection of M. The h and A_V are related by (see [25])

$$g(h(U_1, U_2), V) = g(A_V U_1, U_2) \tag{2.8}$$

The equation of Gauss is given by

$$\begin{aligned} g(R(U_1, U_2)U_3, U_4) = g(\bar{R}(U_1, U_2)U_3, U_4) + g(h(U_1, U_4), h(U_2, U_3)) \\ - g(h(U_1, U_3), h(U_2, U_4)) \end{aligned} \tag{2.9}$$

A submanifold M of a manifold $\bar{M}$ is said to be totally umbilical if

$$h(U_1, U_2) = g(U_1, U_2)L, \tag{2.10}$$

where $U_1, U_2 \in TM$ and $L = \frac{1}{m}\sum_{i=1}^{m} g(e_i, e_i)$, $\{e_i\}_{i=1}^{m}$ is a mean curvature on M. Further, M is called totally geodesic when $h(U_1, U_2) = 0$ and M is minimal in $\bar{M}$ when $L = 0$.

Torseforming vector field on a Riemannian manifold $\bar{M}$ is given by

$$\bar{\nabla}_{U_1}\rho = fU_1 + \pi(U_1)\rho \tag{2.11}$$

Also, 1-form π is called generating form, function f is called conformal scalar of ρ. If π in (1.3) vanishes identically, then the vector field ρ is called concircular.If $f = 1$ and $\pi = 0$, then the vector field ρ is called concurrent. The vector field ρ is called recurrent if it satisfies (1.3) with $f = 0$. Also, if $f = \pi = 0$, the vector field ρin (1.3) is called parallel. Let $\bar{M}$ be a Sasakian manifold provide with a torqued vector field τ and $\phi : M \to \bar{M}$ be an isometric immersion. Then we get

$$\tau = \tau^{\top} + \tau^{\perp}, \tag{2.12}$$

where $\tau^{\top}$and $\tau^{\perp}$ the tangetial and normal components of τ on $\bar{M}$, respectively.

3. The Submanifolds admitting RYS of Sasakian Manifolds

Let M be a submanifold provide with a torqued vector field τ of a Sasakian manifold $\bar{M}$. We have

$$\bar{\nabla}_{U_1}\tau = fU_1 + \pi(U_1)\tau, \quad \pi(\tau) = 0 \tag{3.1}$$

Suppose that ξ is a torqued vector field on $\bar{M}$. Taking $\tau = \xi$ in equation (3.1), we get

$$\bar{\nabla}_{U_1}\xi = fU_1 + \pi(U_1)\xi \quad \pi(\xi) = 0 \tag{3.2}$$

Also, using the inner product with ξ to (3.2), we have

$$\pi(U_1) = -f\eta(U_1) \tag{3.3}$$

Hence the equation (3.2) reduces to

$$\bar{\nabla}_{U_1}\xi = f(U_1 - \eta(U_1\xi)) \tag{3.4}$$

Since torqued vector field τ on the ambient space $\bar{M}$, it follows from (1.3), (2.12) and Gauss and Weingarten formula,

$$\nabla_{U_1}\tau^{\top} + h(U_1, \tau^{\top}) - A_{\tau^{\perp}}U_1 + \nabla^{\perp}_{U_1}\tau^{\perp} = fU_1 + \pi(U_1)\tau^{\top} + \pi(U_1)\tau^{\perp} \tag{3.5}$$

To comparing the tangential and normal components of (3.5),we obtain

$$\left.\begin{aligned} h(U_1, \tau^{\top}) + \nabla^{\perp}_{U_1}\tau^{\perp} &= \pi(U_1)\tau^{\perp}, \\ \nabla_{U_1}\tau^{\top} - A_{\tau^{\perp}}U_1 &= fU_1 + \pi(U_1)\tau^{\top} \end{aligned}\right\} \tag{3.6}$$

If M is totally geodesic submanifold of $\bar{M}$, then the equation (3.6) becomes

$$\nabla_{U_1}\tau^{\top} = fU_1 + \pi(U_1)\tau^{\top}, \tag{3.7}$$

it implies that $\tau^{\top}$ is a torseforming on M.

Theorem 3.1. *Let M be a submanifolds of a Sasakian manifold $\bar{M}$ provide with a torqued vector field τ. The submanifold M is totally geodesic if and only if the tangential component $\tau^{\top}$ of τ is a torse-forming vector field on M whose conformal scalar is the restriction of the torqued function and whose generating form is the restriction of the torqued function of τ on M.*

