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h-admissible Fourier Integral Operators with Complex Phase Function

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Abstract: We study in this work a particular class of h-admissible Fourier integral operators with complex phase function. These operators are bounded on Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and on its dual $\mathcal{S}'(\mathbb{R}^n)$.

Key Words: h-admissible Fourier integral operator, complex phase function, amplitude, boundedness.

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1. Introduction

The theory of h-pseudodifferential operators is well adapted to the study of various semiclassical elliptic differential equations issues. However, when investigating semiclassical equations of the hyperbolic class, this theory falls short, and one is compelled to look at a broader class of operators, the so-called h-Fourier integral operators.

Many authors have worked hard since 1970 to learn more about this type of operator.(see, e.g., [4,5,6,9,8,10,11,13]). The first works on Fourier integral operators deal with local properties. We note that, K. Asada and D. Fujiwara ([4]) have studied for the first time a class of Fourier integral operators defined on $\mathbb{R}^n$.

The h-Fourier integral operators are represented by formulas of the type

$$I_h\left(a,\phi\right)f\left(x\right)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^N}e^{ih^{-1}\phi(x,\xi,y)}a\left(x,\xi,y\right)f\left(y\right)dyd\xi, \tag{1.1}$$

$f\in\mathcal{S}\left(\mathbb{R}^n\right)$ (the Schwartz space). The function a is called the amplitude, the function ϕ is called the phase function and $h\in\left]0,h_0\right]$ is a semiclassical parameter.

According to the theory of Fourier integral operators treated by Hörmander [11], the phase functions in $C^\infty\left(\mathbb{R}^n\times\mathbb{R}^n\setminus 0\right)$ homogenous of degree 1 in the frequency variable ξ and with non-vanishing determinant of the mixed Hessian matrix (i.e. non-degenerate phase functions), while the symbols in $C^\infty\left(\mathbb{R}^n\times\mathbb{R}^n\right)$ such that

$$\left|\partial_\xi^\alpha\partial_x^\beta a\left(x,\xi\right)\right|\leq C_{\alpha,\beta}\left(1+|\xi|\right)^{m-\rho|\alpha|+\delta|\beta|}$$

(i.e. $a\left(x,\xi\right)\in S^m_{\rho,\delta}$), was initiated in the Basic paper of L. Hörmander [11]. Moreover, G. Eskin [7] (in the case $a\in S^0_{1,0}$) and Hörmander [11,12] (in the case $a\in S^0_{\rho,1-\rho}$, $\frac{1}{2}<\rho\leq 1$) proved the local L^2 boundedness of Fourier integral operators with non-degenerate phase functions.

Other symbols and phase functions, on the other hand, were investigated. In [10,15], D. Robert and B. Helffer treated the symbol class Γ^μ_ρ (see below) and they considered phase functions satisfying certain conditions.

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The goal of this paper is to extend the concept of h-admissible Fourier integral operators described in [1,2] by taking into account the phase function with complex values. Then using the same technique as in [3] to demonstrate that the h-admissible Fourier integral operators with complex phase function are well defined and bounded on $\mathcal{S}(\mathbb{R}^n)$ and on its dual. We further show that these class of operators are stable by composition.

When the phase function $\phi(x,y,\xi) = S(x,\xi) - \langle y, \xi \rangle$; where $S \in C^\infty(\mathbb{R}^{2n};\mathbb{C})$; the operator 1.1 will be a special case of h-admissible Fourier integral operators with complex phase function. We'll also make additional assumptions about the phase function S and the amplitude a in this situation.

Now, let us talk about the structure of this paper. In the second section we gave some notations and we recall the background of general class of h-admissible Fourier integral operators. In the third section, the composition of h-admissible Fourier integral operators with complex phase function is described. The last section focuses on the specific case.

2. Preliminaries

2.1. Notations

Unless otherwise specified, $n \in \mathbb{N}$ is assumed throughout the study. In particular $n \neq 0$. For all $x, y, \xi \in \mathbb{R}^n$ we define

$$\langle x, \xi \rangle := \sum_{j=0}^{n} x_j \xi_j \quad \textit{and} \quad \widehat{d_h \xi} := (2\pi h)^{-n} d\xi.$$

First, let us recall a weight function defined by

$$\lambda(x,y,\xi) := \left(1 + |x|^2 + |y|^2 + |\xi|^2\right)^{1/2}.$$

In addition, for two nonnegative quantities A and B, the notation $A \lesssim B$ means that $A \leq CB$ for some unspecified constant $C > 0$, and $A \backsimeq B$ means that $A \lesssim B$ and $B \lesssim A$.

