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Stability Analysis of an Age Structured Population Model With Fractional Time Derivative

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ABSTRACT: In this paper, we analyse the large time behaviour in a fractional nonlinear model of population daynamics with age dependent. We show the existence and uniqueness of the solution by using the method of seperation of variables, and we studied the Ulam-Hyers stability of the model.

Key Words: Age dependence, population dynamics, Ulam-Hyers stability, fractional model.

Contents

1. Introduction

The following model:

$$\dot{P} = \delta P, \tag{1.1}$$

where δ is the growth rate, is considered as the simplest model in the domain of dynamic of populations, and it was introduced by Malthus in [15]. This model does not take into account the effect of overcrowding, for this reason Verhulst proposed the following model [19],

$$\dot{P} = (\delta_0 - \omega_0 P)P, \tag{1.2}$$

where δ_0 and ω_0 are positive constants. In (1.1) the growth rate is given by $(\delta_0 - \omega_0 P)$ it then depends on the total population P, so it takes into account the effect of overcrowding. The models (1.1) and (1.2) have played a considerable role in the domain of dynamic of populations however they do not take into consideration the age of individuals, even though age is one of the most important parameters structuring a population. The first age-structured model was proposed by Lotka and Mckendrick, they assumed that the density of population $u(t,a)$ satisfies the following problem [14],

$$\begin{aligned}
&\frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a} = -\mu(a)u(t,a),\\
&u(t,0) = \int_0^\infty \beta(a)u(t,a)da,\\
&u(0,a) = u_0(a).
\end{aligned}$$

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With,

$$P(t) = \int_0^\infty u(t,a)da.$$

Where μ is the death rate and β is the birth rate. In this work we assumed that the parameters μ and β are given like in Michel Langlais's paper [13]. We then propose the following model,

$$\begin{aligned}
&\frac{\partial^\alpha u(t,a)}{\partial t^\alpha} + \frac{\partial u(t,a)}{\partial a} = -\mu_n(a)u(t,a) - \mu_e(P(t))u(t,a),\\
&P(t) = \int_0^\infty u(t,a),\\
&u(t,0) = \int_0^\infty \beta(a)u(t,a)da.
\end{aligned}$$

Supplemented by the following initial condition,

$$u(0,a) = u_0(a).$$

Where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo derivative of order α. For more details about the parameters used in this model see section 3. Using the method of separation of variables we proved the existence and uniqueness of the solution, and then we studied the Ulam Hyers stability of the model.

2. Preliminaries

In this section we introduce notations, definitions and preliminary facts which are used throughout this paper.
We denote by $X = \mathcal{C}([0,T],\mathbb{R}^+)$ the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}^+$, with the norm $\|P\|_X = \sup\{|P(t)|, t \in [0,T]\}$. We need some basic definitions and properties of the fractional calculus theory. For more details, see [11].

Definition 2.1. *[11]*
The fractional integral of the function $h \in L^1([a,b])$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds,$$

where Γ is the gamma function

Definition 2.2. *[11] For a function h given on the interval $[a,b]$, the Caputo fractional order derivative of h, is given by*

$$^cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ denote the integer part of α.

Theorem 2.3 (The Schauder's theorem)**.** *Let C be a nonempty closed convex subset of a Banach space X. Let $T : C \longrightarrow C$ be a continuous mapping such that $T(C)$ is relatively compact subset of X. Then T has at least one fixed point in C .*

Let ϵ be a positive number, $f : [a,b] \times \mathbb{B} \longrightarrow \mathbb{B}$ be a continuous function with $\mathbb{B}$ is a Banach space and $\varphi : [a,b] \longrightarrow \mathbb{R}^+$ be a continuous function. We consider the following equation

$$^cD^\alpha x(t) = f(t,x(t)), \quad \alpha \in (0,1)(or(1,2)) \quad t \in [a,b], \tag{2.1}$$

and the following inequalities

$$|^cD^\alpha y(t) - f(t,y(t))| \le \epsilon, \quad t \in [a,b), \tag{2.2}$$

$$|^cD^\alpha y(t) - f(t,y(t))| \le \varphi(t), \quad t \in [a,b), \tag{2.3}$$

$$|^cD^\alpha y(t) - f(t,y(t))| \le \epsilon\varphi(t), \quad t \in [a,b). \tag{2.4}$$

