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On Some Properties of $\mathcal{I}^{\mathcal{K}}_{sn}$-Topological Spaces

Ankur Sharmah and Debajit Hazarika

abstract: In this paper, we introduce the notion of $\mathcal{I}^{\mathcal{K}}_{sn}$-open set and show that the family of $\mathcal{I}^{\mathcal{K}}_{sn}$-open sets in a topological space forms a topology. The category of $\mathcal{I}^{\mathcal{K}}$-neighborhood spaces is introduced and several properties are obtained thereafter. Moreover, we obtain a necessary and sufficient condition for the coincidence of the notions "preserving $\mathcal{I}^{\mathcal{K}}$-convergence" and "$\mathcal{I}^{\mathcal{K}}$-continuity" for any mapping defined on X. Several mappings that are defined on a topological space are shown to be coincident in an $\mathcal{I}^{\mathcal{K}}$-sequential space. The entire investigation is performed in the setting of $\mathcal{I}^{\mathcal{K}}$-convergence which further extends the recent developments [11,13,1].

Key Words: $\mathcal{I}^{\mathcal{K}}$-convergence, $\mathcal{I}^{\mathcal{K}}_{sn}$-open sets, $\mathcal{I}^{\mathcal{K}}_{sn}$-neighborhood space and $\mathcal{I}^{\mathcal{K}}$-compatibility.

Contents

1. Introduction

The idea of sequential neighborhood and sequentially open set have been widely used in Point Set Topology to obtain various results [12,13]. Lin [11] studied this concept in ideal topological spaces and introduced an intermediate open set termed as $\mathcal{I}_{sn}$-open set which sits in between open and $\mathcal{I}$-open sets. It is helpful for $\mathcal{I}_{sn}$-open sets over $\mathcal{I}$-open sets that the family of all $\mathcal{I}_{sn}$-open subsets in a topological space forms a topology. For two ideals $\mathcal{I} \subset \mathcal{J}$, the notion of $\mathcal{J}$-open implies $\mathcal{I}$-open but the same has not been achieved for $\mathcal{I}_{sn}$-open sets. Simple observation shows that for two ideals $\mathcal{I}$, $\mathcal{J}$ with $\mathcal{I} \subset \mathcal{J}$, if a subset in X is $\mathcal{J}_{sn}$-open then it is also $\mathcal{I}$-open. If the collections τ, $\tau_{\mathcal{I}}$, $\tau_{\mathcal{I}_{sn}}$ represent the topologies generated by the family of all open sets, $\mathcal{I}$-open sets and $\mathcal{I}_{sn}$-open sets, then $\tau \subset \tau_{\mathcal{I}_{sn}} \subset \tau_{\mathcal{I}}$. Again, if the parent space is a $\mathcal{I}$-sequential space then the corresponding collections τ, $\tau_{\mathcal{I}}$, $\tau_{\mathcal{I}_{sn}}$ coincide. Moreover, it is evident that for a maximal ideal $\mathcal{I}$, the terminologies $\mathcal{I}$-open and $\mathcal{I}_{sn}$-open coincide, i.e, $\tau_{\mathcal{I}_{sn}} = \tau_{\mathcal{I}}$. Further, if $\mathcal{I} = Fin$, then the notions, $\mathcal{I}_{sn}$-open and $\mathcal{I}$-open, merge with the concept of open sets. It is natural to ask the following question.

Question 1.1. *Can we obtain a necessary and sufficient condition for the coincidence of the collections τ, $\tau_{\mathcal{I}}$ and $\tau_{\mathcal{I}_{sn}}$ i.e. $\tau = \tau_{\mathcal{I}_{sn}} = \tau_{\mathcal{I}}$?*

For a topological space and ideal $\mathcal{I}$ of ω, the set of all natural numbers, it is clear that A is $\mathcal{I}_{sn}$-open set of X if and only if $\{n \in \omega : x_n \in A\} \in \mathcal{I}^*$ for each sequence $\{x_n\}$ in X with $x_n \to_{\mathcal{I}} x$. The following diagram represents the relations among several terminologies appeared in [11].

$$\begin{tikzcd}
 & \mathcal{I}\text{-open} \arrow[r] \arrow[d] & \mathcal{I}\text{-neighborhood} \\
\mathcal{I}_{sn}\text{-neighborhood} & \mathcal{I}_{sn}\text{-open} \arrow[l] \arrow[r] & \mathcal{I}\text{-sequential neighborhood}
\end{tikzcd}$$

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In this paper, the entire investigation is performed in the arena of $\mathcal{I}^{\mathcal{K}}$-convergence extending the recent developments [11,13,1]. We introduce the notion of $\mathcal{I}^{\mathcal{K}}_{sn}$-open set which sits in the hierarchy relation as shown follows: open $\rightarrow$ $\mathcal{I}^{\mathcal{K}}_{sn}$-open $\rightarrow$ ($\mathcal{K}_{sn}$-open, $\mathcal{I}^{\mathcal{K}}$-open) $\rightarrow$ $\mathcal{K}$-open set. Several operators for a subset of a topological space are also discussed and it is also shown that the family of $\mathcal{I}^{\mathcal{K}}_{sn}$-open sets of a topological space forms a topology (Theorem 2.13). The notion of $\mathcal{I}^{\mathcal{K}}$-neighborhood space is defined and we discuss several basic properties of it. If f is a function defined on a space X, then the space is an $\mathcal{I}^{\mathcal{K}}$-neighborhood space if and only if every $\mathcal{I}^{\mathcal{K}}$-continuous map defined on it, is an $\mathcal{I}^{\mathcal{K}}_{sn}$-continuous map (Theorem 3.13). Moreover, we term a property of a topological space as "$\mathcal{I}^{\mathcal{K}}$-compatibility" and show that a topological space X satisfying above property gives a neccessary and sufficient condition for the coincidence of the properties "preserving $\mathcal{I}^{\mathcal{K}}$-convergence" and "$\mathcal{I}^{\mathcal{K}}$-continuity" for any mapping defined on X (Theorem 3.11).

Now, we recall some of the results and definitions required to discuss the underlying theory of ideal convergence. A proper ideal is a collection of subset of a set S (containing ϕ but not the whole set S) which is closed under finite union and subset inclusion. The notions of $\mathcal{I}^*$-convergence and $\mathcal{I}$-convergence, where $\mathcal{I}$ is an ideal on ω, were introduced in various spaces in [9,10,4]. One more notion, that is $\mathcal{I}^{\mathcal{K}}$-convergence, was also introduced in [5] which generalizes the above two modes of convergence. An equivalence between "$\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence" was shown in [10,4] as well as a characterization of the ideals $\mathcal{I}$ and $\mathcal{K}$ was obtained in [5] such that the notions $\mathcal{I}^{\mathcal{K}}$-convergence and $\mathcal{I}$-convergence coincide.

Definition 1.2. [5] *A function (generalized sequence) $f : S \rightarrow X$ is $\mathcal{I}^{\mathcal{M}}$-convergent [1] to an element $\xi \in X$ if there is an $M \in \mathcal{I}^*$ such that the function $g : S \rightarrow X$ given by*

$$g(s) = \begin{cases} f(s), & s \in M \\ \xi, & s \notin M \end{cases}$$

is $\mathcal{M}$-convergent to ξ, where $\mathcal{M}$ is a convergence mode via ideal.

If $\mathcal{M} = \mathcal{K}^*$, then $f : S \rightarrow X$ is said to be $\mathcal{I}^{\mathcal{K}^*}$-convergent [5] to an element $\xi \in X$. Also, if $\mathcal{M} = \mathcal{K}$, then $f : S \rightarrow X$ is said to be $\mathcal{I}^{\mathcal{K}}$-convergent [5] to an element $\xi \in X$.

Following results are mentioned in view of their application in the subsequent sections.

