Stability Analysis of an Age Structured Population Model With Fractional Time Derivative

Fatima Cherkaoui, Hiba El Asraoui and Khalid Hilal

ABSTRACT: In this paper, we analyse the large time behaviour in a fractional nonlinear model of population dynamics with age dependent. We show the existence and uniqueness of the solution by using the method of separation of variables, and we studied the Ulam-Hyers stability of the model.

Key Words: Age dependence, population dynamics, Ulam-Hyers stability, fractional model.

Contents

1 Introduction 1

2 Preliminaries 2

3 Formulation of the model 3

4 The characteristic equation 4

5 Assumptions and notations 5

6 Main results 5

6.1 separable solution 5

6.1.1 Necessary and sufficient condition for the existence of separable solution 5

6.1.2 Equation equivalent to the problem (3.1)-(3.4) 6

7 Ulam-Hyers stability results 9

1. Introduction

The following model:

\[ \dot{P} = \delta P, \]  

where \( \delta \) is the growth rate, is considered as the simplest model in the domain of dynamic of populations, and it was introduced by Malthus in [15]. This model does not take into account the effect of overcrowding, for this reason Verhulst proposed the following model [19],

\[ \dot{P} = (\delta_0 - \omega_0 P)P, \]

where \( \delta_0 \) and \( \omega_0 \) are positive constants. In (1.1) the growth rate is given by \( (\delta_0 - \omega_0 P) \) it then depends on the total population \( P \), so it takes into account the effect of overcrowding. The models (1.1) and (1.2) have played a considerable role in the domain of dynamic of populations however they do not take into consideration the age of individuals, even though age is one of the most important parameters structuring a population. The first age-structured model was proposed by Lotka and McKendrick, they assumed that the density of population \( u(t, a) \) satisfies the following problem [14],

\[ \frac{\partial u(t, a)}{\partial t} + \frac{\partial u(t, a)}{\partial a} = -\mu(a)u(t, a), \]

\[ u(t, 0) = \int_0^\infty \beta(a)u(t, a)da, \]

\[ u(0, a) = u_0(a). \]
With,

\[ P(t) = \int_0^\infty u(t, a)\,da. \]

Where \( \mu \) is the death rate and \( \beta \) is the birth rate. In this work we assumed that the parameters \( \mu \) and \( \beta \) are given like in Michel Langlais’s paper \[13\]. We then propose the following model,

\[
\frac{\partial^\alpha u(t, a)}{\partial t^\alpha} + \frac{\partial u(t, a)}{\partial a} = -\mu_n(a)u(t, a) - \mu_e(P(t))u(t, a),
\]

\[ P(t) = \int_0^\infty u(t, a), \]

\[ u(t, 0) = \int_0^\infty \beta(a)u(t, a)\,da. \]

Supplemented by the following initial condition,

\[ u(0, a) = u_0(a). \]

Where \( \frac{\partial^\alpha}{\partial t^\alpha} \) is the Caputo derivative of order \( \alpha \). For more details about the parameters used in this model see section 3. Using the method of separation of variables we proved the existence and uniqueness of the solution, and then we studied the Ulam Hyers stability of the model.

2. Preliminaries

In this section we introduce notations, definitions and preliminary facts which are used throughout this paper. We denote by \( X = \mathcal{C}([0, T], \mathbb{R}^+) \) the Banach space of all continuous functions from \( [0, T] \) into \( \mathbb{R}^+ \), with the norm \( \| P \|_X = \sup\{|P(t)|, t \in [0, T]\} \). We need some basic definitions and properties of the fractional calculus theory. For more details, see \[11\].

**Definition 2.1.** \[11\] The fractional integral of the function \( h \in L^1([a, b]) \) of order \( \alpha \in \mathbb{R}^+ \) is defined by

\[ \Gamma^\alpha_a h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\,ds, \]

where \( \Gamma \) is the gamma function

**Definition 2.2.** \[11\] For a function \( h \) given on the interval \([a, b]\), the Caputo fractional order derivative of \( h \), is given by

\[ {}^cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\,ds, \]

where \( n = [\alpha] + 1 \) and \([\alpha]\) denote the integer part of \( \alpha \).