Definition 2.4. *[20] The equation (2.1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $y \in \mathcal{C}^1([a,b),\mathbb{B})$ of the inequality (2.2) there exists a solution $x \in \mathcal{C}^1([a,b),\mathbb{B})$ of the equation (2.1) with*

$$|y(t) - x(t)| \leq c_f \epsilon, \quad t \in [a,b).$$

Definition 2.5. *[20] The equation (2.1) is generalized Ulam-Hyers stable if there exists $\theta_f \in \mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$, $\theta_f(0) = 0$ such that for each solution $y \in \mathcal{C}^1([a,b),\mathbb{B})$ of the inequality (2.2) there exists a solution $x \in \mathcal{C}^1([a,b),\mathbb{B})$ of the equation (2.1) with*

$$|y(t) - x(t)| \leq \theta_f(\epsilon), \quad t \in [a,b).$$

Definition 2.6. *[20] The equation (2.1) is Ulam-Hyers stable with respect to φ if there exists $c_{f,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in \mathcal{C}^1([a,b),\mathbb{B})$ of the inequality (2.4) there exists a solution $x \in \mathcal{C}^1([a,b),\mathbb{B})$ of the equation (2.1) with*

$$|y(t) - x(t)| \leq c_{f,\varphi} \epsilon \varphi(t), \quad t \in [a,b).$$

Definition 2.7. *[20] The equation (2.1) is generalized Ulam-Hyers-Rassias stable with respect to φ if there exists $c_{f,\varphi} > 0$ such that for each solution $y \in \mathcal{C}^1([a,b),\mathbb{B})$ of the inequality (2.3) there exists a solution $x \in \mathcal{C}^1([a,b),\mathbb{B})$ of the equation (2.1) with*

$$|y(t) - x(t)| \leq c_{f,\varphi} \varphi(t), \quad t \in [a,b).$$

Remark 2.8. *A function $y \in \mathcal{C}^1([a,b),\mathbb{B})$ is a solution of the inequality (2.2) if and only if there exists a function $g \in \mathcal{C}^1([a,b),\mathbb{B})$ such that*

$$|g(t)| \leq \epsilon, \qquad t \in [a,b).$$

$${}^c D^\alpha y(t) = f(t, y(t)) + g(t).$$

One can have similar remarks for the inequalities (2.3) and (2.4).

Lemma 2.9. *[20] Let $z, w : [0,T) \longrightarrow [0,\infty)$ be a continuous functions, where $T \leq \infty$. If w is nondecreasing and there are constants $k \geq 0$ and $\alpha > 0$ such that*

$$z(t) \leq w(t) + k \int_0^t (t-s)^{\alpha-1} z(s) ds, \qquad t \in [0,T).$$

Then

$$z(t) \leq w(t) + \int_0^t \left[\Sigma_{n=1}^{\infty} \frac{(k\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} w(s) \right] ds, \qquad t \in [0,T).$$

If $w(t) = \bar{a}$, constant on $[0,T)$, then the above inequality is reduced to

$$z(t) \leq \bar{a} E_\alpha(k\Gamma(\alpha) t^\alpha), \qquad 0 \leq t < T,$$

where E_α is the Mittag-Leffler function defined by

$$E_\beta(y) := \Sigma_{k=0}^{\infty} \frac{y^k}{\Gamma(k\beta + 1)}, \quad y \in \mathbb{C}, \quad Re(\beta) > 0.$$

3. Formulation of the model

Let $u(t,a)$ be the density of population having at time $t > 0$ the age a. We are going to study existence, uniqueness and Ulam-Hyers stability of this equation

$$\frac{\partial^\alpha u(t,a)}{\partial t^\alpha} + \frac{\partial u(t,a)}{\partial a} = -\mu_n(a) u(t,a) - \mu_e(P(t)) u(t,a). \tag{3.1}$$

where,

$$\frac{\partial^\alpha}{\partial t^\alpha} \text{is the Caputo derivative, with} \quad 0 < \alpha < 1,$$

and

$$P(t) = \int_0^\infty u(t,a)da. \tag{3.2}$$

- $P(t)$ is the total population at time t.
- $\mu_n(a)$ is the probability of dying due to natural causes at age a, $\mu_n(a) \geq 0$.
- One define the function π in the interval $[0, +\infty[$ by