Lemma 1.3. [5, Lemma 3.5] *If $\mathcal{I}$ and $\mathcal{K}$ are two ideals on a set S and $f : S \rightarrow X$ is a function such that $\mathcal{K} - \lim f = a$, then $\mathcal{I}^{\mathcal{K}} - \lim f = a$.*

Proposition 1.4. [1, Proposition 2.1] *Let X be a topological space and $f : S \rightarrow X$ be a function. Let $\mathcal{I}, \mathcal{K}$ be two ideals on S such that $\mathcal{I} \cup \mathcal{K}$ is an ideal. Then*

(i) *$f(s) \rightarrow_{\mathcal{I}^{\mathcal{K}^*}} a$ if and only if $f(s) \rightarrow_{(\mathcal{I} \cup \mathcal{K})^*} a$.*

(ii) *$f(s) \rightarrow_{\mathcal{I}^{\mathcal{K}}} a$ implies $f(s) \rightarrow_{\mathcal{I} \cup \mathcal{K}} a$.*

2. Some results on $\mathcal{I}^{\mathcal{K}}_{sn}$-open sets

Let X be a topological space and $P \subset X$. Then P is an $\mathcal{I}$-sequential neighborhood [11] of a point x in X if for every sequence $\mathcal{I}$-converging to x is $\mathcal{I}$-eventually contained in the set P and then, P is said to be $\mathcal{I}_{sn}$-open [11] in X if P is $\mathcal{I}$-sequential neighborhood of each of its point. As a generalization of Lin's [11] study on $\mathcal{I}_{sn}$-open sets, we introduce $\mathcal{I}^{\mathcal{K}}_{sn}$-open sets as follows:

Definition 2.1. *Let P be a subset of a topological space X and let $\mathcal{I}$, $\mathcal{K}$ be two ideals of ω.*

(a) A sequence $a = \{a_n\}$ is said to be $\mathcal{I}$-eventually in P if there exists an $\mathcal{I}$-tail of $\{a_n\}$ which eventually belongs to P, i.e., $\{n \in \omega : a_n \in P\} \in \mathcal{I}^*$.

(b) P is said to be an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a point $a \in X$ if every sequence which is $\mathcal{I}^{\mathcal{K}}$-convergent to $a \in P$ has an $\mathcal{I}$-tail which is $\mathcal{K}$-eventually in P, i.e., there is an $M \in \mathcal{I}^*$ such that $\{n \in M : a_n \notin P\} \in \mathcal{K}$.

(c) P is said to be an $\mathcal{I}^{\mathcal{K}}_{sn}$-open set of X if P is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of each of its points.

We say that two ideals $\mathcal{I}$ and $\mathcal{J}$ on a set S satisfy the ideality condition if $\mathcal{I}\cup\mathcal{J}$ is a proper ideal [8], i.e., $S \neq I \cup J$, for all $I \in \mathcal{I}$, $J \in \mathcal{J}$. Unless mentioned otherwise, we refer X as a topological space and assume the ideality condition of the ideals $\mathcal{I}$ and $\mathcal{K}$ for $\mathcal{I}^{\mathcal{K}}$-convergence to investigate the notion of $\mathcal{I}^{\mathcal{K}}_{sn}$-open set.

Remark 2.2. *For a topological space X and ideals $\mathcal{I}$, $\mathcal{K}$ of ω,*

(i) If P is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of $a \in P$ then $\{n \in \omega : a_n \in P\} \in \mathcal{K}^*$, for any sequence $\{a_n\}$, $\mathcal{I}^{\mathcal{K}}$-convergent to a. So, it is clear that P is a $\mathcal{K}$-sequential neighborhood of $a \in P$.

(ii) Meanwhile if $\mathcal{K} \subseteq \mathcal{I}$, then the converse of the above result is also true. Assume $P \subset X$ and a sequence $\{a_n\}$ in X with $a_n \to_{\mathcal{K}} a$. Then $\{n \in \omega : a_n \in P\} \in \mathcal{K}^*$. Since $\mathcal{K} \subseteq \mathcal{I}$, we have $\{n \in \omega : a_n \in P\} \in \mathcal{I}^*$ i.e. $\{n \in \omega : a_n \notin P\} \in \mathcal{I}$. Consider $S = \{n \in \omega : a_n \in P\}$, then for the $\mathcal{I}$-tail $\{a_{n_k}\}_{n_k \in S}$, we have $\{n \in S : a_n \in P\} \in \mathcal{K}^*$.

Remark 2.2 suggests that the notions of $\mathcal{K}_{sn}$-open set and $\mathcal{I}^{\mathcal{K}}_{sn}$-open set coincide, if $\mathcal{K} \subseteq \mathcal{I}$. We now have the following question.

Question 2.3. *Is there any relation between the notions of $\mathcal{K}_{sn}$-open and $\mathcal{I}^{\mathcal{K}}_{sn}$-open, provided $\mathcal{I}\cup\mathcal{K}$ is an ideal ?*

Lemma 2.4. [1] *Let $\mathcal{M}_1, \mathcal{M}_2$ be two convergence modes in a topological space X such that $\mathcal{M}_1$-convergence implies $\mathcal{M}_2$-convergence. Then $O \subseteq X$ is $\mathcal{M}_2$-open implies that O is $\mathcal{M}_1$-open.*

Lemma 2.5. *Consider the following conditions for $O \subset X$ and for a sequence $\{a_n\}$ in X, provided the ideals $\mathcal{I}$ and $\mathcal{K}$ satisfy the ideality condition.*

(i) O is open in X.

(ii) O is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X.

(iii) O is $\mathcal{K}_{sn}$-open in X.

(iv) O is $\mathcal{K}$-open in X.

(v) $\{n \in \omega : a_n \in O\} \in \mathcal{K}^*$ with $\mathcal{I}^{\mathcal{K}} - \lim a_n = a \in O$.

(vi) O is an $\mathcal{I}^{\mathcal{K}}$-open in X.

(vii) $\{n \in \omega : a_n \in O\} \notin \mathcal{K}$ with $\mathcal{I}^{\mathcal{K}} - \lim a_n = a \in O$.

(viii) O is $(\mathcal{I}\cup\mathcal{K})_{sn}$-open in X.

(ix) O is $\mathcal{I}\cup\mathcal{K}$-open in X.

(x) O is sequentially open in X.

Then the following implications hold.

$$\begin{tikzcd}
 & \text{(viii)} \arrow[dr] & \text{(iv)} \arrow[ddr] & & \\
\textit{(i)} \arrow[ur] \arrow[r] \arrow[dr] & \text{(iii)} \arrow[ur] & \text{(ix)} \arrow[u] \arrow[d] & \text{(v)} \arrow[ul] \arrow[r] \arrow[dl] & \text{(x)} \\
 & \text{(ii)} \arrow[u] \arrow[urr] \arrow[r] & \text{(vi)} \arrow[r, leftrightarrow] & \text{(vii)} \arrow[ur] &
\end{tikzcd}$$

Proof. Some trivial proofs are ommited and the rest follows as given below.

(i) $\implies$ (ii). Suppose O is open in X and the sequence $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in X$. There exists $M \in \mathcal{I}^*$ such that $\{n \in M : a_n \notin O\} \in \mathcal{K}$. That is $\{a_n\}$ has an $\mathcal{I}$-tail $\{a_n\}_{n\in M}$ which is $\mathcal{K}$-eventually in O. Thus, O is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open set of X.

(ii) $\implies$ (iii). Suppose that O is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open set of X. Consider a sequence $\{a_n\}$ such that $a_n \to_{\mathcal{K}} a \in X$. Then $a_n \to_{\mathcal{I}^{\mathcal{K}}} a$. So, there exists a set $M \in \mathcal{I}^*$ such that $\{n \in M : a_n \notin O\} \in \mathcal{K}$. Therefore, $\{n \in M : a_n \in O\} \in \mathcal{K}^*$. Again, $\{n \in M : a_n \in O\} \subset \{n \in \omega : a_n \in O\}$. Hence, $\{n \in \omega : a_n \in O\} \in \mathcal{K}^*$. Thus, O is $\mathcal{K}_{sn}$-open in X.

(iii) $\implies$ (iv). It follows immediately by Lemma 2.1 of [11].

(iv) $\implies$ (vii). It follows immediately by Lemma 3.6 of [13].

(vii) $\implies$ (x). Since every convergent sequence in a topological space X is $\mathcal{I}^{\mathcal{K}}$-convergent, so, by definition, every $\mathcal{I}^{\mathcal{K}}$-closed set in X is sequentially closed and therefore, every $\mathcal{I}^{\mathcal{K}}$-open set in X is sequentially open.