**Theorem 2.3** (The Schauder’s theorem). Let \( C \) be a nonempty closed convex subset of a Banach space \( X \). Let \( T : C \rightarrow C \) be a continuous mapping such that \( T(C) \) is relatively compact subset of \( X \). Then \( T \) has at least one fixed point in \( C \).

Let \( \epsilon \) be a positive number, \( f : [a, b] \times \mathbb{B} \rightarrow \mathbb{B} \) be a continuous function with \( \mathbb{B} \) is a Banach space and \( \varphi : [a, b] \rightarrow \mathbb{R}^+ \) be a continuous function. We consider the following equation

\[ {}^cD^\alpha x(t) = f(t, x(t)), \quad \alpha \in (0, 1)(or(1, 2)) \quad t \in [a, b], \tag{2.1} \]

and the following inequalities

\[ |{}^cD^\alpha y(t) - f(t, y(t))| \leq \epsilon, \quad t \in [a, b], \tag{2.2} \]

\[ |{}^cD^\alpha y(t) - f(t, y(t))| \leq \varphi(t), \quad t \in [a, b], \tag{2.3} \]

\[ |{}^cD^\alpha y(t) - f(t, y(t))| \leq \epsilon\varphi(t), \quad t \in [a, b]. \tag{2.4} \]
2.3
2.4

Definition 2.4. [20] The equation (2.1) is Ulam-Hyers stable if there exists a real number \( c_f > 0 \) such that for each \( \epsilon > 0 \) and for each solution \( y \in C^1([a,b), B) \) of the inequality (2.2) there exists a solution \( x \in C^1([a,b), B) \) of the equation (2.1) with

\[
|y(t) - x(t)| \leq c_f \epsilon, \quad t \in [a,b).
\]

Definition 2.5. [20] The equation (2.1) is generalized Ulam-Hyers stable if there exists \( \theta_f \in C(R^+, R^+) \), \( \theta_f(0) = 0 \) such that for each solution \( y \in C^1([a,b), B) \) of the inequality (2.2) there exists a solution \( x \in C^1([a,b), B) \) of the equation (2.1) with

\[
|y(t) - x(t)| \leq \theta_f(\epsilon), \quad t \in [a,b).
\]

Definition 2.6. [20] The equation (2.1) is Ulam-Hyers stable with respect to \( \varphi \) if there exists \( c_{f, \varphi} > 0 \) such that for each \( \epsilon > 0 \) and for each solution \( y \in C^1([a,b), B) \) of the inequality (2.4) there exists a solution \( x \in C^1([a,b), B) \) of the equation (2.1) with

\[
|y(t) - x(t)| \leq c_{f, \varphi} \epsilon \varphi(t), \quad t \in [a,b).
\]

Definition 2.7. [20] The equation (2.1) is generalized Ulam-Hyers-Rassias stable with respect to \( \varphi \) if there exists \( c_{f, \varphi} > 0 \) such that for each solution \( y \in C^1([a,b), B) \) of the inequality (2.3) there exists a solution \( x \in C^1([a,b), B) \) of the equation (2.1) with

\[
|y(t) - x(t)| \leq c_{f, \varphi} \epsilon \varphi(t), \quad t \in [a,b).
\]

Remark 2.8. A function \( y \in C^1([a,b), B) \) is a solution of the inequality (2.2) if and only if there exists a function \( g \in C^1([a,b), B) \) such that

\[
|g(t)| \leq \epsilon, \quad t \in [a,b).
\]

\[cD^\alpha y(t) = f(t, y(t)) + g(t).\]

One can have similar remarks for the inequalities (2.3) and (2.4).