  $$\pi(a) = \exp(-\int_0^a \mu_n(\sigma)d\sigma).$$

  $\pi(a)$ is the probability of living to age a. One remarks that: $\pi(0) = 1$, is decreasing and $\lim_{a\to+\infty} \pi(a) = 0$.
- $\mu_e(P)$ is the probability of death due to environmental factors. μ_e is a function of the total population.
- The birth low is given by

  $$u(t,0) = \int_0^\infty \beta(a)u(t,a)da. \tag{3.3}$$

- β is called the birth-modulus which represents the fertility of the population One assumes that β has a compact support in $[0, \infty[$, so that

  $$supp\beta \subset [0, A],$$

  where $A = \max\{a, \beta(a) > 0\} < \infty$, is the minopause age.

We give the initial condition:

$$u(0,a) = u_0(a). \tag{3.4}$$

4. The characteristic equation

We can look for a solution of (3.1) in the form

$$u(t,a) = E_\alpha(rt^\alpha)\gamma(a). \tag{4.1}$$

r represents the pure population growth effect.
Substitution of (4.1) into (3.1) gives

$$\gamma(a)rE_\alpha(rt^\alpha) + E_\alpha(rt^\alpha)\frac{d\gamma}{da} = -\mu_n E_\alpha(rt^\alpha)\gamma(a),$$

thus

$$\frac{d\gamma}{da} = -[\mu_n + r]\gamma(a),$$

and so

$$\gamma(a) = \gamma(0)\exp(-ra - \int_0^a \mu_n(s)ds), \tag{4.2}$$

with this $\gamma(a)$ in (4.1) the resulting $u(t,a)$ when inserted into (3.3) gives

$$E_\alpha(rt^\alpha)\gamma(0) = \int_0^\infty \beta(a)E_\alpha(rt^\alpha)\gamma(0)\exp[-ra - \int_0^a \mu_n(s)ds]da,$$

and hence, on conceling $E_\alpha(rt^\alpha)\gamma(0)$,

$$1 = \int_0^\infty \beta(a)\exp[-ra - \int_0^a \mu_n(s)ds]da.$$

Thus

$$\int_0^\infty \beta(a)\pi(a)e^{-ra}da = 1. \tag{4.3}$$

5. Assumptions and notations

(H1) We assume that

$$0 \le \beta(a) \le \beta_1 < \infty \qquad on \quad [0,\infty[,$$

where β_1 is a real constant, and $\int_0^\infty \beta^2(a)da < \infty$.

(H2) We assume that

$$0 \le \mu_n(a) \le \mu_1 < \infty \qquad on \quad [0,\infty[,$$

where μ_1 is a real constant.

(H3) We assume that

$$0 \le \mu_e(P) \le \bar{\mu} \qquad on \quad [0,\infty[,$$

where $\bar{\mu}$ is a minimum mortality rate.

6. Main results

6.1. separable solution

We first define the notion of separable solution of the problem (3.1), (3.2), (3.3) and (3.4) which is a solution that can be written as:

$$u(t,a) = \psi(a)P(t), \tag{6.1}$$

with the normalisation

$$\int_0^\infty \psi(a)da = 1. \tag{6.2}$$

6.1.1. Necessary and sufficient condition for the existence of separable solution. For the existence of r^*, the root of the characteristic equation we assume that the hypothesis **(H1)** and **(H2)** are satisfied. Consider the function

$$\phi(r) = \int_0^\infty \beta(a)\pi(a)e^{-ra}da.$$

The function is continuous, and satisfies,

$$\lim_{r\to-\infty}\phi(r) = +\infty \qquad \text{and} \qquad \lim_{r\to+\infty}\phi(r) = 0.$$

And according to the intermediate value theorem, there exists a unique real number r^*, such that $\phi(r^*) = 1$.

Lemma 6.1. *Let r^* be the root of (4.3). There exists a non-trivial separable solution of the problem (3.1)-(3.4), if and only if*

$$\int_0^\infty \pi(a)e^{-r^*a}da < +\infty. \tag{6.3}$$

Proof.
Let $u(t,a) = \psi(a)P(t)$ be a separable solution, then $\int_0^\infty \psi(a) = 1$.
However, $\psi(a) = \psi_0\pi(a)\exp(-r^*a)$, thus

$$\psi_0 \int_0^\infty \pi(a)\exp(-r^*a)da = 1.$$

Therefore,

$$\psi_0^{-1} = \int_0^\infty \pi(a)\exp(-r^*a)da < \infty.$$

Inversely, if $\int_0^\infty \pi(a)\exp(-r^*a)da < \infty$. Let

$$\psi_0 = \left(\int_0^\infty \pi(a)\exp(-r^*a)da\right)^{-1},$$

and

$$\psi(a) = \psi_0\pi(a)\exp(-r^*a).$$

It remains to show the existence of a function $P(t)$, such that the function $u(t,a) = \psi(a)P(t)$ will be a separable solution. We will define this function in the following paragraph.