Furthermore, we define the semiclassical Fourier transform $\mathcal{F}_h$ and its inverse $\mathcal{F}_h^{-1}$ by

$$\mathcal{F}_h f(\xi) := \int_{\mathbb{R}^n} e^{-ih^{-1}\langle x,\xi\rangle} f(x)\, dx, \quad \textit{and} \quad \mathcal{F}_h^{-1} f(x) := \int_{\mathbb{R}^n} e^{ih^{-1}\langle x,\xi\rangle} f(\xi)\, \widehat{d_h \xi},$$

where $f \in \mathcal{S}(\mathbb{R}^n)$. And we scale partial derivatives with respect to a variable $x \in \mathbb{R}^n$ with the factor $-i$ are denoted by

$$D_x^\alpha := (-i)^{|\alpha|} \partial_x^\alpha := (-i)^{|\alpha|} \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index and $|\alpha| = \sum_{j=1}^n \alpha_j$ is the length of α.

Considering two Freshet spaces E and F, the set $\mathcal{L}(E,F)$ contains of all bounded linear operators $L : E \to F$. If $E = F$, we also just write $\mathcal{L}(E)$.

2.2. A general class of h-admissible Fourier integral operators

Definition 2.1. *Let Ω be an open set in $\mathbb{R}^{3n}$, $\mu \in \mathbb{R}$ and $\rho \in [0,1]$. The space of amplitudes $\Gamma_\rho^\mu(\Omega)$ is the set of smooth functions $a : \Omega \to \mathbb{C}$ such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_y^\gamma a(x,\xi,y)\right| \leq C_{\alpha\beta\gamma} \lambda^{\mu - \rho(|\alpha|+|\beta|+|\gamma|)}(x,\xi,y) \tag{2.1}$$

uniformly in Ω for all $(\alpha,\beta,\gamma) \in \mathbb{N}^{3n}$. Moreover, let for all $k \in \mathbb{N}$

$$|a|_k^{\mu,\rho} := \max_{|\alpha|+|\beta|+|\gamma| \leq k} \sup_{(x,\xi,y)\in\Omega} \lambda^{-\mu+\rho(|\alpha|+|\beta|+|\gamma|)}(x,\xi,y) \left|\partial_x^\alpha \partial_\xi^\beta \partial_y^\gamma a(x,\xi,y)\right|$$

be the associated sequence of semi-norms.

Remark 2.2. *Instead of $\Gamma_\rho^\mu\left(\mathbb{R}^{3n}\right)$, we use Γ_ρ^μ for short.*

Now, we take an interest in giving a sense of the integral defined by:

$$[I_h(a,\phi)f](x)=\iint e^{ih^{-1}\phi(x,\xi,y)}a(x,\xi,y)f(y)\,dy\widehat{d_h\xi},\quad f\in S(\mathbb{R}^n) \tag{2.2}$$

where $a\in\Gamma_\rho^\mu$ and $\phi=\varphi+i\psi$ be a complex phase function which satisfies the following conditions

$(H1)$ $\phi\in C^\infty\left(\mathbb{R}^{3n};\mathbb{C}\right)$

$(H2)$ For all $(\alpha,\beta,\gamma)\in\mathbb{N}^{3n}$ and $(x,\xi,y)\in\mathbb{R}^{3n}$ such that

$$\left|\partial_x^\alpha\partial_\xi^\beta\partial_y^\gamma\phi(x,\xi,y)\right|\lesssim\lambda^{(2-|\alpha|-|\beta|-|\gamma|)}(x,\xi,y);$$

$(H3)$ For all $(x,\xi,y)\in\mathbb{R}^{3n}$ such that

$$\lambda(x,\xi,y)\backsimeq\lambda(\partial_y\phi,\partial_\xi\phi,y).$$

$(H'3)$ For all $(x,\xi,y)\in\mathbb{R}^{3n}$ such that

$$\lambda(x,\xi,y)\backsimeq\lambda(x,\partial_\xi\phi,\partial_x\phi).$$

$(H4)$ For all $(x,\xi,y)\in\mathbb{R}^{3n}$ $\quad\mathrm{Im}(\phi(x,\xi,y))=\psi(x,\xi,y)\geq 0.$

Remark 2.3. *The phase function $\phi(x,y,\xi)=\langle x-y,\xi\rangle$ clearly fulfills the hypotheses $(H1),(H2),(H3)$ and $(H'3)$.*

The following definitions are given to generalize the concept of h-admissible operators [15].

