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The Chevalley–Jordan Decomposition and Spectral Projections of Complex Matrices

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ABSTRACT: In this paper, a novel and simple method for obtaining the Chevalley–Jordan decomposition and the spectral projections of matrices is presented. Our method is direct and elementary, it gives tractable and manageable formulas with minimum mathematical prerequisites. Moreover, knowing only some associated matrices of the matrix, we can simply provide the minimal polynomial of this matrix.

Key Words: The Chevalley–Jordan decomposition, spectral projections, minimal polynomial.

Contents

1. Introduction

The Chevalley–Jordan decomposition of a complex matrix plays a central role in linear algebra. In a recent work, the Chevalley–Jordan decomposition is proved to be a powerful technique in electroencephalography signals [1]. This decomposition is a further development of Jordan canonical form. For any square complex matrix the Chevalley–Jordan decomposition exists and unique, it expresses any square complex matrix as the sum of its commuting diagonalizable part and nilpotent part. This decomposition can be described if the matrix has its Jordan normal form (which is practically difficult to compute), but it may computed even if the Jordan normal form does not.

The main purpose of this work is to propose a new and simple method for determining explicitly the Chevalley–Jordan decomposition and the spectral projections of matrices in a direct way.

Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be distinct elements of $\mathbb{C}$ and $m_1, m_2, \ldots, m_s$ be nonnegative integers. For any polynomial $P(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s}$, we denote by $L_{jk_j}(x)[P]$ the following polynomial

$$L_{jk_j}(x)[P] = P_j(x)(x-\alpha_j)^{k_j} \sum_{i=0}^{m_j-1-k_j} \frac{1}{i!} g_j^{(i)}(\alpha_j)(x-\alpha_j)^i, \tag{1.1}$$

where $1 \le j \le s,\ 0 \le k_j \le m_j - 1$,

$$P_j(x) = \prod_{i=1, i\neq j}^{s} (x-\alpha_i)^{m_i} = \frac{P(x)}{(x-\alpha_j)^{m_j}}, 1 \le j \le s$$

and

$$g_j(x) = (P_j(x))^{-1}.$$

Here $g_j^{(i)}(x)$ denotes the ith derivative of $g_j(x)$.

These polynomials are of great importance, they are used in [4] to invert the confluent Vandermonde matrix and used in [5] for computing the exponential of complex matrices.

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2. Main results

In this section, we will be interested in determining explicit formulas for the Chevalley–Jordan decomposition and the spectral projections of complex matrices. Our contribution does not consist only in giving these formulas, but also in making their determination much practical and expressing them in an elegant representations.
Now, we start by formulating the following important result.

Theorem 2.1. *Let A be a $k \times k$ matrix, and let $\chi_A(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s}$, $\alpha_1 = 0$, be its characteristic polynomial (possibly $m_1 = 0$). Then for every $n \in \mathbb{N}$, we have*

$$A^n = \sum_{j=0}^{m_1-1} \delta_{nj} B_{1j} + \sum_{j=0}^{m_2-1} \binom{n}{j} \alpha_2^{n-j} B_{2j} + \cdots + \sum_{j=0}^{m_s-1} \binom{n}{j} \alpha_s^{n-j} B_{sj}, \tag{2.1}$$

where $B_{jk_j} = L_{jk_j}(A)[\chi_A]$ and δ_{nj} denotes the Kronecker symbol.

Proof. Using Theorem 2.9 of [5], one can easily prove this result. □

In the present approach, one gains a basic construction of the so-called spectral decomposition of A. The approach makes the determination of the spectral decomposition of any square matrix more practical than the usual method of partial fraction decomposition. Using Lagrange polynomials W. A. Harris et al. [2] have derived the spectral decomposition of a matrix with simple eigenvalues. Here, we generalize this result to any matrix using a generalization of Hermite's interpolation formula given by A. Spitzbart [6].

Using the previous theorem we find a new method to calculate the Chevalley–Jordan decomposition and the spectral projections of A at the same time.

Theorem 2.2. *Let A be a $k \times k$ matrix, and let $\chi_A(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s}$, $\alpha_1 = 0$, be its characteristic polynomial (possibly $m_1 = 0$). Then the Chevalley–Jordan decomposition of A is*

$$A = \boldsymbol{D} + \boldsymbol{N}, \tag{2.2}$$

where

$$\boldsymbol{N} = B_{11} + B_{21} + \cdots + B_{s1} \tag{2.3}$$

and

$$\boldsymbol{D} = \alpha_2 B_{20} + \alpha_3 B_{30} + \cdots + \alpha_s B_{s0}, \tag{2.4}$$

where $B_{jr} = L_{jr}(A)[\chi_A]$ for $j = 1, 2, \ldots, s$ and $r = 0, 1$. Moreover, $B_{10}, B_{20}, \ldots, B_{s0}$ are the spectral projections of A at $\alpha_1, \alpha_2, \ldots, \alpha_s$, respectively.