6.1.2. Equation equivalent to the problem (3.1)-(3.4). Let u be a separable solution to the problem (3.1)-(3.4). By replacing u with $\psi \times P$, in the equation (3.1), we obtain

$$\frac{1}{P(t)}{}^C D^\alpha P(t) + \mu_e(P(t)) = \frac{-1}{\psi(a)}\frac{d\psi(a)}{da} - \mu_n(a).$$

In this equality the left term depends only on a, while the right term depends only on t. Then, these two terms are constant and equal to a constant r. This leads to the following two equations

$$\frac{-1}{\psi(a)}\frac{d\psi(a)}{da} - \mu_n(a) = r, \tag{6.4}$$

and

$$\frac{1}{P(t)}D^\alpha P(t) + \mu_e(P(t)) = r. \tag{6.5}$$

The equation (6.4) can be written as:

$$\frac{\psi(a)}{da} = -\psi(a)(\mu_n(a) + r), \qquad \text{on} \quad [0,\infty[.$$

This equation has the solution

$$\psi(a) = \psi_0\pi(a)e^{-ra}. \tag{6.6}$$

From (6.2), we deduce that

$$\psi_0 = \left(\int_0^\infty \pi(a)\exp(-ra)da\right)^{-1}. \tag{6.7}$$

The condition (3.3) can be written as

$$\psi(0)P(t) = P(t)\int_0^\infty \beta(a)\psi(a)da = P(t)\int_0^\infty \beta(a)\psi(a)\pi(a)\exp(-ra)da.$$

Thus, $\int_0^\infty \beta(a)\pi(a)\exp(-ra)da = 1$, and by uniqueness of the solution of the characteristic equation (4.3)

$$r = r^*.$$

Hence, the solution of the equation (6.4) is

$$\psi(a) = \psi_0\pi(a)e^{-r^*a},$$

where ψ_0 is given by (6.7).

Study of equation (6.5).

Let $r = r^*$. The equation (6.5) can be written as

$$\begin{cases} {}^C D^\alpha P(t) &= & P(t)(r^* - \mu_e(P(t))) = f(P(t)), \quad t \in [0,T], \\ P(0) &= & P_0, \end{cases} \tag{6.8}$$

where $f : X \longrightarrow \mathbb{R}^+$ is a continuous function and satisfies the Lipschitz condition i.e $\exists L > 0$ such that

(H4) $\|f(P) - f(Q)\| \leq L\|P - Q\| \quad \text{for all} \quad P, Q \in X.$

Lemma 6.2. *The function $P \in X$ is a solution of the equation*

$$\begin{cases} {}^C D^\alpha P(t) &= & f(P(t)), \quad t \in [0,T], \\ P(0) &= & P_0 \in X, \end{cases}$$

if and only if P satisfies the integral equation

$$P(t) = P_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(P(s))ds. \tag{6.9}$$

Proof.

Assume that $P \in X$ is a solution of problem (6.9). Applying the Caputo fractional operator of the order α, we obtain the first equation in (6.8).

Again, substituting $t = 0$ in (6.9) we have

$$P(0) = P_0.$$

Conversely, we have ${}^C D^\alpha P(t) = f(P(t))$, so we get

$$\begin{aligned} P(t) &= P(0) + I^\alpha f(P(t)), \\ &= P_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(P(s))ds. \end{aligned}$$

Theorem 6.3. *Assume that hypothesis **(H4)**and $\left(\frac{LT^\alpha}{\Gamma(\alpha+1)}\right) < 1$ hold. Then, the fractional problem (6.8) has a unique solution defined on $[0,T]$.*

Proof.