(i) $\implies$ (viii) and (viii) $\implies$ (ix). As $\mathcal{I}$ and $\mathcal{K}$ satisfy ideality condition, so, $\mathcal{I} \cup \mathcal{K}$ is an ideal. So, by Lemma 2.1 of [11], O is $(\mathcal{I} \cup \mathcal{K})_{sn}$-open in X. Thus, O is $\mathcal{I} \cup \mathcal{K}$-open in X.

(ix) $\implies$ (vi) and (vi) $\implies$ (vii). By above Lemma 2.4 and Corollary 2.4 of [2], results follows directly. □

Lemma 2.6. *Let X be a topological space. If a sequence $\{a_n\}$ in X is $\mathcal{I}^{\mathcal{K}}$-convergent to $\xi \in X$ and a sequence $\{b_n\}$ is $\mathcal{K}$-equivalent to $\{a_n\}$, i.e., $\{n \in \omega : a_n \neq b_n\} \in \mathcal{K}$, then $\{b_n\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to $\xi \in X$.*

Proof. Suppose that $a_n \to_{\mathcal{I}^{\mathcal{K}}} \xi$. Then there exists $M \in \mathcal{I}^*$ such that $\{a_n\}_{n\in M}$ is $\mathcal{K}$-convergent to ξ. For any open set O containing ξ, $\{n \in M : a_n \notin O\} \in \mathcal{K}$. Since $\{n \in \omega : a_n \neq b_n\} \in \mathcal{K}$, therefore $\{n \in M : b_n \notin O\} \in \mathcal{K}$. That is $b_n \to_{\mathcal{I}^{\mathcal{K}}} \xi$. □

Proposition 2.7. *The following statements hold in a topological space X.*

(i) If $Y \subset X$ and P is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X, then $P \cap Y$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in the subspace Y. Similar result also holds for $\mathcal{I}^{\mathcal{K}}$-open, $\mathcal{I}^{\mathcal{K}}_{sn}$-closed and $\mathcal{I}^{\mathcal{K}}$-closed subsets of X.

(ii) If Y is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X and P is $\mathcal{I}^{\mathcal{K}}$-open ($\mathcal{I}^{\mathcal{K}}_{sn}$-open) in the subspace Y, P is $\mathcal{I}^{\mathcal{K}}$-open ($\mathcal{I}^{\mathcal{K}}_{sn}$-open) in X.

(iii) If Y is $\mathcal{I}^{\mathcal{K}}$-closed in X and P is $\mathcal{I}^{\mathcal{K}}$-closed in Y, then P is $\mathcal{I}^{\mathcal{K}}$-closed in X.

Proof. Let $Y \subset X$ is a subspace of a topological space X and $O \subset X$.

(i) Let $\{a_n\}$ be an $\mathcal{I}^{\mathcal{K}}$-convergent sequence in Y such that $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in Y$ and let P be an open set in X containing a. Then there exists a set $M \in \mathcal{I}^*$ such that $\{n \in M : a_n \notin P\} \in \mathcal{K}$. Again, $\{n \in M : a_n \notin P\} = \{n \in \omega : a_n \notin P \cap Y\} \in \mathcal{K}$. Therefore, the sequence $\{a_n\}$ in X has an $\mathcal{I}$-tail which is $\mathcal{K}$-convergent to $a \in P$. Thus, the sequence $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in X$.

Suppose that P is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of X. Let $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in P \cap Y$, then $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in X$. So, there exists $M \in \mathcal{I}^*$ such that $\{n \in M : a_n \in P\} = \{n \in M : a_n \in P \cap Y\} \in \mathcal{K}^*$. That is $\{a_n\}$ has an $\mathcal{I}$-tail which is $\mathcal{K}$-eventually in $P \cap Y$. Hence, $P \cap Y$ is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of Y.

Suppose that P is an $\mathcal{I}^{\mathcal{K}}$-open subset of X and let $\{a_n\}$ be a sequence in Y which is $\mathcal{I}^{\mathcal{K}}$-convergent to $a \in P \cap Y$. Then, $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in X$. Then there exists $M \in \mathcal{I}^*$ such that $\{n \in \omega : a_n \in P\} \notin \mathcal{K}$. That implies $\{n \in \omega : a_n \in P \cap Y\} \notin \mathcal{K}$. So, by Lemma 2.5, $P \cap Y$ is an $\mathcal{I}^{\mathcal{K}}$-open subset of Y.

(ii) Suppose that Y is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X. Consider a sequence $\{a_n\}$ in X such that $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in P$. Since $a \in Y$, we observe that $\{n \in \omega : a_n \notin Y\} \in \mathcal{K}$. Now, for $A = \{n \in \omega : a_n \notin Y\}$, consider a sequence $\{b_n\}$ defined as $b_n = a_n$, if $n \in A^{\complement}$ and $b_n = a$, otherwise. Since, $\{b_n\}$ is $\mathcal{K}$-equivalent to $\{a_n\}$, so, by Lemma 2.6, $b_n \to_{\mathcal{I}^{\mathcal{K}}} a \in P$.

Again, if P is an $\mathcal{I}^{\mathcal{K}}$-open subset of Y, that implies $\{n \in \omega : a_n \in P\} \supset \{n \in \omega : b_n \in P\} \notin \mathcal{K}$. Hence, P is an $\mathcal{I}^{\mathcal{K}}$-open subset of X.

Similarly, if P is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of Y, then $\{n \in \omega : a_n \in P\} \supset \{n \in \omega : b_n \in P\} \in \mathcal{K}^*$. Thus, P is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of X.

(iii) Suppose that Y is $\mathcal{I}^{\mathcal{K}}$-closed in X and P is $\mathcal{I}^{\mathcal{K}}$-closed in Y. Let $\{a_n\}$ be a sequence in P with $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in X$. Since, the set Y is an $\mathcal{I}^{\mathcal{K}}$-closed subset of X, so $a \in Y$. Again, P is an $\mathcal{I}^{\mathcal{K}}$-closed subset of Y, so, $a \in P$. Hence, P is an $\mathcal{I}^{\mathcal{K}}$-closed subset of X.

□

Now, for $\mathcal{K} = Fin$, the notions of $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a point x, $\mathcal{I}^{\mathcal{K}}$-open, $\mathcal{I}^{\mathcal{K}}_{sn}$-open set, $\mathcal{I}^{\mathcal{K}}$-closed and $\mathcal{I}^{\mathcal{K}}_{sn}$-closed set are termed as $\mathcal{I}^*$-sequential neighborhood of a point x, $\mathcal{I}^*$-open, $\mathcal{I}^*_{sn}$-open set, $\mathcal{I}^*$-closed and $\mathcal{I}^*_{sn}$-closed set respectively. Again, the convergent modes $\mathcal{I}^{\mathcal{K}^*}$ and $(\mathcal{I} \cup \mathcal{K})^*$ are equivalent [1], so the notions $\mathcal{I}^{\mathcal{K}^*}_{sn}$-sequential neighborhood and $(\mathcal{I} \cup \mathcal{K})^*_{sn}$-neighborhood of a point x coincides, provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.

Remark 2.8. *The notions of $\mathcal{I}^{\mathcal{K}^*}_{sn}$-open sets and $(\mathcal{I} \cup \mathcal{K})^*_{sn}$-open sets in a topological space coincide, provided $\mathcal{I} \cup \mathcal{K}$ is an ideal.*

A significant difficulty in the theory of $\mathcal{I}$-convergence is that the family of all $\mathcal{I}$-open subsets of a topological space may not form a topology. It actually corresponds to a "generalised topology" (closed under arbitrary union but may fail to be closed under intersections). This inadequateness further translates to the context of $\mathcal{I}^{\mathcal{K}}$-open subsets in a topological space. But, if the ideal $\mathcal{J}$ is assumed to be a maximal ideal, then the family of all the $\mathcal{I}^{\mathcal{J}}$-open subsets of a topological space forms a topology ([2], Theorem 2.8 and 2.9). Here, we obtain that the family of all the $\mathcal{I}^{\mathcal{K}}_{sn}$-open subsets in a topological space forms a topology and hereby extend the result by Lin [11] for $\mathcal{I}_{sn}$-open sets. To establish that first let's discuss some notations and results that are used in the subsequent segments.