Lemma 2.9. [20] Let \( z, w : [0,T) \rightarrow [0,\infty) \) be a continuous functions, where \( T < \infty \). If \( w \) is nondecreasing and there are constants \( k \geq 0 \) and \( \alpha > 0 \) such that

\[
z(t) \leq w(t) + k \int_0^t (t - s)^{\alpha - 1} z(s) ds, \quad t \in [0,T).
\]

Then

\[
z(t) \leq w(t) + \int_0^t \left[ \sum_{n=1}^{\infty} \frac{(k \Gamma(\alpha))^n}{\Gamma(n \alpha)} (t - s)^{n \alpha - 1} w(s) \right] ds, \quad t \in [0,T).
\]

If \( w(t) = \bar{a} \), constant on \([0,T)\), then the above inequality is reduced to

\[
z(t) \leq \bar{a} E_{\alpha}(k \Gamma(\alpha) t^{\alpha}), \quad 0 \leq t < T,
\]

where \( E_{\alpha} \) is the Mittag-Leffler function defined by

\[
E_{\beta}(y) := \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k \beta + 1)}, \quad y \in \mathbb{C}, \quad Re(\beta) > 0.
\]

3. Formulation of the model

Let \( u(t, a) \) be the density of population having at time \( t > 0 \) the age \( a \). We are going to study existence, uniqueness and Ulam-Hyers stability of this equation

\[
\frac{\partial^\alpha u(t, a)}{\partial t^\alpha} + \frac{\partial u(t, a)}{\partial a} = - \mu_n(a) u(t, a) - \mu_c(P(t)) u(t, a).
\]
where,
\[
\frac{\partial^\alpha}{\partial t^\alpha} \text{ is the Caputo derivative, with } 0 < \alpha < 1,
\]
and
\[
P(t) = \int_0^\infty u(t, a) da. \quad (3.2)
\]

- \(P(t)\) is the total population at time \(t\).
- \(\mu_n(a)\) is the probability of dying due to natural causes at age \(a\), \(\mu_n(a) \geq 0\).
- One define the function \(\pi\) in the interval \([0, +\infty[\) by
\[
\pi(a) = \exp(-\int_0^a \mu_n(\sigma) d\sigma).
\]
\(\pi(a)\) is the probability of living to age \(a\). One remarks that: \(\pi(0) = 1\), is decreasing and \(\lim_{a \to +\infty} \pi(a) = 0\).
- \(\mu_e(P)\) is the probability of death due to environmental factors. \(\mu_e\) is a function of the total population.
- The birth low is given by
\[
u(t, 0) = \int_0^\infty \beta(a)u(t, a) da. \quad (3.3)
\]
- \(\beta\) is called the birth-modulus which represents the fertility of the population One assumes that \(\beta\) has a compact support in \([0, \infty]\), so that
\[
\text{supp}\beta \subset [0, A],
\]
where \(A = \max\{a, \beta(a) > 0\} < \infty\), is the minopause age.

We give the initial condition:
\[
u(0, a) = u_0(a). \quad (3.4)
\]

4. The characteristic equation

We can look for a solution of \((3.1)\) in the form
\[
u(t, a) = E_\alpha(rt^\alpha)\gamma(a). \quad (4.1)
\]
\(r\) represents the pure population growth effect.
Substitution of \((4.1)\) into \((3.1)\) gives
\[
\gamma(a)rE_\alpha(rt^\alpha) + E_\alpha(rt^\alpha)\frac{d\gamma}{da} = -\mu_nE_\alpha(rt^\alpha)\gamma(a),
\]
thus
\[
\frac{d\gamma}{da} = -[\mu_n + r]\gamma(a),
\]
and so
\[
\gamma(a) = \gamma(0)\exp(-ra - \int_0^a \mu_n(s) ds), \quad (4.2)
\]
with this $\gamma(a)$ in (4.1) the resulting $u(t, a)$ when inserted into (3.3) gives

$$E_\alpha(r^\alpha)\gamma(0) = \int_0^\infty \beta(a)E_\alpha(r^\alpha)\gamma(0) \exp[-ra - \int_0^a \mu_n(s)ds]da,$$

and hence, on canceling $E_\alpha(r^\alpha)\gamma(0)$,

$$1 = \int_0^\infty \beta(a) \exp[-ra - \int_0^a \mu_n(s)ds]da.$$ 

Thus

$$\int_0^\infty \beta(a)\pi(a)e^{-ra}da = 1.$$  \hfill (4.3)