Definition 2.4. *We call h-admissible symbol of weight (μ,ρ), every application $a(h)$ of $]0,h_0]$ in Γ_ρ^μ, such that for all $N\in\mathbb{N}$*

$$a(h)=\sum_{j=0}^N h^j a_j+h^{N+1}r_{N+1}(h),$$

where $a_j\in\Gamma_\rho^{\mu-2\rho j}$, and $\{r_{N+1}(h),h\in]0,h_0]\}$ is bounded in $\Gamma_\rho^{\mu-2\rho(N+1)}$.

Definition 2.5. *We call h-admissible Fourier integral operator, every C^∞ application A of $]0,h_0]$ in $\mathcal{L}\left(\mathcal{S}(\mathbb{R}^n);L^2(\mathbb{R}^n)\right)$, for which there exists a sequence $(a_j)_j\in\Gamma_0^\mu$ satisfying for all $N\in\mathbb{N}$ and N large enough*

$$A(h)=\sum_{j=0}^N h^j I_h(a_j,\phi)+h^{N+1}R_{N+1}(h) \tag{2.3}$$

where

$$[I_h(a_j,\phi)f](x)=\iint e^{ih^{-1}\phi(x,\xi,y)}a_j(x,\xi,y)f(y)dy\widehat{d_h\xi},$$

$$\sup_{h\in(0,h_0]}\|R_{N+1}(h)\|_{\mathcal{L}(L^2(\mathbb{R}^n))}<\infty$$

To provide the right-hand side of (2.2) some context, we consider $g\in\mathcal{S}\left(\mathbb{R}^{3n}\right)$ and $g(0,0,0)=1$. If $a\in\Gamma_0^\mu$, we define

$$a_r(x,\xi,y)=g\left(\frac{x}{r},\frac{\xi}{r},\frac{y}{r}\right)a(x,\xi,y),\quad r>0.$$

Theorem 2.6. *Let ϕ be a phase function satisfying $(H1),(H2),(H3)$ and $(H'3)$. then*

1. *For all $f\in\mathcal{S}(\mathbb{R}^n)$, $\lim_{r\to+\infty}[I_h(a_r,\phi)f](x)$ exists for every point $x\in\mathbb{R}^n$ and is independent of the choice of the function g. We then set*

$$[I_h(a,\phi)f](x):=\lim_{r\to+\infty}[I_h(a_r,\phi)f](x)\quad\forall x\in\mathbb{R}^n.$$

2. *$I_h(a,\phi)\in\mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ and $I_h(a,\phi)\in\mathcal{L}(\mathcal{S}'(\mathbb{R}^n))$.*

Proof: See [3] □

3. Composition of two h-admissible Fourier integral operators

We derive a theorem in this section demonstrating that these types of operators with complex phase functions are stable by composition

Theorem 3.1. *Let ϕ_1, ϕ_2 be two complex phases functions satisfying $(H1), (H2), (H3)$ and $(H'3)$. Set*

$$\phi(x,\xi,y) = \phi_1(x,\xi_1,z) + \phi_2(z,\xi_2,y),$$

with $x,y \in \mathbb{R}^n, \xi = (\xi_1, z, \xi_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^n \times \mathbb{R}^{N_2}$. Then ϕ verifies $(H1), (H2), (H3)$ and $(H'3)$, for all $a_1 \in \Gamma_0^{\mu_1}, a_2 \in \Gamma_0^{\mu_2}$, we have

$$I_h(a_1,\phi_1) I_h(a_2,\phi_2) = I_h(a_1 \times a_2, \phi) \tag{3.1}$$

with

$$(a_1 \times a_2)(x,\xi,y) = a_1(x,\xi_1,z) a_2(z,\xi_2,y). \tag{3.2}$$

Proof: (H_1) and (H_2) are immediate to verify for ϕ. So we will prove the hypothesis (H_3).