We define a subset U of X by,

$$U = \{P \in X / \|P\| \leq N\},$$

where,

$$N = \|P_0\| + \frac{1}{\Gamma(\alpha+1)}\|f\|T^\alpha.$$

It is clear that U satisfies the hypothesis of Theorem 2.3.
By application of Lemma 6.2, (6.8) is equivalent to the integral equation,

$$P(t) = P_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s) f(P(s)) ds, \; t \in [0, T].$$

Define the operator $A : U \to U$, by

$$AP(t) = P_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(P(s)) ds.$$

Then our equation is transformed into the operator equation as,

$$P(t) = AP(t), \quad \forall t \in [0, T].$$

- We show that A is continuous in U.
  Let $(P_n)_n$ be a sequence in U converging to a point $P \in U$. Then, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{n\longrightarrow\infty} AP_n(t) &= \lim_{n\longrightarrow\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(P_n(s)) ds + P_0 \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lim_{n\longrightarrow\infty} f(P_n(s)) ds + P_0 \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(P(s)) ds + P_0 \\ &= AP(t). \end{aligned}$$

  For all $t \in [0, T]$. This shows that A is a continuous operator on U.

- A is a compact operator on U.
  First, we show that $A(U)$ is uniformly bounded set in X.
  Let $P \in U$. Then for all $t \in [0, T]$,

$$|AP(t)| \leq |P_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(P(s))| ds.$$

$$\|AP\| \leq \|P_0\| + \frac{1}{\Gamma(\alpha+1)} \|f\|_\infty T^\alpha < \infty.$$

  Thus, $\|AP\| < \infty$ for all $P \in U$.
  This shows that A is uniformly bounded on U.
  Next, we show that $A(U)$ is equicontinuous set on X.
  For any $P \in U$, let $\epsilon > 0$, there exists $\eta = \frac{\Gamma(\alpha+1)\epsilon}{4\|f\|}$.
  For all $t_1, t_2 \in [0, T]$, $|t_2 - t_1| < \eta$ we have

$$\|AP(t_2) - AP(t_1)\| \leq \frac{1}{\Gamma(\alpha+1)} \|f\| (t_2^\alpha - t_1^\alpha).$$

  - Case $0 < t_1 < \eta$, $t_2 < 2\eta$

$$t_2^\alpha - t_1^\alpha \leq 2^\alpha \eta^\alpha \leq 2^\alpha \eta \leq 4\eta.$$

  - Case $0 < t_1 < t_2 \leq \eta$.

$$t_2^\alpha - t_1^\alpha \leq t_2^\alpha < \eta^\alpha \leq \alpha\eta \leq 4\eta.$$

Thus,

$$\|AP(t_2) - AP(t_1)\| \leq \frac{1}{\Gamma(\alpha+1)}\|f\| \times 4\frac{\Gamma(\alpha+1)\epsilon}{4\|f\|} = \epsilon.$$

Hence, for all $t_2, t_1 \in [0, T]$ and for all $P \in U$, $\forall\epsilon > 0$, $\exists\eta > 0$,

$$|t_2 - t_1| < \eta \Longrightarrow \|AP(t_2) - AP(t_1)\| \leq \epsilon.$$

This shows that $A(U)$ is an equicontinuous set on X.
Then, by the Arzela-Ascoli theorem, A is a continuous and compact operator on U.
Thus by Schauder theorem, the operator A has a fixed point.

Now, we must show that A has a unique fixed point.
First, we prove that for all $j \in \mathbb{N}$ and $t \in [0, T]$, we have,

$$\|A^j P - A^j Q\| \leq \Big(\frac{LT^\alpha}{\Gamma(\alpha+1)}\Big)^j \|P - Q\|,$$

for $j = 0$, the inequality is trivial.
We suppose that the inequality holds for $j-1$ and we prove that it holds for j.
Let $t \in [0, T]$.

$$\begin{aligned} |A^j P(t) - A^j Q(t)| &= |A(A^{j-1}(P(t)) - A(A^{j-1}Q(t))|, \\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(A^{j-1}P(s)) - f(A^{j-1}Q(s))|ds. \end{aligned}$$

Thus,

$$\begin{aligned} \|A^j P - A^j Q\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|A^{j-1}P - A^{j-1}Q\|, \\ &\leq \frac{LT^\alpha}{\Gamma(\alpha+1)}\|A^{j-1}P - A^{j-1}\|, \\ &\leq \Big(\frac{LT^\alpha}{\Gamma(\alpha+1)}\Big)^j \|P - Q\|. \end{aligned}$$

Hence, the operator A satisfies the assumptions of weissinger's fixed point theorem with $\alpha_j = \Big(\frac{LT^\alpha}{\Gamma(\alpha+1)}\Big)^j$, we can therefore deduce the uniqueness of the solution of our equation.