5. Assumptions and notations

(H1) We assume that

$$0 \leq \beta(a) \leq \beta_1 < \infty \quad \text{on} \quad [0, \infty[,$$

where $\beta_1$ is a real constant, and $\int_0^\infty \beta^2(a)da < \infty$.

(H2) We assume that

$$0 \leq \mu_n(a) \leq \mu_1 < \infty \quad \text{on} \quad [0, \infty[,$$

where $\mu_1$ is a real constant.

(H3) We assume that

$$0 \leq \mu_e(P) \leq \bar{\mu} \quad \text{on} \quad [0, \infty[,$$

where $\bar{\mu}$ is a minimum mortality rate.

6. Main results

6.1. separable solution

We first define the notion of separable solution of the problem (3.1), (3.2), (3.3) and (3.4) which is a solution that can be written as:

$$u(t, a) = \psi(a)P(t),$$ \hfill (6.1)

with the normalisation

$$\int_0^\infty \psi(a)da = 1.$$ \hfill (6.2)

6.1.1. Necessary and sufficient condition for the existence of separable solution. For the existence of $r^*$, the root of the characteristic equation we assume that the hypothesis (H1) and (H2) are satisfied. Consider the function

$$\phi(r) = \int_0^\infty \beta(a)\pi(a)e^{-ra}da.$$ 

The function is continuous, and satisfies,

$$\lim_{r \to -\infty} \phi(r) = +\infty \quad \text{and} \quad \lim_{r \to +\infty} \phi(r) = 0.$$ 

And according to the intermediate value theorem, there exists a unique real number $r^*$, such that $\phi(r^*) = 1.$
Lemma 6.1. Let $r^*$ be the root of (4.3). There exists a non-trivial separable solution of the problem (3.1)-(3.4), if and only if

$$\int_0^\infty \pi(a)e^{-r^*a}da < +\infty. \quad (6.3)$$

Proof.

Let $u(t,a) = \psi(a)P(t)$ be a separable solution, then $\int_0^\infty \psi(a) = 1$.

However, $\psi(a) = \psi_0\pi(a)\exp(-r^*a)$, thus

$$\psi_0\int_0^\infty \pi(a)\exp(-r^*a)da = 1.$$ 

Therefore,

$$\psi_0^{-1} = \int_0^\infty \pi(a)\exp(-r^*a)da < \infty.$$ 

Inversely, if $\int_0^\infty \pi(a)\exp(-r^*a)da < \infty$. Let

$$\psi_0 = \left(\int_0^\infty \pi(a)\exp(-r^*a)da\right)^{-1},$$

and

$$\psi(a) = \psi_0\pi(a)\exp(-r^*a).$$

It remains to show the existence of a function $P(t)$, such that the function $u(t,a) = \psi(a)P(t)$ will be a separable solution. We will define this function in the following paragraph.

6.1.2. Equation equivalent to the problem (3.1)-(3.4). Let $u$ be a separable solution to the problem (3.1)-(3.4). By replacing $u$ with $\psi \times P$, in the equation (3.1), we obtain

$$\frac{1}{P(t)}D_\alpha P(t) + \mu_\epsilon(P(t)) = \frac{-1}{\psi(a)}\frac{d\psi(a)}{da} - \mu_n(a).$$