Lin et. al. introduced the concept of G-hull and G-kernel [12] for a given method G on a set X. In a space X, the corresponding notions $\mathcal{I}$-hull and $\mathcal{I}$-kernel of a subset P is defined as the set $[P]_{\mathcal{I}} = \{x \in X$: there exists a sequence $\{a_n\}$ in P with $a_n \to_{\mathcal{I}} a\}$ and $(P)_{\mathcal{I}} = \{a \in X :$ there exists no sequence in $X \setminus P$ with $a_n \to_{\mathcal{I}} a\}$ respectively.

The $\mathcal{I}^{\mathcal{K}}$-hull and $\mathcal{I}^{\mathcal{K}}$-kernel of a subset $P \subset X$ are denoted by $[P]_{\mathcal{I}^{\mathcal{K}}}$ and $(P)_{\mathcal{I}^{\mathcal{K}}}$ respectively and are defined as follows:

$$\begin{aligned} [P]_{\mathcal{I}^{\mathcal{K}}} &= \{a \in X : \text{there exists a sequence } \{a_n\} \text{ in } P \text{ with } a_n \to_{\mathcal{I}^{\mathcal{K}}} a\}. \\ (P)_{\mathcal{I}^{\mathcal{K}}} &= \{a \in X : \text{there is no sequence } \{a_n\} \text{ in } X \setminus P,\ \mathcal{I}^{\mathcal{K}}\text{-converges to } a\}. \end{aligned} \tag{2.1}$$

It is easy to check from Lemma 1.3 that the set inequalities $[P]_{\mathcal{K}} \subset [P]_{\mathcal{I}^{\mathcal{K}}}$ and $(P)_{\mathcal{I}^{\mathcal{K}}} \subset (P)_{\mathcal{K}}$ hold. Again, if the ideals $\mathcal{I}$, $\mathcal{K}$ satisfy ideality condition, then Proposition 1.4 suggest that $[P]_{\mathcal{I}^{\mathcal{K}}} \subset [P]_{\mathcal{I} \cup \mathcal{K}}$ and $(P)_{\mathcal{I} \cup \mathcal{K}} \subset (P)_{\mathcal{I}^{\mathcal{K}}}$. Therefore, for two given ideals $\mathcal{I}$, $\mathcal{K}$ on ω, the Lemma 2.6 [11], Lemma 1.3 and Proposition 1.4 lead to the following set inequality for a subset P in a topological space X.

$$P^o \subset (P)_{\mathcal{I} \cup \mathcal{K}} \subset (P)_{\mathcal{I}^{\mathcal{K}}} \subset (P)_{\mathcal{K}} \subset P \subset [P]_{\mathcal{K}} \subset [P]_{\mathcal{I}^{\mathcal{K}}} \subset [P]_{\mathcal{I} \cup \mathcal{K}} \subset \bar{P}. \tag{2.2}$$

Again, by Theorem 3.5 [12], we observe that the result $[P]_{\mathcal{I}^{\mathcal{K}}} = X \setminus (X \setminus P)_{\mathcal{I}^{\mathcal{K}}}$ holds for a subset P in a space X. On the other hand, two operators $\mathcal{I}_{sn}$-interior and $\mathcal{I}_{sn}$-closure were also introduced by Lin [11] in a topological space. For $P \subset X$, the sets defined as $(P)_{\mathcal{I}_{sn}} = \{a \in X : P$ is $\mathcal{I}$-sequential neighborhood of $a\}$ and $\{a \in X$: if O is an $\mathcal{I}$-sequential neighborhood of a, then $O \cap P \neq \phi\}$ respectively, represents the $\mathcal{I}_{sn}$-interior and $\mathcal{I}_{sn}$-closure of P in X.

In this segment, following the line of work [1], we study the operators $\mathcal{I}^{\mathcal{K}}_{sn}$-interior and $\mathcal{I}^{\mathcal{K}}_{sn}$-closure of a subset P in a topological space X.

Definition 2.9. *Let X be a topological space and $P \subset X$. Let $\mathcal{I}$, $\mathcal{K}$ be two ideals on ω. Then, the operators $\mathcal{I}^{\mathcal{K}}_{sn}$-closure and $\mathcal{I}^{\mathcal{K}}_{sn}$-interior of P are defined as follows:*

$$[P]_{\mathcal{I}^{\mathcal{K}}_{sn}} = \{a \in X : \text{ If } O \text{ is an } \mathcal{I}^{\mathcal{K}}\text{-sequential neighborhood of } a \text{ , then } O \cap P \neq \phi\}.$$
$$(P)_{\mathcal{I}^{\mathcal{K}}_{sn}} = \{a \in X : \ P \text{ is an } \mathcal{I}^{\mathcal{K}}\text{-sequential neighborhood of } a\}.$$

Again, we observe that a set P is an $\mathcal{I}^{\mathcal{K}}_{sn}$-closed subset in X if and only if $P = [P]_{\mathcal{I}^{\mathcal{K}}_{sn}}$ and also, a set P is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of X if and only if $P = (P)_{\mathcal{I}^{\mathcal{K}}_{sn}}$.

For a given subset P in a topological space X, we have $[P]_{\mathcal{I}^{\mathcal{K}}_{sn}} = X \setminus (X \setminus P)_{\mathcal{I}^{\mathcal{K}}_{sn}}$: consider, $a \in (X \setminus P)_{\mathcal{I}^{\mathcal{K}}_{sn}}$, that means $X \setminus P$ is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a. But $(X \setminus P) \cap P = \phi$, thus, $a \notin [P]_{\mathcal{I}^{\mathcal{K}}_{sn}}$. Therefore, $[P]_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset X \setminus (X \setminus P)_{\mathcal{I}^{\mathcal{K}}_{sn}}$. Conversely, let $a \notin [P]_{\mathcal{I}^{\mathcal{K}}_{sn}}$. Then there exists an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a such that $O \cap P = \phi$. Therefore, $O \subset X \setminus P$ i.e. $(X \setminus P)$ is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a. So, $a \in (X \setminus P)_{\mathcal{I}^{\mathcal{K}}_{sn}} \implies a \notin X \setminus (X \setminus P)_{\mathcal{I}^{\mathcal{K}}_{sn}}$. Thus, $X \setminus (X \setminus P)_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset [P]_{\mathcal{I}^{\mathcal{K}}_{sn}}$.

Lemma 2.10. *Let X be a topological space and $\mathcal{I}$, $\mathcal{K}$ be two ideals of ω. Then, for $P \subset X$, we obtain*

(i) $[P]_{\mathcal{K}_{sn}} \subset [P]_{\mathcal{I}^{\mathcal{K}}_{sn}}$ and also, $(P)_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset (P)_{\mathcal{K}_{sn}}$.

(ii) $[P]_{\mathcal{I}^{\mathcal{K}}} \subset [P]_{\mathcal{I}^{\mathcal{K}}_{sn}}$ and also, $(P)_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset (P)_{\mathcal{I}^{\mathcal{K}}}$.

Proof. Consider $P \subset X$, where X is a topological space. Then

(i) This result follows from Lemma 2.5 and Remark 2.2.

(ii) Consider $a \in [P]_{\mathcal{I}^{\mathcal{K}}}$, then there exists a sequence $\{a_n\}$ in P such that $a_n \to_{\mathcal{I}^{\mathcal{K}}} a$. Let O be an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a, then $\{n \in \omega : a_n \in O\} \in \mathcal{K}^*$. Again, $\{n \in \omega : a_n \in O\} \subset P \cap O (\neq \phi)$. Hence, $a \in [P]_{\mathcal{I}^{\mathcal{K}}_{sn}}$.

Suppose that $a \in (P)_{\mathcal{I}^{\mathcal{K}}_{sn}}$, then P is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a. If possible, consider a sequence $\{a_n\}$ in $X \setminus P$ with $a_n \to_{\mathcal{I}^{\mathcal{K}}} a$. But then $\{n \in \omega : a_n \in P\} = \phi$, which is a contradiction to the fact that P is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a. Hence, $a \in (P)_{\mathcal{I}^{\mathcal{K}}}$.