7. Ulam-Hyers stability results

We consider equation (6.8) and the inequality (2.3). We suppose that,

- **(H5)** $\phi \in C([1, +\infty), \mathbb{R}^+)$ is continuous, nondecreasing, and there exists $\lambda_\phi > 0$ such that

$$\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\phi(s)ds \leq \lambda_\phi \phi(t),\ \forall t \in [a, +\infty).$$

We obtain the following generalised Ulam-Hyers-Rassias stable results.

Theorem 7.1. *Under the conditions **(H4)** and **(H5)** the equation (6.8) is generalized Ulam-Hyers-Rassias stable.*

Proof.
Let $Q \in C([1, +\infty), \mathbb{R}^+)$ be a solution of the inequality (2.3) Denote by P the unique solution of

$$\begin{cases} {}^C D^\alpha P(t) = f(P(t)), \\ P(0) = Q(0). \end{cases}$$

Then, we have:

$$P(t) = Q(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(P(s))ds. \ t \in [0,T].$$

Q is a solution of the inequality (2.3), then there exists $g \in C([0,T], \mathbb{R}^+)$ such that,

$$(i) \ \ |g(t)| \leq \phi(t), \ t \in [0,T].$$
$$(ii) \ \ {}^C D^\alpha Q(t) = f(Q(t)) + g(t), \ t \in [0,T].$$

Integrating (ii) from 0 to t we have

$$Q(t) = Q(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(Q(s))ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds,$$

By differential inequality (2.3), we have

$$\begin{aligned} |Q(t) - Q(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(Q(s))ds| &= |\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds|, \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s)|ds, \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha 1} \phi(s)ds, \\ &\leq \lambda_\alpha \phi(t), \ t \in [0,T]. \end{aligned}$$

From this relation it follows:

$$\begin{aligned} |Q(t) - P(t)| = &|Q(t) - Q(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(Q(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(Q(s))ds - \frac{1}{\Gamma(\alpha} \int_0^t (t-s)^{\alpha-1} f(P(s))ds| \\ &\leq |Q(t) - Q(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(Q(s) - f(P(s))|ds \\ &\leq \lambda_\phi \phi(t) + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |Q(s) - P(s)|ds \\ &\leq \lambda_\phi \phi(t) + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |Q(s) - P(s)|ds. \end{aligned}$$

Let $Z(t) = |Q(t) - P(t)|$. Then

$$Z(t) \leq W(t) + k \int_0^t (t-s)^{\alpha-1} Z(s)ds,$$

with $W(t) = \lambda_\phi \phi(T) = cst$ and $k = \frac{L}{\Gamma(\alpha)}$.
By Lemma 2.9,

$$Z(t) \leq W(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(k\Gamma(\alpha))^n}{\Gamma(n\alpha)} (t-s)^{\alpha n-1} \Big] W(s)ds.$$

Since $W(t) = cst$, then

$$Z(t) \leq \lambda_\phi \phi(T) E_\alpha(LT^\alpha) \qquad 0 \leq t < T.$$

Thus the equation (6.8) is generalised Ulam-Hyres-Rassias stable.

References

1. Allen, L.J.S. Discrete and continuous models of populations and epidemics. Journal of Mathematical Systems, Estimation, and Control, 1(3): 335-369 (1991).

2. Barril, C., Calsina,À. and Rippol, J. A practical approach to $\mathcal{R}_0$ in continuous time ecological models, Math. Meth. Appl. Sci., 41(2017)8432-8445.

3. Busenberg, S., Iannelli, M., Thieme, H. Global behavior of an age-structured epidemic model. SIAM J. Math. Anal., 22(4): 1065-1080 (1991).

4. Cha, Y., Iannelli, M., Milner, E. Existence and uniqueness of endemic states for the age-structured SIR epidemic model. Maththematical Biosciences, 150: 177-190 (1998).

5. Diekmann, O., Heesterbeek, J.A.P.: Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation. Wiley, Chichester (2000).