In this equality the left term depends only on $a$, while the right term depends only on $t$. Then, these two terms are constant and equal to a constant $r$. This leads to the following two equations

$$\frac{-1}{\psi(a)}\frac{d\psi(a)}{da} - \mu_n(a) = r, \quad (6.4)$$

and

$$\frac{1}{P(t)}D_\alpha P(t) + \mu_\epsilon(P(t)) = r. \quad (6.5)$$

The equation (6.4) can be written as:

$$\frac{\psi(a)}{da} = -\psi(a)(\mu_n(a) + r), \quad \text{on} \quad [0, \infty[.$$ 

This equation has the solution

$$\psi(a) = \psi_0\pi(a)e^{-r^*a}. \quad (6.6)$$

From (6.2), we deduce that

$$\psi_0 = \left(\int_0^\infty \pi(a)\exp(-ra)da\right)^{-1}. \quad (6.7)$$
The condition (3.3) can be written as
\[ \psi(0)P(t) = P(t) \int_0^\infty \beta(a)\psi(a) da = P(t) \int_0^\infty \beta(a)\psi(a)\pi(a)\exp(-ra) da. \]

Thus, \( \int_0^\infty \beta(a)\pi(a)\exp(-ra) da = 1 \), and by uniqueness of the solution of the characteristic equation (4.3)
\[ r = r^*. \]

Hence, the solution of the equation (6.4) is
\[ \psi(a) = \psi_0\pi(a)e^{-r^*a}, \]
where \( \psi_0 \) is given by (6.7).

**Study of equation (6.5).**

Let \( r = r^* \). The equation (6.5) can be written as
\[
\begin{cases}
C^D_\alpha P(t) = P(t)(r^* - \mu_P(t)) = f(P(t)), & t \in [0, T], \\
P(0) = P_0,
\end{cases}
\]
where \( f : X \rightarrow \mathbb{R}^+ \) is a continuous function and satisfies the Lipschitz condition i.e \( \exists L > 0 \) such that

(\( H_4 \) \( \| f(P) - f(Q) \| \leq L \| P - Q \| \) for all \( P, Q \in X. \)

**Lemma 6.2.** The function \( P \in X \) is a solution of the equation
\[
\begin{cases}
C^D_\alpha P(t) = f(P(t)), & t \in [0, T], \\
P(0) = P_0, & P_0 \in X
\end{cases}
\]
if and only if \( P \) satisfies the integral equation
\[ P(t) = P_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(P(s)) ds. \]  

**Proof.**

Assume that \( P \in X \) is a solution of problem (6.9). Applying the Caputo fractional operator of the order \( \alpha \), we obtain the first equation in (6.8). Again, substituting \( t = 0 \) in (6.9) we have
\[ P(0) = P_0. \]

Conversely, we have \( C^D_\alpha P(t) = f(P(t)) \), so we get
\[ P(t) = P(0) + I^\alpha f(P(t)), \]
\[ = P_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(P(s)) ds. \]

**Theorem 6.3.** Assume that hypothesis (\( H_4 \)) and \( \left( \frac{LT^\alpha}{\Gamma(\alpha+1)} \right) < 1 \) hold. Then, the fractional problem (6.8) has a unique solution defined on \( [0, T] \).

**Proof.**

We define a subset \( U \) of \( X \) by,
\[ U = \{ P \in X/ \| P \| \leq N \}, \]
where,
\[ N = \| P_0 \| + \frac{1}{\Gamma(\alpha + 1)} \| f \| T^\alpha. \]
It is clear that $U$ satisfies the hypothesis of Theorem 2.3.

By application of Lemma 6.2, (6.8) is equivalent to the integral equation,

$$P(t) = P_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)f(P(s))ds, \ t \in [0, T].$$

Define the operator $A : U \to U$, by

$$AP(t) = P_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}f(P(s))ds.$$

Then our equation is transformed into the operator equation as,

$$P(t) = AP(t), \ \forall t \in [0, T].$$

• We show that $A$ is continuous in $U$.