We suppose that the submanifold M admits a RYS in Theorem 3.1. From (3.6), we have the following cases:
Case I: Let we take $\tau^{\top} \in \gamma(D)$, then from (2.2),(2.7),(2.8) and (3.6) we obtain

$$g(\nabla_{U_1}\tau^{\top}, \xi) = g(fU_1, \xi), \tag{3.8}$$

where $TM = D \oplus Span\ \xi$, for any $U_1 \in \gamma(TM)$. Since the Riemannian metric g is non-degenere, we have

$$\nabla_{U_1}\tau^{\top} = fU_1 \tag{3.9}$$

This shows that the vector field $\tau^\top$ is a concircular on M.
On the other hand, from the Lie derivative and (3.9), we have

$$(L_{\tau^\top} g)(U_1, U_2) = g(\nabla_{U_1}\tau^\top, U_2) + g(\nabla_{U_2}\tau^\top, U_1) = 2fg(U_1, U_2), \tag{3.10}$$

for any $U_1 \in \gamma(TM)$, which means that the vector field $\tau^\top$ is conformal killing. Also from (1.2)and (3.10), we get

$$S(U_1, U_2) = \frac{1}{2p}[qr - 2(\lambda + f)]g(U_1, U_2), \tag{3.11}$$

where S is the Ricci tensor of M. Hence it is Einstein.
Case II: Suppose we take $\tau^\top \in \gamma(D)$, then it follows from (3.8), we have

$$g(\nabla_{U_1}\tau^\top, \xi) = 0, \tag{3.12}$$

for any $U_1 \in \gamma(D)$. This shows that $\tau^\top$ is a parallel vector field on distribution D. Thus $\tau^\top$ is a D-killing vector field.

Further, in view of (1.2)and (3.12), we acquire the Ricci tensor S^D of the distribution D

$$S^D(U_1, U_2) = \frac{1}{2p}[qr - 2\lambda]g(U_1, U_2) \tag{3.13}$$

Hence the distribution D is an Einstein.
Case III: Suppose we take ξ instead of $\tau^\top$ in (3.7), we have

$$\nabla_{U_1}\xi = fU_1 + \pi(U_1)\xi \tag{3.14}$$

Taking the inner product of ξ in (3.14), we obtain

$$g(\nabla_{U_1}\xi, \xi) = f\eta(U_1) + \pi(U_1), \tag{3.15}$$

it implies that

$$\pi(U_1) = -fU_1 \tag{3.16}$$

We observe that $\pi(\xi) \neq 0$, so ξ is a torseforming on M. In view of (3.6), we state that

Corollary 3.2. *Let M be a submanifold of a Sasakian manifold $\bar{M}$ provide with a torqued vector field τ. If M is $\tau^\perp$- umblical, then $\tau^\top$ is a torseforming on M.*

Suppose we take ξ instead of $\tau^\top$ in (3.6), we have

$$\nabla_{U_1}\xi - A_{\tau^\perp}U_1 = fU_1 + \pi(U_1)\xi \tag{3.17}$$

In view of (2.2), (2.4) and (3.17), we obtain

$$A_{\tau^\perp}U_1 = -\varphi U_1 - fU_1 - \pi(U_1)\xi \tag{3.18}$$

If we use the equations (2.1),(2.8) and (3.18), then we have

$$g(h(U_1, U_2), \tau^\perp) = -g(\varphi U_1, U_2) - fg(U_1, U_2) - \pi(U_1)\eta(U_2) \tag{3.19}$$

Interchanging the roles of U_1 and U_2 in (3.19) gives

$$g(h(U_2, U_1), \tau^\perp) = -g(\varphi U_2, U_1) - fg(U_2, U_1) - \pi(U_2)\eta(U_1) \tag{3.20}$$

We know that h and g are symmetric, so from (3.19) and (3.20), it follows that

$$2g(h(U_1, U_2), \tau^\perp) = -2fg(U_1, U_2) - \pi(U_1)\eta(U_2) - \pi(U_2)\eta(U_1), \tag{3.21}$$

for any $U_1, U_2 \in \gamma(TM)$.

Further, the definition of Lie-derivative together with (3.14) and (3.18) yields

$$L_\xi g(U_1, U_2) = g(\nabla_{U_1}\xi, U_2) + g(\nabla_{U_2}\xi, U_1) = 0 \tag{3.22}$$

Now, from (1.2) and (3.22) we obtain

$$S(U_1, U_2) = \frac{1}{2p}(qr - 2\lambda)g(U_1, U_2) \tag{3.23}$$

Hence, it follows that

Theorem 3.3. *If $\bar{M}$ is a Sasakian submanifold provide with a torqued vector field τ and M be a submanifold admitting a RYS of $\bar{M}$, then (M, g, ξ, λ) is Einstein.*

Contracting (3.23), we obtain

$$\lambda = \frac{r[nq - 2p]}{2} \tag{3.24}$$

Theorem 3.4. *If $\bar{M}$ is a Sasakian submanifold provide with a torqued vector field τ, then RYS is*

(i) expanding for $r[nq - 2p] > 0$,

(ii) shrinking for $r[nq - 2p] < 0$, and

(iii) steady for $r[nq - 2p] = 0$.

If $p = 0$, then from (3.24), one has

$$\lambda = \frac{nqr}{2} \tag{3.25}$$

Corollary 3.5. *If $\bar{M}$ is a Sasakian submanifold provide with a torqued vector field τ, then q-Yamabe soliton is*

(i) expanding for $nqr > 0$,

(ii) shrinking for $nqr < 0$, and

(iii) steady for $nqr = 0$.