□

The following inequalities represent the hierarchy relation of different operators on a set P in the topological space X.

$$P^o \subset (P)_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset (P)_{\mathcal{K}_{sn}} \subset (P)_{\mathcal{K}} \subset P \subset [P]_{\mathcal{K}} \subset (P)_{\mathcal{I}^{\mathcal{K}}} \subset [P]_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset [P]_{\mathcal{K}_{sn}} \subset \bar{P}. \tag{2.3}$$

From the Lemma 2.10(ii) and Inequality 2.3, we may now ask the following question:

Question 2.11. *For $P \subset X$, what is the relation between the operators $(P)_{\mathcal{I}^{\mathcal{K}}}$ and $(P)_{\mathcal{K}_{sn}}$ in an arbitrary topological space X?*

Lemma 2.12. *Let X be a topological space and P, $Q \subset X$. Then $(P \cap Q)_{\mathcal{I}^{\mathcal{K}}_{sn}} = (P)_{\mathcal{I}^{\mathcal{K}}_{sn}} \cap (Q)_{\mathcal{I}^{\mathcal{K}}_{sn}}$ and also, for a family of subsets $(P_\lambda)_{\lambda \in \Lambda}$ in X, the result $[\bigcup_{\lambda \in \Lambda} P_\lambda]_{\mathcal{I}^{\mathcal{K}}_{sn}} = \bigcup_{\lambda \in \Lambda} [P_\lambda]_{\mathcal{I}^{\mathcal{K}}_{sn}}$ holds.*

Proof. Suppose that $a \in (P \cap Q)_{\mathcal{I}^{\mathcal{K}}_{sn}}$, then $P \cap Q$ is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a. So, P and Q are $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a, that means $a \in (P)_{\mathcal{I}^{\mathcal{K}}_{sn}} \cap (Q)_{\mathcal{I}^{\mathcal{K}}_{sn}}$. Hence, $(P \cap Q)_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset (P)_{\mathcal{I}^{\mathcal{K}}_{sn}} \cap (Q)_{\mathcal{I}^{\mathcal{K}}_{sn}}$. For converse part, assume that $a \in (P)_{\mathcal{I}^{\mathcal{K}}_{sn}} \cap (Q)_{\mathcal{I}^{\mathcal{K}}_{sn}}$. So, it is clear that P, Q are $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a. Then there exists $M_1, M_2 \in \mathcal{I}^*$ such that the sets $\{n \in M_1 : a_n \notin P\} \in \mathcal{K}$ and $\{n \in M_2 : a_n \notin Q\} \in \mathcal{K}$. Consider $M = M_1 \cap M_2 \in \mathcal{I}^*$ such that the respective sets $\{n \in M : a_n \notin P\} \in \mathcal{K}$ and $\{n \in M : a_n \notin Q\} \in \mathcal{K}$. Again, $\{n \in M : a_n \notin P \cap Q\} = \{n \in M : a_n \notin Q\} \cup \{n \in M : a_n \notin Q\} \in \mathcal{K}$. Clearly, $P \cap Q$ is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a. So, $a \in (P \cap Q)_{\mathcal{I}^{\mathcal{K}}_{sn}}$. Hence, $(P)_{\mathcal{I}^{\mathcal{K}}_{sn}} \cap (Q)_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset (P \cap Q)_{\mathcal{I}^{\mathcal{K}}_{sn}}$.

For $[\bigcup_{\lambda \in \Lambda} P_\lambda]_{\mathcal{I}^{\mathcal{K}}_{sn}} = \bigcup_{\lambda \in \Lambda} [P_\lambda]_{\mathcal{I}^{\mathcal{K}}_{sn}}$, suppose that $a \in [\bigcup_{\lambda \in \Lambda} P_\lambda]_{\mathcal{I}^{\mathcal{K}}_{sn}}$ and O be an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a. Then $O \cap (\bigcup_{\lambda \in \Lambda} P_\lambda) \neq \phi \iff O \cap P_\lambda \neq \phi$, for some $\lambda \in \Lambda \iff a \in \bigcup_{\lambda \in \Lambda} [P_\lambda]_{\mathcal{I}^{\mathcal{K}}_{sn}}$. □

Now, we state our main conclusion as following result.

Proposition 2.13. *Let X be a topological space and $\mathcal{I}$, $\mathcal{K}$ be two ideals. Then*

(1) Arbitrary union of $\mathcal{I}^{\mathcal{K}}_{sn}$-open subsets of X is $\mathcal{I}^{\mathcal{K}}_{sn}$-open.

(2) Finite intersection of $\mathcal{I}^{\mathcal{K}}_{sn}$-open subsets of X is $\mathcal{I}^{\mathcal{K}}_{sn}$-open.

Proof. Consider a topological space X and the ideals $\mathcal{I}$, $\mathcal{K}$ on ω.

(1) Let $\{P_\lambda\}_{\lambda\in\Lambda}$ be a family of $\mathcal{I}^{\mathcal{K}}_{sn}$-open subsets in X. Then, using Lemma 2.12 and Equation 2.3, it is easy to check that

$$(\bigcup_{\lambda\in\Lambda} P_\lambda)_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset (\bigcup_{\lambda\in\Lambda} P_\lambda)_{\mathcal{K}_{sn}} \subset \bigcup_{\lambda\in\Lambda} P_\lambda = \bigcup_{\lambda\in\Lambda} (P_\lambda)_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset (\bigcup_{\lambda\in\Lambda} P_\lambda)_{\mathcal{I}^{\mathcal{K}}_{sn}}.$$

Hence, $(\bigcup_{\lambda\in\Lambda} P_\lambda)_{\mathcal{I}^{\mathcal{K}}_{sn}} = \bigcup_{\lambda\in\Lambda} P_\lambda$. Thus, $(\bigcup_{\lambda\in\Lambda} P_\lambda)_{\mathcal{I}^{\mathcal{K}}_{sn}}$ is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subsets in X.

(2) Let P, Q be two subset $\mathcal{I}^{\mathcal{K}}_{sn}$-open subsets in X. Then, by Lemma 2.12, $(P\cap Q)_{\mathcal{I}^{\mathcal{K}}_{sn}} = P_{\mathcal{I}^{\mathcal{K}}_{sn}} \cap Q_{\mathcal{I}^{\mathcal{K}}_{sn}} = P\cap Q$. Hence, $P\cap Q$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X.

□

Remark 2.14. *Let X be an ideal topological space $(X,\tau,\ \mathcal{I},\ \mathcal{K})$ where $\mathcal{I}$ and $\mathcal{K}$ be two ideals on ω. Then*

(i) The collection of subsets of X, we defined as

$$\tau_{\mathcal{I}^{\mathcal{K}}_{sn}} = \{P \subset X : P = (P)_{\mathcal{I}^{\mathcal{K}}_{sn}}\}$$

is the $\mathcal{I}^{\mathcal{K}}_{sn}$-open topology obtained from the topology τ and the ideals $\mathcal{I}$, $\mathcal{K}$. Also, the space $(X,\tau_{\mathcal{I}^{\mathcal{K}}_{sn}})$ is said to be an $\mathcal{I}^{\mathcal{K}}_{sn}$-topological space.

(ii) The space $(X,\tau_{\mathcal{I}^{\mathcal{K}}_{sn}})$ is the $\mathcal{I}^{\mathcal{K}}$-sequential coreflection of X. In particular, if $\mathcal{I} = \mathcal{K} = Fin$, then $(X,\tau_{\mathcal{I}^{\mathcal{K}}_{sn}})$ coincides with sX [13], the sequential coreflection of X.

3. $\mathcal{I}^{\mathcal{K}}$-neighborhood space and $\mathcal{I}^{\mathcal{K}}$-compatible spaces

In general, the family of $\mathcal{I}^{\mathcal{K}}$-open sets in a topological space X may not represent a sub-base for a topology (Counter example if possible). But the collection $S_{\mathcal{I}^{\mathcal{K}}} = \{\bigcap_{i=1}^{n} O_i : O_i$ is a $\mathcal{I}^{\mathcal{K}}$-open subset of X$\}$, serves as a sub-base for a topology, say $\tau_{\mathcal{I}^{\mathcal{K}}}$. We say $\tau_{\mathcal{I}^{\mathcal{K}}}$ is the topology generated by all the $\mathcal{I}^{\mathcal{K}}$-open sets in X.

By Lemma 2.5, the following set inequalities can be easily deduced.

$$\tau \subset \tau_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset \tau_{\mathcal{K}_{sn}}, \tau_{\mathcal{I}^{\mathcal{K}}} \subset \tau_{\mathcal{K}}$$

In the above inequalities, we may observe that the collections $\tau_{\mathcal{K}_{sn}}$ and $\tau_{\mathcal{I}^{\mathcal{K}}}$ are seemingly unrelatable which is left open for further study.