Let $(P_n)_n$ be a sequence in $U$ converging to a point $P \in U$. Then, by the Lebesgue dominated convergence theorem,

$$\lim_{n \to \infty} AP_n(t) = \lim_{n \to \infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}f(P_n(s))ds + P_0$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \lim_{n \to \infty} f(P_n(s))ds + P_0$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}f(P(s))ds + P_0$$

$$= AP(t).$$

For all $t \in [0, T]$. This shows that $A$ is a continuous operator on $U$.

• $A$ is a compact operator on $U$.

First, we show that $A(U)$ is uniformly bounded set in $X$.

Let $P \in U$. Then for all $t \in [0, T]$,

$$|AP(t)| \leq |P_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}|f(P(s))|ds.$$

$$\|AP\| \leq \|P_0\| + \frac{1}{\Gamma(\alpha+1)} \|f\|_\infty T^\alpha < \infty.$$

Thus, $\|AP\| < \infty$ for all $P \in U$.

This shows that $A$ is uniformly bounded on $U$.

Next, we show that $A(U)$ is equicontinuous set on $X$.

For any $P \in U$, let $\epsilon > 0$, there exists $\eta = \frac{\Gamma(\alpha+1)\epsilon}{4\|f\|}$. For all $t_1, t_2 \in [0, T], |t_2 - t_1| < \eta$ we have

$$\|AP(t_2) - AP(t_1)\| \leq \frac{1}{\Gamma(\alpha+1)} \|f\| |(t_2^\alpha - t_1^\alpha)|.$$

- Case $0 < t_1 < \eta, \ t_2 < 2\eta$ \hfill $t_2^\alpha - t_1^\alpha \leq 2^\alpha \eta^\alpha \leq 2^\alpha \eta \leq 4\eta$.

- Case $0 < t_1 < t_2 \leq \eta$ \hfill $t_2^\alpha - t_1^\alpha \leq \eta^\alpha \leq \alpha \eta \leq 4\eta$. 

Thus,
\[
\|AP(t_2) - AP(t_1)\| \leq \frac{1}{\Gamma(\alpha + 1)} \|f\| \times \frac{\Gamma(\alpha + 1)\epsilon}{4\|f\|} = \epsilon.
\]

Hence, for all \(t_2, t_1 \in [0, T]\) and for all \(P \in U\), \(\forall \epsilon > 0\), \(\exists \eta > 0\),
\[
|t_2 - t_1| < \eta \implies \|AP(t_2) - AP(t_1)\| \leq \epsilon.
\]

This shows that \(A(U)\) is an equicontinuous set on \(X\).

Then, by the Arzela-Ascoli theorem, \(A\) is a continuous and compact operator on \(U\).

Thus by Schauder theorem, the operator \(A\) has a fixed point.

Now, we must show that \(A\) has a unique fixed point.

First, we prove that for all \(j \in \mathbb{N}\) and \(t \in [0, T]\), we have,
\[
\|A^j P - A^j Q\| \leq \left( \frac{LT^\alpha}{\Gamma(\alpha + 1)} \right)^j \|P - Q\|,
\]
for \(j = 0\), the inequality is trivial.

We suppose that the inequality holds for \(j - 1\) and we prove that it holds for \(j\).

Let \(t \in [0, T]\).
\[
|A^j P(t) - A^j Q(t)| = |A(A^{j-1}(P(t)) - A(A^{j-1}Q(t))|,
\]
\[
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(A^{j-1}P(s)) - f(A^{j-1}Q(s))| ds.
\]

Thus,
\[
\|A^j P - A^j Q\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|A^{j-1}P - A^{j-1}Q\|,
\]
\[
\leq \frac{LT^\alpha}{\Gamma(\alpha + 1)} \|A^{j-1}P - A^{j-1}Q\|,
\]
\[
\leq \left( \frac{LT^\alpha}{\Gamma(\alpha + 1)} \right)^j \|P - Q\|.
\]

Hence, the operator \(A\) satisfies the assumptions of weissinger’s fixed point theorem with \(\alpha_j = \left( \frac{LT^\alpha}{\Gamma(\alpha + 1)} \right)^j\),
we can therefore deduce the uniqueness of the solution of our equation.