6. Diekmann, O., Heesterbeek, J. A. P., Metz, J. A. J., On the deflnition and the computation of the basic reproduction ratio R in models for infectious diseases in heterogeneous populations, J. Math. Biol., 28 (1990), 365-382.

7. Dietz, K., Schenzle, D.: Proportionate mixingmodels for age-dependent infection transmission. J. Math. Biol. 22, 117-120 (1985).

8. Dunford, N. and Schwartz, J. T. Linear Operators Part I: General Theory, New York: Interscience publishers, 1958.

9. El-Doma, M. Analysis of an age-dependent SIS epidemic model with vertical transmission and proportionate mixing assumption. Math. Comput. Model., 29: 31-43 (1999).

10. Greenhalgh, D. Analytical threshold and stability results on age-structured epidemic models with vaccination. Theoretical Population Biology, 33: 266-290 (1988).

11. Greenhalgh, D., Dietz, K.: Some bounds on estimates for reproductive ratios derived from the age-specific force of infection. Math. Biosci. 124, 9-57 (1994).

12. Gurtin, M.E. and MacCamy, R. C., Product Solutions and Asymptotic Behavior for Age-Dependent, Dispersing Populations, (1981).

13. Iannelli, M., Mathematical Theory of Age-Structured Population Dynamics, Giardini Editori e Stampatori in Pisa, (1995).

14. Inaba, H. on a new perspective of the basic reproduction number in heterogeneous environments. J.Math.Biol., 65(2012) 309-348.

15. Inaba, H. Threshold and stability results for an age-structured epidemic model. J. Math. Biol., 28: 149-175(1990).

16. Inaba, H. A semigroup approach to the strong ergodic theorem of the multistate stable population process, Math. Popul. Studies, 1 (1988), 49-77.

17. Inaba, H. Age-Structured Population Dynamics in Demography and Epidemiology, Springer, Singapore, (2017).

18. Kermack, W.O., McKendrick, A.G.: Contributions to the mathematical theory of epidemics I. Proc. R. Soc. 115, 700-721 (1927).

19. Kilbas, A.A., Marichev, O.I. and Samko, S.G. Fractional integrals and derivatives : Theory and applications, (1993).

20. Kilbas, A. A., Srivastava, H. H., Trujillo, J. J. Theory and Applications of Fractional Differential Equations, (2006).

21. Krasnoselskii, M.A. Positive Solutions of Operator Equations. Groningen, Noordhoff, (1964).

22. Langlais, M. Large time behavior in a nonlinear age-dependent population dynamics problem with spatial diffusion, (1988).

23. Lotka,A.J. Elements of Physical Biology. Baltimore: Williams and Wilkins. (Republished as Elements of Mathematical Biology. New York: Dover 1956).

24. Marek, I. Frobenius theory of positive operators: comparison theorems and applications. SIAM J. Appl. Math., 19(3): 607-628 (1970).

25. May, R.M., Anderson, R.M. Endemic infections in growing populations. Math. Biosci., 77: 141-156 (1985).

26. Mckendrick, A. Applications of mathematics to medical problems. Proc. Edinburgh Math. Soc., 44: 98-130 (1926).

27. Miller, K. S. and Ross, B. An Introduction to the Fractional Calculus and Fractional Differential Equations, (1993).

28. Sawashima, I. On spectral properties of some positive operators. Nat. Sci. Dep. Ochanomizu Univ., 15 : 53-64 (1964).

29. Shenghai, Z.: On age-structured SIS epidemic model for time dependent population. Acta Math. Appl. Sin. 15, 45-53 (1999).

30. Smith, H. L. and Thieme, H. R. Dynamical Systems and Population Persistence, Graduate Studies in Mathematics 118, Amer. Math. Soc. Providence, Rhode Island, (2011).

31. Tudor, D.W. An age-dependent epidemic model with applications to measles. Math. Biosci., 73: 131-147 (1985).

32. Webb, G.B. Theory of Nonlinear Age-dependent Population Dynamics. New York and Basel: Marcel Dekker, (1985).

33. Yang, H.M. Directly transmitted infections modeling considering an age-structured contact rate. Math. Comput. Model., 29: 39-48 (1999).

34. Ye, H, Gao, J, Ding, Y: A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. 328(2), 1075-1081 (2007).

35. Zhou Y., Jiao F., Existence of mild solutions for fractional neutral evolution equations, Comput. Math. Appl., (2010), 59(3), 1063-1077.

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