Proposition 3.1. *Let $(X,\tau,\mathcal{I},\mathcal{K})$ be an ideal topological space and μ be a finer topology of X such that μ contains each $\mathcal{I}^{\mathcal{K}}$-open subset of X. Then, the following conditions are equivalent.*

(i) (X,τ) and (X,μ) have the same $\mathcal{I}^{\mathcal{K}}$-convergent sequences,

(ii) μ contains only the $\mathcal{I}^{\mathcal{K}}_{sn}$-open sets i.e. $\mu = \tau_{\mathcal{I}^{\mathcal{K}}_{sn}}$.

Assumption of one of the above implies that (X,μ) is $\mathcal{I}^{\mathcal{K}}$-sequential space.

Proof. (i) $\iff$ (ii). Suppose that both (X,τ) and (X,μ) have the same $\mathcal{I}^{\mathcal{K}}$-convergent sequences. Then, it is easy to check that (X,τ) and (X,μ) have the same $\mathcal{I}^{\mathcal{K}}_{sn}$-open subsets. We claim that $\mu = \tau_{\mathcal{I}^{\mathcal{K}}_{sn}}$. Since the family μ contains each $\mathcal{I}^{\mathcal{K}}$-open subset of (X,τ), so by Lemma 2.5, $\tau_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset \mu$. For the reverse inequality, consider $P \in \mu$, then by Lemma 2.5, P is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of (X,μ). Therefore, P is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of (X,τ) and thus, $P \in \tau_{\mathcal{I}^{\mathcal{K}}_{sn}}$.

Let us assume that $\mu = \tau_{\mathcal{I}^{\mathcal{K}}_{sn}}$. Since, $\tau \subset \tau_{\mathcal{I}^{\mathcal{K}}_{sn}}$, so, if $\{a_n\}$ be a sequence in X such that $a_n \to_{\mathcal{I}^{\mathcal{K}}} a$ in $(X,\tau_{\mathcal{I}^{\mathcal{K}}_{sn}})$, then $a_n \to_{\mathcal{I}^{\mathcal{K}}} a$ in (X,τ). For converse result, assume that $a_n \to_{\mathcal{I}^{\mathcal{K}}} a$ in (X,τ) and let $a \in P \in \tau_{\mathcal{I}^{\mathcal{K}}_{sn}}$. Then P is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a. Therefore, there exists a subset $M \in \mathcal{I}^*$ such that $\{n \in \omega : a_n \notin P\} \in \mathcal{K}$. This implies $a_n \to_{\mathcal{I}^{\mathcal{K}}} a$ in $(X,\tau_{\mathcal{I}^{\mathcal{K}}_{sn}})$.

Consider that (X,τ) and (X,μ) have the same $\mathcal{I}^{\mathcal{K}}$-convergent sequence. So, their corresponding $\mathcal{I}^{\mathcal{K}}$-open subsets are also same i.e. $\tau_{\mathcal{I}^{\mathcal{K}}} = \mu_{\mathcal{I}^{\mathcal{K}}}$. By our assumption, $\tau_{\mathcal{I}^{\mathcal{K}}} \subset \mu \subset \mu_{\mathcal{I}^{\mathcal{K}}} = \tau_{\mathcal{I}^{\mathcal{K}}}$ i.e. $\mu = \mu_{\mathcal{I}^{\mathcal{K}}}$. Thus, (X,μ) is $\mathcal{I}^{\mathcal{K}}$-sequential. □

Definition 3.2. *Let $(X,\tau,\mathcal{I},\mathcal{K})$ be an ideal topological space. Then*

(i) X is said to be an $\mathcal{I}^{\mathcal{K}}$-neighborhood space if a subset P of X is $\mathcal{I}^{\mathcal{K}}$-open if and only if $P = (P)_{\mathcal{I}^{\mathcal{K}}_{sn}}$.

(ii) X is said to be possesses the property "$\mathcal{I}^{\mathcal{K}}$-compatibility", if any set P is open whenever it is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X.

The term "$\mathcal{I}^{\mathcal{K}}$-compatiblility" describes the ingenuity of the notions like $\mathcal{I}_{sn}$-open and $\mathcal{I}^{\mathcal{K}}_{sn}$-open which is inclined towards the notion of open sets by virtue of their definitions. In a space, coincidence of the class of $\mathcal{I}^{\mathcal{K}}_{sn}$-open subsets with the topology explains the compatibility of the space with the ideals $\mathcal{I}$ and $\mathcal{K}$.

Since, $P = (P)_{\mathcal{I}^{\mathcal{K}}_{sn}}$ if and only if P is $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of X. In that way, X is an $\mathcal{I}^{\mathcal{K}}$-neighborhood space if $\tau_{\mathcal{I}^{\mathcal{K}}_{sn}} = \tau_{\mathcal{I}^{\mathcal{K}}}$. Again, for a topological space (X,τ), if $\tau_{\mathcal{I}^{\mathcal{K}}} = \tau$, then X is an $\mathcal{I}^{\mathcal{K}}$-sequential space. Summarily, we get that X is an $\mathcal{I}^{\mathcal{K}}$-sequential space if and only if X is an $\mathcal{I}^{\mathcal{K}}$-neighborhood space and possesses the property "$\mathcal{I}^{\mathcal{K}}$-compatibility".

Lemma 3.3. *Suppose that (X,τ) is a topological space. Then*

(i) X is an $\mathcal{I}^{\mathcal{K}}$-sequential space if and only if any $\mathcal{I}^{\mathcal{K}}$-neighborhood of each point is a neighborhood of the point in X.

(ii) X is an $\mathcal{I}^{\mathcal{K}}$-neighborhood space if and only if $\mathcal{I}^{\mathcal{K}}$-neighborhood of each point is an $\mathcal{I}^{\mathcal{K}}_{sn}$-neighborhood of the point in X.

Theorem 3.4. *Every first countable space possesses the property "$\mathcal{I}^{\mathcal{K}}$-compatibility".*

Proof. Let X be a first countable topological space and $P \subset X$ is not open in X. Then there exists $a \in P$ such that each neighborhood of a has non empty intersection with $X \setminus P$. Consider a countable basis $\{O_i : i \in \omega\}$ of a. Then for each $n \in \omega$ choose $a_n \in (X \setminus P) \cap (\bigcap_{i=1}^{n} O_i)$. Then, the sequence $\{a_n\}$ is $\mathcal{K}$-convergent to a. That implies $a_n \to_{\mathcal{I}^{\mathcal{K}}} a$. In essence, P is not $\mathcal{I}^{\mathcal{K}}$-open in X. □

Proposition 3.5. *Quotient spaces of a space possessing the property "$\mathcal{I}^{\mathcal{K}}$-compatibility" also possesses the same.*

Proof. Consider a quotient space $(X/\sim,\tau_q)$, where $X/\sim$ be the set of equivalence classes with the equivalence relation $\sim$ in the topological space X and the topology τ_q comprises of the set $P \subset X/\sim$; P is open if and only if $q^{-1}(P)$ is open in X where $q : X \to X/\sim$ is the projection mapping.

Suppose that $P \subset X/\sim$ is not open. Then $q^{-1}(P)$ is not open in X. Then, there exists a sequence $\{a_n\}$ in $X \setminus q^{-1}(P)$, $\mathcal{I}^{\mathcal{K}}$-converging to $a \in q^{-1}(P)$. Since q is continuous, $\{q(a_n)\}$ in $Y \setminus P$ is $\mathcal{I}^{\mathcal{K}}$-convergent to $q(a) \in P$. Thus, P is not $\mathcal{I}^{\mathcal{K}}_{sn}$-open in $(X,\sim)$.