### 7. Ulam-Hyers stability results

We consider equation \((6.8)\) and the inequality \((2.3)\). We suppose that,

- \((H5)\) \(\phi \in C([1, +\infty), \mathbb{R}^+)\) is continuous, nondecreasing, and there exists \(\lambda_\phi > 0\) such that
\[
\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \phi(s) ds \leq \lambda_\phi \phi(t), \ \forall t \in [a, +\infty).
\]

We obtain the following generalised Ulam-Hyers-Rassias stable results.

**Theorem 7.1.** Under the conditions \((H4)\) and \((H5)\) the equation \((6.8)\) is generalized Ulam-Hyers-Rassias stable.

**Proof.**

Let \(Q \in C([1, +\infty), \mathbb{R}^+)\) be a solution of the inequality \((2.3)\) Denote by \(P\) the unique solution of
\[
\begin{cases}
\frac{CD^\alpha P(t) = f(P(t)),}{P(0) = Q(0).}
\end{cases}
\]
Then, we have:

\[ P(t) = Q(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(P(s))ds, \quad t \in [0, T]. \]

Q is a solution of the inequality (2.3), then there exists \( g \in C([0, T], \mathbb{R}^+) \) such that,

\[
(i) \quad |g(t)| \leq \phi(t), \quad t \in [0, T].
(ii) \quad C D^\alpha Q(t) = f(Q(t)) + g(t), \quad t \in [0, T].
\]

Integrating (ii) from 0 to \( t \) we have

\[ Q(t) = Q(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(Q(s))ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds, \]

By differential inequality (2.3), we have

\[
|Q(t) - Q(0)| - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(Q(s))ds | \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds |
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s)|ds,
\]

\[
\leq \lambda_\phi \phi(t), \quad t \in [0, T].
\]

From this relation it follows:

\[
|Q(t) - P(t)| = |Q(t) - Q(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(Q(s))ds |
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(Q(s))ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(P(s))ds |
\]

\[
\leq |Q(t) - Q(0)| - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(Q(s) - f(P(s))|ds
\]

\[
\leq \lambda_\phi \phi(t) + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |Q(s) - P(s)|ds
\]

\[
\leq \lambda_\phi \phi(t) + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |Q(s) - P(s)|ds.
\]

Let \( Z(t) = |Q(t) - P(t)|. \) Then

\[ Z(t) \leq W(t) + k \int_0^t (t-s)^{\alpha-1} Z(s)ds, \]

with \( W(t) = \lambda_\phi \phi(T) = cst \) and \( k = \frac{L}{\Gamma(\alpha)} \).

By Lemma 2.9,

\[ Z(t) \leq W(t) + \int_0^t \left[ \sum_{n=1}^{\infty} \frac{(k\Gamma(\alpha)n)^n}{\Gamma(n\alpha)} (t-s)^{\alpha n-1} \right] W(s)ds. \]

Since \( W(t) = cst \), then

\[ Z(t) \leq \lambda_\phi \phi(T) E_\alpha(LT^\alpha) \quad 0 \leq t < T. \]

Thus the equation (6.8) is generalised Ulam-Hyers-Rassias stable.
References


Fatima Cherkaoui,
Laboratory LMACS,
FST of Beni Mellal, Sultan Moulay Slimane University
Morocco.
E-mail address: cherkaoui2310@gmail.com

and

Hiba El Asraoui,
Laboratory LMACS,
FST of Beni Mellal, Sultan Moulay Slimane University
Morocco.
E-mail address: hiba.elasraoui@hotmail.com

and

Khalid Hilal,
Laboratory LMACS,
Faculty of Science and Technology of Beni Mellal,
Sultan Moulay Slimane University, Beni Mellal, Morocco.
E-mail address: hilalkhalid2005@yahoo.fr