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Product of Generalized Derivations of Order 2 with Derivations Acting on Multilinear Polynomials with Centralizing Conditions *

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ABSTRACT: Let R be a prime ring with $char(R) \neq 2$. Suppose that $f(x_1, \ldots, x_n)$ be a noncentral multilinear polynomial over C, G be a nonzero generalized derivation of R and d a nonzero derivation of R. In this paper we describe all possible forms of G in the case

$$G^2(f(\xi))d(f(\xi)) \in C$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$.

Key Words: Prime ring, derivation, generalized derivation.

Contents

1. Introduction

Throughout this paper, R always denotes an associative prime ring, extended centroid C, and U its Utumi quotient ring. It is proven that C is a field, when R is prime ring. Readers are provided [2,4] for more details about U and C. An additive map d on R is said to be derivation if:

$$d(xy) = d(x)y + xd(y) \quad \text{for all} \quad x, y \in R.$$

In [8] Brešar introduced a new notion by extending the concept of derivation, named generalized derivation. An additive map F on R is said to be generalized derivation if there exists a derivation d on R such that:

$$F(xy) = F(x)y + xd(y) \quad \text{for all} \quad x, y \in R.$$

The derivation d involves in the definition of generalized derivation F is called the associated derivation of F. A polynomial $f \in C[x_1, \ldots, x_n]$ is said to be multilinear if it is linear in every x_i, $1 \leq i \leq n$. During last three decades there has been a lot of studies on generalized derivation (see [1,3,5,6,7,10,11,12,14,17,23,24]) on different subsets of R.

In [19], Lee and Shiue showed that if R is a prime ring, $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C and d a nonzero derivation of R such that $d(u)u \in C$ for all $u \in f(R)$, then $char(R) = 2$ and R satisfies s_4.

In [5], Demir and Argaç considered a similar situation where the derivation is replaced by generalized derivation and the evaluations are taken over a non zero right ideal of R. More precisely they proved: Let R be a noncommutative prime ring and F is a generalized derivation on R such that $F(u)u \in C$ for all $u \in f(\rho)$, where ρ is a right ideal of R. Then $F(x) = ax$, where $a \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued on R, except when $char(R) = 2$ and R satisfies s_4.

In [14], it is proved that if F_1 and F_2 are generalized derivations of a prime ring R having $char(R) \neq 2$, such that $F_1(x)F_2(x) = 0$ for all $x \in R$, then there exist elements $p, q \in U$ such that $F_1(x) = xp$ and

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$F_2(x) = qx$ for all $x \in R$ and $pq = 0$ except when atleast one F_i is zero. Moreover the above identity is studied by Carini et al [3] by taking the multilinear polynomial and studied the structures of F_1 and F_2.

Furthermore, Eroğlu and Argaç [10] determined all possible structures of F by considering $F^2(u)u \in C$ for all $u \in f(R)$ and F is a generalized derivation of R.

More recently, Yadav [24] described all possible forms of the maps when $F^2(u)d(u) = 0$ for all $u \in f(R)$, where F is generalized derivation of R and d is a nonzero derivation of R. He proved the following:

Let R be a noncommutative prime ring of $char(R) \neq 2$, U be its Utumi quotient ring, C be its extended centroid and $f(x_1, \dots, x_n)$ be a noncentral multilinear polynomial over C. Suppose that d is a nonzero derivation of R and G is a generalized derivation on R. If

$$G^2(f(\xi))d(f(\xi)) = 0$$

for all $\xi = (\xi_1, \dots, \xi_n) \in R^n$, then one of the following holds:

1. there exist $a \in U$ such that $G(x) = ax$ for all $x \in R$ with $a^2 = 0$;
2. there exist $a \in U$ such that $G(x) = xa$ for all $x \in R$ with $a^2 = 0$.

In this article we extend Yadav's result [24] in central case. More precisely, we study the following:

Theorem 1.1. *Let R be a noncommutative prime ring of $char(R) \neq 2$, U be its Utumi quotient ring, C be its extended centroid and $f(x_1, \dots, x_n)$ be a noncentral multilinear polynomial over C. Suppose that d is a nonzero derivation of R and G is a generalized derivation on R. If*

$$G^2(f(\xi))d(f(\xi)) \in C$$

for all $\xi = (\xi_1, \dots, \xi_n) \in R^n$, then one of the following holds:

1. *there exists $a \in U$ such that $G(x) = ax$ for all $x \in R$ with $a^2 = 0$;*
2. *there exists $a \in U$ such that $G(x) = xa$ for all $x \in R$ with $a^2 = 0$.*

2. When derivations are inner

We dedicate this section to prove the main theorem in case both the generalized derivation G and the derivation d are inner, that is, there exist $a, b, c \in U$ such that $G(x) = ax + xb$ and $d(x) = [c, x]$ for all $x \in R$. Then $G^2(f(\xi))d(f(\xi)) \in C$ for all $\xi = (\xi_1, \dots, \xi_n) \in R^n$ implies

$$a^2 f(\xi)cf(\xi) + 2af(\xi)bcf(\xi) + f(\xi)b^2cf(\xi) - a^2 f(\xi)^2 c - 2af(\xi)bf(\xi)c - f(\xi)b^2 f(\xi)c \in C.$$

This gives

$$\begin{aligned} & a'f(\xi)cf(\xi)^2 + 2af(\xi)pf(\xi)^2 + f(\xi)p'f(\xi)^2 \\ & + f(\xi)a'f(\xi)^2c + 2f(\xi)af(\xi)bf(\xi)c + f(\xi)^2b'f(\xi)c \\ & - a'f(\xi)^2cf(\xi) - 2af(\xi)bf(\xi)cf(\xi) - f(\xi)b'f(\xi)cf(\xi) \\ & - f(\xi)a'f(\xi)cf(\xi) - 2f(\xi)af(\xi)pf(\xi) - f(\xi)^2p'f(\xi) = 0 \end{aligned} \tag{2.1}$$

for all $\xi = (\xi_1, \dots, \xi_n) \in R^n$, where $a' = a^2$, $b' = b^2$, $p = bc$ and $p' = b^2c$.