□

Proposition 3.6. *If X locally possesses the property of coincidence of $\mathcal{I}^{\mathcal{K}}_{sn}$-open set with open set, then X possesses the property "$\mathcal{I}^{\mathcal{K}}$-compatibility".*

Proof. Suppose that X locally possesses the property of coincidence of $\mathcal{I}^{\mathcal{K}}_{sn}$-open set with open set. Then, without loss of generality, for each $x \in X$, there exists an open set V containing x such that the subspace topologies $\tau_{\mathcal{I}^{\mathcal{K}}_{sn}(V)}$ and $\tau_{(V)}$ in V coincides. We claim that X possesses the property "$\mathcal{I}^{\mathcal{K}}$-compatibility". Consider O is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of X and that way, $O \cap V$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-open set in V. So, $O \cap V$ is open set in V and therefore, O is open in X. Thus, X possesses the property "$\mathcal{I}^{\mathcal{K}}$-compatibility". □

Proposition 3.7. *If a topological space X possesses the property "$\mathcal{I}^{\mathcal{K}}$-compatibility", then the $\mathcal{I}^{\mathcal{K}}_{sn}$-open subspaces of X also possesses the same.*

Proof. Let Y be an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subspace of a topological space X. Consider an $\mathcal{I}^{\mathcal{K}}_{sn}$-open set O in Y, we claim that O is open in Y. Then by Proposition 2.7(ii), O is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X. Therefore, O is open in X which implies O is open in Y. Thus, the $\mathcal{I}^{\mathcal{K}}_{sn}$-open and open sets coincides in the subspace Y. □

Theorem 3.8. *A topological space (X, τ), (a) being $\mathcal{I}^{\mathcal{K}}$-neighborhood space or (b) possessing "$\mathcal{I}^{\mathcal{K}}$-compatibility" are*

(i) hereditary with respect to $\mathcal{I}^{\mathcal{K}}$-open ($\mathcal{I}^{\mathcal{K}}$-closed) subspaces

(ii) and preserved by disjoint topological sums.

Proof. (i) (a) Suppose that Y is $\mathcal{I}^{\mathcal{K}}$-open in X and Let P is $\mathcal{I}^{\mathcal{K}}$-open in Y. By Lemma 2.5, Y being an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset in X, Proposition 2.7 implies that P is $\mathcal{I}^{\mathcal{K}}$-open in X. Further, P is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of X. Thus, $P = P \cap Y$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in Y.

Let Y be $\mathcal{I}^{\mathcal{K}}$-closed in X and let P be $\mathcal{I}^{\mathcal{K}}$-closed in Y. Then, by Proposition 2.7 P is $\mathcal{I}^{\mathcal{K}}$-closed in X. Then, Lemma 2.5 suggest that P is $\mathcal{I}^{\mathcal{K}}_{sn}$-closed in X. Thus, by Proposition 2.7 P is $\mathcal{I}^{\mathcal{K}}_{sn}$-closed in Y.

(b) Suppose that Y be an $\mathcal{I}^{\mathcal{K}}$-open subspace of X and consider P be an $\mathcal{I}^{\mathcal{K}}_{sn}$-open in Y. Then, Y is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subspace of X. So, by Proposition 2.13, $P = P \cap Y$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X. That implies P is open in X. Hence, by Proposition 2.7, $P = P \cap Y$ is open in Y.

Consider that Y be $\mathcal{I}^{\mathcal{K}}$-closed in X and F be an $\mathcal{I}^{\mathcal{K}}_{sn}$-closed set in Y. Then, Y being an $\mathcal{I}^{\mathcal{K}}_{sn}$-closed in X, by Proposition 2.7, we have $F = F \cap Y$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-closed in X i.e. F is closed in X. But Y is closed in X, that implies $F = F \cap Y$ is closed in Y.

(ii) Consider a family of topological spaces $\{X_\alpha\}_{\alpha \in \lambda}$ and $X = \oplus_{\alpha \in \Delta} X_\alpha$.

(a) Suppose that for each $\alpha \in \Lambda$, X_α is an $\mathcal{I}^{\mathcal{K}}$-neighborhood space. Let P be a $\mathcal{I}^{\mathcal{K}}$-open subset of X. Then by Proposition 2.7, $P \cap X_\alpha$ is $\mathcal{I}^{\mathcal{K}}$-open in X_α, for each $\alpha \in \Lambda$. Then, $P \cap X_\alpha$ is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of X_α for each $\alpha \in \Lambda$. Then, by the definition of topological sum and Proposition 2.13 (i), $P = \bigcup_{\lambda \in \Lambda}(P \cap X_\alpha)$ is an $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset of X.

(b) Suppose that for each $\alpha \in \Lambda$, X_α possesses property "$\mathcal{I}^{\mathcal{K}}$-compatibility". Let P be $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X. Then by Proposition 2.7, $P \cap X_\alpha$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X_α i.e $P \cap X_\alpha$ is open in X_α for each $\alpha \in \Lambda$. Then $P = \bigcup_{\lambda \in \Lambda}(P \cap X_\alpha)$ is open in X.

□

The following result identifies the notion of $\mathcal{I}^{\mathcal{K}}_{sn}$-continuity and the property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence as the same.

Theorem 3.9. *Let X, Y be topological spaces with $P \subset X$ and $f : X \to Y$ be a mapping. Then* *(i)* f *is* $\mathcal{I}^{\mathcal{K}}_{sn}$*-continuous.*

(ii) $f^{-1}(P)$ *is an* $\mathcal{I}^{\mathcal{K}}_{sn}$*-closed subset in* X *whenever* P *is an* $\mathcal{I}^{\mathcal{K}}_{sn}$*-closed subset in* Y*.*

(iii) $f([P]_{\mathcal{I}^{\mathcal{K}}_{sn}}) \subset [f(P)]_{\mathcal{I}^{\mathcal{K}}_{sn}}$*, for each* $P \subset X$*.*

(iv) O *is an* $\mathcal{I}^{\mathcal{K}}$*-sequential neighborhood of* $b \in Y$ *and* $a \in f^{-1}(b)$*. Then* $f^{-1}(O)$ *is an* $\mathcal{I}^{\mathcal{K}}$*-sequential neighborhood of* $a \in X$*.*

(v) f *preserves* $\mathcal{I}^{\mathcal{K}}$*-convergence of sequences.*

Proof. (i) $\implies$ (v). Consider, f is $\mathcal{I}^{\mathcal{K}}_{sn}$-continuous and $a_n \to_{\mathcal{I}^{\mathcal{K}}} a \in X$. Consider an open set P containing $f(a)$ in Y. Then P is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in Y. According to our assumption, $f^{-1}(P)$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X. Then, $\{n \in \omega : a_n \notin f^{-1}(P)\} \in \mathcal{K}$ i.e. $\{n \in \omega : f(a_n) \notin P\} \in \mathcal{K}$. Hence, the sequence $\{f(a_n)\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to $f(a)$.

(v) $\implies$ (iv). Suppose that f possesses the property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence. Let P be an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of $b \in Y$ and $a \in f^{-1}(b)$. Consider a sequence $\{a_n\}$, $\mathcal{I}^{\mathcal{K}}$-convergent to $a \in f^{-1}(P)$. Then, the sequence $f(a_n) \to_{\mathcal{I}^{\mathcal{K}}} f(a) \in P$. Now, $\{n \in \omega : a_n \notin f^{-1}(P)\} = \{n \in \omega : f(a_n) \notin P\} \in \mathcal{K}$. Therefore, $f^{-1}(P)$ is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a in X.

(iv) $\implies$ (iii). Let $P \subset X$ and $a \in [P]_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset X$. Let O be an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of $f(a)$ in Y. By condition (iv), $f^{-1}(O)$ is an $\mathcal{I}^{\mathcal{K}}$-sequential neighborhood of a in X and $f^{-1}(O) \cap P \neq \phi$. Hence, $O \cap f(P) \neq \phi$. Therefore, $f(a) \in [f(P)]_{\mathcal{I}^{\mathcal{K}}_{sn}}$.

(iii) $\implies$ (ii). Let P be $\mathcal{I}^{\mathcal{K}}_{sn}$-closed in Y. Then $f([f^{-1}(P)]_{\mathcal{I}^{\mathcal{K}}_{sn}}) \subset [f(f^{-1}(P))]_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset [P]_{\mathcal{I}^{\mathcal{K}}_{sn}} = P$. In essence, $[f^{-1}(P)]_{\mathcal{I}^{\mathcal{K}}_{sn}} \subset f^{-1}(P)$ i.e. $f^{-1}(P) = [f^{-1}(P)]_{\mathcal{I}^{\mathcal{K}}_{sn}}$. Thus, $f^{-1}(P)$ is an $\mathcal{I}^{\mathcal{K}}_{sn}$-closed subset in X.