Proof. To obtain

$$A = \boldsymbol{D} + \boldsymbol{N}$$

and

$$I = B_{10} + B_{20} + \cdots + B_{s0}, \tag{2.5}$$

it suffices to take $n = 1$ and $n = 0$ in Formula (2.1), respectively. Since $B_{j0}, B_{j1}; j = 1, 2, \ldots, s$, are polynomials of the matrix A, we have $\boldsymbol{DN} = \boldsymbol{ND}$.

On the other hand, it is clear that $B_{j0}B_{i0} = B_{i0}B_{j0} = 0, i \neq j$. Multiplying both sides of (2.5) by B_{j0}, we get $B_{j0}^2 = B_{j0}, j = 1, 2, \ldots, s$.

Furthermore, the matrix B_{j0} is diagonalizable, then so is $\alpha_j B_{j0}$. The fact that $\alpha_j B_{j0}$ and $\alpha_i B_{i0}$ commute assures then that $\boldsymbol{D}$ is diagonalizable.

To complete the proof, it remains to show that the matrix $\boldsymbol{N}$ is nilpotent. To see this, it suffices to verify, using Formula (1.1), that each B_{j1} is nilpotent. □

Remark 2.3. *We can immediately find that $B_{j0}, j = 1, 2, \ldots, s$, are the spectral projections of A by using Proposition* 3.1 *and Corollary* 3.8 *of [3].*

The following examples are illustrations of Theorem 2.2.

Example 2.4. *Let us consider the following matrix*

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

It is clear that the characteristic polynomial of A is $\chi_A(x) = (x-2)^2(x-3)$. The Chevalley–Jordan decomposition of this matrix is

$$A = \boldsymbol{D} + \boldsymbol{N},$$

where

$$\boldsymbol{D} = 2B_{10} + 3B_{20} \quad \textit{and} \quad \boldsymbol{N} = B_{11}.$$

A trivial calculation yields

$$B_{10} = L_{10}(A)[\chi_A] = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_{11} = L_{11}(A)[\chi_A] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_{20} = L_{20}(A)[\chi_A] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\boldsymbol{D} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = A \quad \textit{and} \quad \boldsymbol{N} = 0.$$

We deduce that A is diagonalizable.

Example 2.5. *Consider the matrix*

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & -2 & 0 & -3 \\ 2 & 3 & 0 & 3 \\ 1 & 5 & -1 & 6 \end{pmatrix}.$$

The characteristic polynomial of A is $\chi_A(x) = (x-2)(x-1)^3$. The Chevalley–Jordan decomposition of this matrix is

$$A = \boldsymbol{D} + \boldsymbol{N},$$

where

$$\boldsymbol{D} = 2B_{10} + B_{20} \quad \textit{and} \quad \boldsymbol{N} = B_{21}.$$

On the other hand, we have

$$\begin{cases} B_{10} & = & (A-I)^3 \\ B_{20} & = & (A-2I)(-A^2+A-I) \\ B_{21} & = & -A(A-2I)(A-I) \end{cases}$$

Simple calculation gives

$$B_{10} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_{20} = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_{21} = \begin{pmatrix} -1 & -2 & 1 & -2 \\ 0 & -3 & 0 & -3 \\ 1 & 2 & -1 & 2 \\ 1 & 5 & -1 & 5 \end{pmatrix}.$$

Then

$$\boldsymbol{D} = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

and

$$\boldsymbol{N} = \begin{pmatrix} -1 & -2 & 1 & -2 \\ 0 & -3 & 0 & -3 \\ 1 & 2 & -1 & 2 \\ 1 & 5 & -1 & 5 \end{pmatrix}.$$

The following Theorem shows that knowing only the associated matrices B_{ij} of A, we can simply provide the minimal polynomial of the matrix A.

Theorem 2.6. *Let A be a matrix and let $\chi_A(x) = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_s)^{m_s}$ be its characteristic polynomial with $\alpha_1 = 0$ (possibly $m_1 = 0$). Then,*

1. *The index of α_i is the greatest integer j such that $B_{ij-1} \neq 0$.*
2. *The index of α_i is 1 if and only if $B_{i1} = 0$.*
3. *A is diagonalizable if and only if $B_{i1} = 0, i = 1, 2, \ldots, s$.*

Proof. Clearly Formula (2.1) is a $\mathcal{P}$-canonical form of A (see [3]) and the result follows then from Corollary 3.6., Theorem 4.1. of [3] and Theorem 2.1 above. □

Let us consider the same matrix of Example (2.5)

Example 2.7. *We have*

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & -2 & 0 & -3 \\ 2 & 3 & 0 & 3 \\ 1 & 5 & -1 & 6 \end{pmatrix}.$$

Since $B_{21} \neq 0$, then using (3) *of the last theorem the matrix A is not diagonalizable.*
It is easy to verify that

$$B_{22} = -(A - 2I)(A - I)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -3 & -6 & 3 & -6 \\ 0 & 0 & 0 & 0 \\ 3 & 6 & -3 & 6 \end{pmatrix}.$$

using (1) *of the last theorem the minimal polynomial of A is $(x-2)(x-1)^3$.*

3. Conclusion

We have presented a new and elegant method to facilitate the computation of the Chevalley–Jordan decomposition and the spectral projections of matrices. We can also provide the minimal polynomial of matrices.

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