To show (H_3) or (H_3'), it suffices to show that ϕ satisfies:

for all $x,y,z \in \mathbb{R}^n$ and $\xi_i \in \mathbb{R}^{N_i}$ such that

$$\lambda(x,\xi_1,y,\xi_2,z) \lesssim \lambda\left(z, \partial_z\phi_2, \partial_y\phi_1 + \partial_y\phi_2, \partial_{\xi_1}\phi_1, \partial_{\xi_2}\phi_2\right). \tag{3.3}$$

Applying the property (H_3) to ϕ_1 and ϕ_2 we get that

$$\lambda(x,\xi_1,y,\xi_2,z) \lesssim \lambda\left(\partial_y\phi_1, \partial_{\xi_1}\phi_1, y, \partial_{\xi_2}\phi_2, \partial_z\phi_2, z\right). \tag{3.4}$$

We have also

$$\lambda(y) \lesssim \lambda\left(\partial_{\xi_2}\phi_2, \partial_z\phi_2, z\right), \tag{3.5}$$

from (H_3) applied to ϕ_2, we get

$$|\partial_y\phi_2| \lesssim \lambda(y,\xi_2,z), \tag{3.6}$$

and

$$\lambda(y,\xi_2,z) \lesssim \lambda\left(\partial_{\xi_2}\phi_2, \partial_z\phi_2, z\right), \tag{3.7}$$

from (H_2) and (H_3) applied to ϕ_2.

Finally, we note that

$$|\partial_y\phi_1| \leq |\partial_y\phi_1 + \partial_y\phi_2| + |\partial_y\phi_2|. \tag{3.8}$$

The inequalities (3.4)-(3.8) imply (3.3).

Let's have a look at the composition formulas now. Let's start by talking about sequences of functions $(i \in \{1,2\})$

$$\chi_{i,p}(x,\xi_i,y) = e^{-p^{-1}\left(|x|^2+|\xi_i|^2+|y|^2\right)}; (x,\xi_i,y) \in \mathbb{R}^n \times \mathbb{R}^{N_i} \times \mathbb{R}^n$$

It is clear that (3.1) is satisfied for

$$a_{1,p} = a_1\chi_{1,p},$$

and

$$a_{2,p} = a_2\chi_{2,p}.$$

But

$$\chi_{1,p}(x,\xi_1,y)\chi_{2,p}(y,\xi_2,z) = e^{-p^{-1}\left(|x|^2+2|y|^2+|\xi_1|^2+|\xi_2|^2+|z|^2\right)}.$$

Hence it results that

$$\lim_{p\to\infty}\left(I_h(a_{1,p}a_{2,p},\phi)f\right)(x) = \left(I_h(a_1a_2,\phi)f\right)(x), \tag{3.9}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$.

On the other hand, the proof of theorem 2.6 shows that, for all $l \in \mathbb{N}$ and $j \in \{1,2\}$, an integer $M_{j,l}$, such that, for all f in $\mathcal{S}(\mathbb{R}^n)$ and $p \geq 1$, we obtain

$$\left\|I_h(a_{j,p},\phi_j) f\right\|_{B_l} \lesssim \|f\|_{B_{M_{j,l}}}, \tag{3.10}$$

where $B_l(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n), x^\alpha D_x^\beta u \in L^2(\mathbb{R}^n), |\alpha| + |\beta| \leq l\right\}.$

We first infer from (3.10) that, for all fixed f_0 in $\mathcal{S}(\mathbb{R}^n)$, $g_p = I_h(a_{2,p}, \phi_2) f_0$ describes a bounded of $\mathcal{S}(\mathbb{R}^n)$ when p varies. $\mathcal{S}(\mathbb{R}^n)$ being Montel space, we can extract a subsequence, suppose that g_p converges in $\mathcal{S}(\mathbb{R}^n)$ to $g = I_h(a_1, \phi_2) f_0$, but we have

$$K_1 \leq K_2 + K_3, \tag{3.11}$$

where

$$K_1 = \|I_h(a_{1,p}, \phi_1) g_p - I_h(a_1, \phi_1) g\|_{B_l},$$

and

$$K_2 = \|(I_h(a_{1,p}, \phi_1) - I_h(a_1, \phi_1)) g\|_{B_l},$$

and

$$K_3 = \|I_h(a_{1,p}, \phi_1)(g_p - g)\|_{B_l}.$$

Even if we re-extract a subsequence, we can assume

$$I_h(a_{1,p}, \phi_1) g \to I_h(a_1, \phi_1) g, \text{ in } \mathcal{S}(\mathbb{R}^n). \tag{3.12}$$

As can be seen from (3.10)-(3.12) that, for all l, leaves to extract a subsequence, we have

$$I_h(a_{1,p}, \phi_1) I_h(a_{2,p}, \phi_2) f_0 \to I_h(a_1, \phi_1) I_h(a_2, \phi_2) f_0 \text{ in } B_l. \tag{3.13}$$

□

4. Main results

We will study at a particular case of the complex phase function *phi* in this section, which is highly useful for solving Cauchy issues [14]. Let

$$\phi(x, y, \xi) = S(x, \xi) - \langle y, \xi \rangle,$$

and suppose that S satisfies:

(G_1) $S \in C^\infty\left(\mathbb{R}_x^n \times \mathbb{R}_\xi^n; \mathbb{C}\right)$, where $S = S_1 + iS_2$.