(ii) $\implies$ (i). Let P is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in Y. Then $Y \setminus P$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-closed in Y. $f^{-1}(Y \setminus P)$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-closed in X. By condition (ii), $X \setminus f^{-1}(P)$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-closed in X. $f^{-1}(P)$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-open subset in X. □

In the line of Lin's [11] interpretation in the context of ideal convergence, we term the property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence of a mapping as the $\mathcal{I}^{\mathcal{K}}_{sn}$-continuity of the map.

Theorem 3.10. [1, Theorem 3.11] *Every continuous function possesses the property of preserving* $\mathcal{I}^{\mathcal{K}}$*-convergence.*

Since continuous map possesses the property of preserving $\mathcal{I}^{\mathcal{K}}$-convergence and also, if a map preserves $\mathcal{I}^{\mathcal{K}}$-convergence then it is $\mathcal{I}^{\mathcal{K}}$-continuous [2]. Subsequently, a mapping f is continuous $\implies$ $\mathcal{I}^{\mathcal{K}}_{sn}$-continuous $\implies$ $\mathcal{I}^{\mathcal{K}}$-continuous.

Theorem 3.11. *The following are equivalent for any spaces* X*,* Y *and function* $f : X \to Y$*.*

(i) X *possesses "*$\mathcal{I}^{\mathcal{K}}$*-compatibility".*

(ii) f *is continuous if and only if* f *preserves* $\mathcal{I}^{\mathcal{K}}$*-convergence of sequences.*

Proof. Suppose that X possesses the property "$\mathcal{I}^{\mathcal{K}}$-compatibility". Then, by Theorem 3.10, f is continuous implies that it preserves $\mathcal{I}^{\mathcal{K}}$-convergence. If possible, let f be not continuous. Then, there exists an open set $O \subset Y$ (i.e. $\mathcal{I}^{\mathcal{K}}_{sn}$-open) such that $f^{-1}(O)$ is not open in X. Since X possesses $\mathcal{I}^{\mathcal{K}}$-compatibility, so $f^{-1}(O)$ is not $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X. Then, there exists a sequence $\{a_n\}$ in X which is $\mathcal{I}^{\mathcal{K}}$-converges to $b \in f^{-1}(O)$ and $\{n \in \omega : a_n \notin f^{-1}(O)\} \notin \mathcal{K}$. Similarly, the sequence $\{f(a_n)\}$ in Y, we have that $\{n \in \omega : f(a_n) \notin O\} \notin \mathcal{K}$. But, O is $\mathcal{I}^{\mathcal{K}}_{sn}$-open, therefore $\{f(a_n)\}$ is not $\mathcal{I}^{\mathcal{K}}$-converges to $f(b) \in O$.

Conversely, suppose that (X, τ) does not possesses the property "$\mathcal{I}^{\mathcal{K}}$-compatibility". Consider the family of all $\mathcal{I}^{\mathcal{K}}_{sn}$-open sets as $\tau_{\mathcal{I}^{\mathcal{K}}_{sn}}$. Then by Proposition 2.13, $\tau_{\mathcal{I}^{\mathcal{K}}_{sn}}$ forms a finer topology than τ. Consider the identity map $i_d : (X,\tau) \to (X, \tau_{\mathcal{I}^{\mathcal{K}}_{sn}})$ which is not continuous. Suppose $\{a_n\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to b in (X,τ). Since each open set in $(X, \tau_{\mathcal{I}^{\mathcal{K}}_{sn}})$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-open set in (X,τ). So, O contains a $\mathcal{K}$-tail of $\{a_n\}$, i.e. $\{n \in \omega : a_n \notin O\} \in \mathcal{K}$. Hence, $\{f(a_n)\}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to b in $(X, \tau_{\mathcal{I}^{\mathcal{K}}_{sn}})$. This is a contradiction. Thus, (X,τ) possesses "$\mathcal{I}^{\mathcal{K}}$-compatibility". □

Corollary 3.12. *The following statements are equivalent for any spaces X, Y and function $f : X \to Y$; the space X possesses the property "$\mathcal{I}$-compatibility" if and only if the notions of continuity of f coincides with the property of preserving $\mathcal{I}$-convergence.*

Theorem 3.13. *A topological space X is an $\mathcal{I}^{\mathcal{K}}$-neighborhood space if and only if every $\mathcal{I}^{\mathcal{K}}$-continuous map is $\mathcal{I}^{\mathcal{K}}_{sn}$-continuous.*

Proof. Consider X to be an $\mathcal{I}^{\mathcal{K}}$-neighborhood space. Let $f : X \to Y$ is $\mathcal{I}^{\mathcal{K}}$-continuous. Let O be $\mathcal{I}^{\mathcal{K}}_{sn}$-open in Y i.e. O is $\mathcal{I}^{\mathcal{K}}$-open in Y. Since f is $\mathcal{I}^{\mathcal{K}}$-continuous, $f^{-1}(O)$ is $\mathcal{I}^{\mathcal{K}}$-open in X. By our consideration, $f^{-1}(O)$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X.

For the converse, assume that X is not an $\mathcal{I}^{\mathcal{K}}$-neighborhood space. Then there is an $\mathcal{I}^{\mathcal{K}}$-open subset O in X such that O is not $\mathcal{I}^{\mathcal{K}}_{sn}$-open. Let $Y = (\{a,b\},\tau)$, where τ comprises of the subsets $\phi, \{a\}, Y$ and $\{a\}$ is open if and only if the set U is closed in X. Define a mapping $q : X \to Y$ as $q(x) = a$ for $x \in U$ and $q(x) = b$ for $x \in X \setminus U$. We claim that q is $\mathcal{I}^{\mathcal{K}}$-continuous. It is clear that inverse image of each $\mathcal{I}^{\mathcal{K}}$-open subset $\phi, \{a\}, Y$ are $\mathcal{I}^{\mathcal{K}}$-open in X. Now, if possible assume that $\{b\}$ is $\mathcal{I}^{\mathcal{K}}$-open in Y. Then $\{b\}$ must be open in Y, otherwise; if not there exists a sequence $\{b_n\}$ in Y with $b_n = a$ for each n such that $\{b_n\}$ is convergent to b. Thus, $y_n \to_{\mathcal{I}^{\mathcal{K}}} b$. But then $\{n \in \omega : b_n \in \{b\}\} = \phi \in \mathcal{K}$ which is a contradiction by Lemma 2.5. Thus, $\{b\}$ is $\mathcal{I}^{\mathcal{K}}$-open in Y implies that $\{b\}$ is open in Y. Hence, O is $\mathcal{I}^{\mathcal{K}}$-closed in X or $q^{-1}(\{b\}) = X \setminus O$ is $\mathcal{I}^{\mathcal{K}}$-open in X. On the other hand, $\{a\}$ is open $\implies$ $\{a\}$ is $\mathcal{I}^{\mathcal{K}}_{sn}$-open in Y. But $q^{-1}(\{a\}) = O$ is not $\mathcal{I}^{\mathcal{K}}_{sn}$-open in X. □

From Theorem 3.9, Theorem 3.11 and Theorem 3.13, we summarise the following result.

Theorem 3.14. *Let X, Y be topological spaces and f be a function from X to Y. Then, X is an $\mathcal{I}^{\mathcal{K}}$-sequential space if and only if notions of continuity, $\mathcal{I}^{\mathcal{K}}_{sn}$-continuity and $\mathcal{I}^{\mathcal{K}}$-continuity coincide.*

4. Conclusion and future scope

This article mainly focused on discussing the notion of $\mathcal{I}^{\mathcal{K}}_{sn}$-open sets and contemporary extensions of several notions closely related to an open set in space. With suitable assumptions for ideals and for the parent space, we have drawn some conclusions and marked some open problems for further study. It can be interesting if a fresh convergence mode can be introduced via $\mathcal{I}^{\mathcal{K}}_{sn}$-open sets which correspond to a new topology in the space. In the meantime, the idea of $\mathcal{I}^{\mathcal{K}}$-compatibility for spaces leads to an open query of finding particular compatible ideals for specific topological spaces.

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A. Sharmah and D. Hazarika,
Department of Mathematical Sciences, Tezpur University, Napam 784028, Assam, India
E-mail address: `ankurs@tezu.ernet.in, debajit@tezu.ernet.in`