If $q = 0$, then from (3.24), one has

$$\lambda = -rp \tag{3.26}$$

Corollary 3.6. *If $\bar{M}$ is a Sasakian submanifold provide with a torqued vector field τ, then p-Ricci soliton is*

(i) expanding for $rp < 0$,

(ii) shrinking for $rp > 0$, and

(iii) steady for $rp = 0$.

4. *RYS* in Sasakian space-form with torqued vector field

The submanifolds admitting a *RYS* in Sasakian space-form $\bar{M}(c)$ provide with torqued vector field τ. From the definition of Ricci tensor, we have

$$S(U_2, U_3) = \sum_{i=1}^{n-1} g(R(e_i, U_2)U_3, e_i) + g(R(\xi, U_2)U_3, \xi), \tag{4.1}$$

where R is the Riemannian curvature tensor of the submanifold M.

Taking $U_1 = U_4 = e_i$ in (2.9) and using (2.1),(2.3),(2.7) and (2.10), we obtain

$$\begin{aligned}\sum_{i=1}^{n-1} g(R(e_i, U_2)U_3, e_i) = &\frac{c+3}{4}[(n-2)g(U_2, U_3) + \eta(U_2)\eta(U_3)] \\ &+ \frac{c-1}{4}[3g(U_2, U_3) - (n+2)\eta(U_2)\eta(U_3)] \\ &+ [(n-2)g(U_2, U_3) + \eta(U_2)\eta(U_3)] \parallel H \parallel^2 \end{aligned} \tag{4.2}$$

Putting $U_1 = U_4 = \xi$ in (2.9), we get

$$g(R(\xi, U_2)U_3, \xi) = g(\bar{R}(\xi, U_2)U_3, \xi) = \eta(U_2)\eta(U_3) - g(U_2, U_3) \tag{4.3}$$

Using (4.2) and (4.3) in (4.1), then we have

$$\begin{aligned} S(U_2, U_3) = &\left[\frac{c(n+1) + 3n - 7}{4} + (n-2) \parallel H \parallel^2\right] g(U_2, U_3) \\ &+ \left[\frac{5 - c(n+1) - n}{4} + \parallel H \parallel^2\right] \eta(U_2)\eta(U_3) \end{aligned} \tag{4.4}$$

Hence we state that

Theorem 4.1. *Let M be an n-dimensional submanifold of Sasakian space form $\bar{M}(c)$. If M is totally umbilical and the mean curvature $\parallel H \parallel$ is constant, then M is η Einstein.*

If we take $U_2 = U_3 = \xi$ in (3.22), then we have

$$S(\xi, \xi) = \frac{1}{2p}(qr - 2\lambda) \tag{4.5}$$

Similarly, if we put $U_2 = U_3 = \xi$ in (4.4), then we obtain

$$S(\xi, \xi) = \frac{(n-1)\left[1 + \parallel H \parallel^2\right]}{2} \tag{4.6}$$

In view of (4.5) and (4.6), we obtain

$$\lambda = \frac{qr - p(n-1)\left[1 + 2 \parallel H \parallel^2\right]}{2} \tag{4.7}$$

This leads to the following result:

Theorem 4.2. *Let M be a totally umbilical submanifold of Sasakian spaceform provide with a torqued vector field τ and admitting a RYS. Then the RYS is*

(i) exapanding if $qr > p(n-1)\left[1 + 2 \parallel H \parallel^2\right]$,

(ii) shrinking if $qr < p(n-1)\left[1 + 2 \parallel H \parallel^2\right]$, and

(iii) steady if $qr = p(n-1)\left[1 + 2 \parallel H \parallel^2\right]$.

Putting $p = 0$ in (4.7), we obtain

$$\lambda = \frac{qr}{2} \tag{4.8}$$

Corollary 4.3. *Let M be a totally umbilical submanifold of Sasakian spaceform provide with a torqued vector field τ and admitting a RYS. Then q-Yamabe soliton is*

(i) exapanding if $qr > 0$,

(ii) shrinking if $qr < 0$*, and*

(iii) steady if $qr = 0$.

Suppose we take $q = 0$ in (4.7), we get

$$\lambda = \frac{p(n-1)\left[1 + 2 \parallel H \parallel^2\right]}{2} \tag{4.9}$$

Corollary 4.4. *Let M be a totally umbilical submanifold of Sasakian spaceform provide with a torqued vector field τ and admitting RYS. Then p-Ricci soliton is*

(i) exapanding if $p(n-1)[1 + 2 \parallel H \parallel^2] > 0$,

(ii) shrinking if $p(n-1)[1 + 2 \parallel H \parallel^2] < 0$*, and*

(iii) steady if $p(n-1)[1 + 2 \parallel H \parallel^2] = 0$.

Acknowledgments

The authors would like to thank the referees for their invaluable comments and suggestions which led to the improvement of the manuscript.

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