(G_2) For all $(\alpha, \beta) \in \mathbb{N}^{2n}$, we have

$$\left|\partial_x^\alpha \partial_\xi^\beta S(x, \xi)\right| \lesssim \lambda^{2-|\alpha|-|\beta|}(x, \xi)$$

(G_3) There exists $\delta_0 > 0$ such that

$$\inf_{(x,\xi)\in\mathbb{R}^{2n}} \left|det \frac{\partial^2 S}{\partial x \partial \xi}(x, \xi)\right| \geq \delta_0;$$

(G_4) For all $(x, \xi) \in \mathbb{R}^{2n}$, $\quad S_2(x, \xi) \geq 0$.

Proposition 4.1. *If S satisfies $(G_1), (G_2), (G_3)$ and (G_4). Then the phase function*

$$\phi(x, y, \xi) = S(x, \xi) - \langle y, \xi \rangle$$

satisfies $(H_1), (H_2), (H_3), (H_3')$ and (H_4).

Proof: $(H_1), (H_2)$ and (H_4) are trivially satisfied.

The following lemma is required to prove (H_3) and (H_3')

Lemma 4.2. *Let's assume that S satisfies $(G_1),(G_2),(G_3)$ and (G_4), then S satisfies:*

$$\begin{cases} |x| \lesssim \lambda(\xi, \partial_\xi S), & \forall (x,\xi) \in \mathbb{R}^{2n}, \\ |\xi| \lesssim \lambda(x, \partial_x S), & \forall (x,\xi) \in \mathbb{R}^{2n}. \end{cases} \tag{4.1}$$

In addition, we have for all $(x,\xi),(x',\xi') \in \mathbb{R}^{2n}$,

$$|x-x'| + |\xi-\xi'| \lesssim \left[|(\partial_\xi S)(x,\xi) - (\partial_\xi S)(x',\xi')| + |\xi-\xi'| \right]. \tag{4.2}$$

Proof: From $(G2)$ and $(G3)$ and using the global inversion theorem we see that the functions f_x and g_ξ defined by

$$f_x : \xi \to \partial_x S(x,\xi), \qquad g_\xi : x \to \partial_\xi S(x,\xi),$$

are diffeomorphisms from $\mathbb{R}^n$ to $\mathbb{C}^n$. And the functions k_1 and k_2 defined by

$$k_1 : (x,\xi) \to (x, \partial_x S(x,\xi)), \qquad k_2 : (x,\xi) \to (\xi, \partial_\xi S(x,\xi)),$$

are diffeomorphisms from $\mathbb{R}^{2n}$ to $\mathbb{R}^n \times \mathbb{C}^n$.

Indeed,

$$k_1'(x,\xi) = \begin{pmatrix} I_n & \frac{\partial^2 S}{\partial x^2}(x,\xi) \\ 0 & \frac{\partial^2 S}{\partial x \partial \xi}(x,\xi) \end{pmatrix},$$

$$k_2'(x,\xi) = \begin{pmatrix} 0 & \frac{\partial^2 S}{\partial x \partial \xi}(x,\xi) \\ I_n & \frac{\partial^2 S}{\partial x^2}(x,\xi) \end{pmatrix}.$$

and

$$|\det k_1'(x,\xi)| = |\det k_2'(x,\xi)| = \left| \det \frac{\partial^2 S}{\partial x \partial \xi}(x,\xi) \right| \geq \delta_0 > 0 \quad \forall (x,\xi) \in \mathbb{R}^{2n}$$

Then

$$\left\| (k_i'(x,\xi))^{-1} \right\| = \left| \det \frac{\partial^2 S}{\partial x \partial \xi}(x,\xi) \right|^{-1} \left\| {}^t Co\left[k_i'(x,\xi)\right] \right\|, \quad \forall i \in \{1,2\}$$

where $Co[k_i'(x,\xi)]$ is the cofactor matrix of $k_i'(x,\xi)$ for all $i \in \{1,2\}$. By (G_3), we know that $\|{}^t Co[k_i'(x,\xi)]\|$ $\forall i \in \{1,2\}$ are uniformly bounded. From (G_2) and (G_3), it follows that $\left\| (f_x^{-1})' \right\|, \left\| \left(g_\xi^{-1}\right)' \right\|$ are uniformly bounded on $\mathbb{R}^n$ and $\left\| (k_2^{-1})' \right\|$ is uniformly bounded on $\mathbb{R}^{2n}$,

where

$$\psi_2(x,\xi) = (\xi, \partial_\xi S(x,\xi))$$

Thus (G_3) and the Taylor's theorem lead to the following estimate:

There exist $M, N > 0$, such that for all $(x,\xi),(x',\xi') \in \mathbb{R}^{2n}$,

$$\begin{aligned} |\xi| &= \left| f_x^{-1}(f_x(\xi)) - f_x^{-1}(f_x(0)) \right| \\ &\leq M \left| \partial_x S(x,\xi) - \partial_x S(x,0) \right| \\ &\leq C_4 \lambda(x, \partial_x S) \\ &\lesssim \lambda(x, \partial_x S), \end{aligned}$$

$$\begin{aligned} |x| &= \left| g_\xi^{-1}(g_\xi(\xi)) - g_\xi^{-1}(g_\xi(0)) \right| \\ &\leq N \left| \partial_\xi S(x,\xi) - \partial_\xi S(0,\xi) \right| \\ &\leq C_5 \lambda(\partial_\xi S, \xi) \\ &\lesssim \lambda(\partial_\xi S, \xi), \end{aligned}$$

$$\begin{aligned}\left|(x,\xi)-\left(x',\xi'\right)\right| &= \left|k_2^{-1}\left(k_2\left(x,\xi\right)\right)-k_2^{-1}\left(k_2\left(x',\xi'\right)\right)\right|\\ &\leq C_5\left|\left(\xi,\partial_\xi S\left(x,\xi\right)\right)-\left(\xi',\partial_\xi S\left(x',\xi'\right)\right)\right|\\ &\lesssim \left|\left(\xi,\partial_\xi S\left(x,\xi\right)\right)-\left(\xi',\partial_\xi S\left(x',\xi'\right)\right)\right|.\end{aligned}$$

□

From (4.1) we have

$$\begin{aligned}\lambda\left(x,y,\xi\right) &\leq \lambda\left(x,\xi\right)+\lambda\left(y\right)\\ &\leq C_6\left(\lambda\left(\xi,\partial_\xi S\right)+\lambda\left(y\right)\right),\end{aligned}$$

with $C_6>0$.

Also, we have $\partial_{y_j}\phi=-\xi_j$; and $\partial_{\xi_j}\phi=\partial_{\xi_j}S-y_j$, so

$$\begin{aligned}\lambda\left(\xi,\partial_\xi S\right) &= \lambda\left(\partial_y\phi,\partial_\xi\phi+y\right)\\ &\leq 2\lambda\left(\partial_y\phi,\partial_\xi\phi,y\right),\end{aligned}$$

which finally gives for some $C_7>0$,

$$\begin{aligned}\lambda\left(x,\xi,y\right) &\leq 2C_6\lambda\left(\partial_y\phi,\partial_\xi\phi,y\right)\\ &\leq \frac{1}{C_7}\lambda\left(\partial_y\phi,\partial_\xi\phi,y\right)\\ &\lesssim \lambda\left(\partial_y\phi,\partial_\xi\phi,y\right),\end{aligned}$$

and from (4.1) , we get

$$\lambda\left(x,\xi,y\right)\gtrsim\lambda\left(\partial_y\phi,\partial_\xi\phi,y\right).$$

So

$$\lambda\left(x,\xi,y\right)\simeq\lambda\left(\partial_y\phi,\partial_\xi\phi,y\right).$$

By a similar argument we can prove $\left(H_3'\right)$.

□

As a consequence of the previous calculus we obtain a result of boundedness of h-admissible Fourier integral [1,2,16] operators with complex phase function in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.

Theorem 4.3. *Let I_h be a h-admissible Fourier integral operator of the form*

$$I_hu\left(x\right)=\int e^{ih^{-1}S(x,\xi)}a\left(x,\xi\right)\mathcal{F}_hu\left(\xi\right)\widehat{d_h\xi},$$

where $a\in\Gamma^m(\mathbb{R}^{2n})$; $h\in]0;h0]$ and S satisfies $(G1)$; $(G2)$; $(G3)$ and $(G4)$. Then I_h can be extended to a bounded operator from $\mathcal{S}(\mathbb{R}^n)$ into itself, and from $\mathcal{S}'(\mathbb{R}^n)$ into itself.

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