Proposition 2.1. *Let C be a field and $R = M_m(C)$ be the ring of all $m \times m$ matrices over C, $m \geq 2$. Suppose that char $(R) \neq 2$ and $f(x_1, \dots, x_n)$ a noncentral multilinear polynomial over C. If a, b and $c \in R$ such that (2.1) holds for all $\xi = (\xi_1, \dots, \xi_n) \in R^n$, then either a or b or c are scalar matrices.*

Proof. By our assumption (2.1) is a generalized polynomial identity of R. Suppose that all of a, b and c are not scalar matrices.

Case-I: Suppose that C is infinite field.

As we assumed $a \notin C.I_m$ and $b \notin C.I_m$ and $c \notin C.I_m$. By [11, Lemma 1.5] there exists an invertible matrix P in $M_m(C)$ such that PaP^{-1}, PbP^{-1} and PcP^{-1} have all non-zero entries. Clearly R satisfies

$$\begin{aligned} &Pa'P^{-1}f(\xi)PcP^{-1}f(\xi)^2 + 2PaP^{-1}f(\xi)PpP^{-1}f(\xi)^2 \\ &+ f(\xi)Pp'P^{-1}f(\xi)^2 + f(\xi)Pa'P^{-1}f(\xi)^2PcP^{-1} \\ &+ 2f(\xi)PaP^{-1}f(\xi)PbP^{-1}f(\xi)PcP^{-1} + f(\xi)^2Pb'P^{-1}f(\xi)PcP^{-1} \\ &- Pa'P^{-1}f(\xi)^2PcP^{-1}f(\xi) - 2PaP^{-1}f(\xi)PbP^{-1}f(\xi)PcP^{-1}f(\xi) \\ &- f(\xi)Pb'P^{-1}f(\xi)PcP^{-1}f(\xi) - f(\xi)Pa'P^{-1}f(\xi)PcP^{-1}f(\xi) \\ &- 2f(\xi)PaP^{-1}f(\xi)PpP^{-1}f(\xi) - f(\xi)^2Pp'P^{-1}f(\xi) = 0 \end{aligned} \tag{2.2}$$

for all $\xi = (\xi_1, \dots, \xi_n) \in R^n$. By hypothesis $f(x_1, \dots, x_n)$ is non central valued. Hence by [18] (see also [20]), there exist matrices $\xi_1, \dots, \xi_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(\xi_1, \dots, \xi_n) = \gamma e_{ij}$, with $i \neq j$. We replace this value of $f(\xi_1, \dots, \xi_n)$ in (2.2), we get

$$\begin{aligned} &2e_{ij}PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1} - 2PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1}e_{ij} \\ &- e_{ij}Pb'P^{-1}e_{ij}PcP^{-1}e_{ij} - e_{ij}Pa'P^{-1}e_{ij}PcP^{-1}e_{ij} \\ &- 2e_{ij}PaP^{-1}e_{ij}PpP^{-1}e_{ij} = 0 \end{aligned} \tag{2.3}$$

Now multiplying by e_{ij} in (2.3) from right side, we get $2e_{ij}PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1}e_{ij} = 0$, this implies $e_{ij}PaP^{-1}e_{ij}PbP^{-1}e_{ij}PcP^{-1}e_{ij} = 0$, as $char(R) \neq 2$. This is a contradiction as PaP^{-1}, PbP^{-1} and PcP^{-1} have all non-zero entries.

Case-II: When C is finite field.
Let K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Since multilinear polynomial $f(x_1, \dots, x_n)$ is non-central-valued on R, so it is also non-central-valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned} \phi(\xi_1, \dots, \xi_n) = \; &a'f(\xi)cf(\xi)^2 + 2af(\xi)pf(\xi)^2 + f(\xi)p'f(\xi)^2 \\ &+ f(\xi)a'f(\xi)^2c + 2f(\xi)af(\xi)bf(\xi)c + f(\xi)^2b'f(\xi)c \\ &- a'f(\xi)^2cf(\xi) - 2af(\xi)bf(\xi)cf(\xi) - f(\xi)b'f(\xi)cf(\xi) \\ &- f(\xi)a'f(\xi)cf(\xi) - 2f(\xi)af(\xi)pf(\xi) - f(\xi)^2p'f(\xi) \end{aligned}$$

which is a generalized polynomial identity for R. Moreover, it is a multi-homogeneous of multi-degree $(2, \dots, 2)$ in the indeterminates $\xi_1, \dots, \xi_n$.

Hence the complete linearization of $\phi(\xi_1, \dots, \xi_n)$ is a multilinear generalized polynomial $\Theta(\xi_1, \dots, \xi_n, s_1, \dots, s_n)$ in $2n$ indeterminates, moreover

$$\Theta(\xi_1, \dots, \xi_n, s_1, \dots, s_n) = 2^n \phi(\xi_1, \dots, \xi_n).$$

Clearly the multilinear polynomial $\Theta(\xi_1, \dots, \xi_n, s_1, \dots, s_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $char(C) \neq 2$ we obtain $\phi(\xi_1, \dots, \xi_n) = 0$ for all $\xi_1, \dots, \xi_n \in \overline{R}$ and then conclusion follows from above when C was infinite.

□

Proposition 2.2. *Let R be a prime ring of char $(R) \neq 2$, C the extended centroid of R and $f(x_1, \dots, x_n)$ a non-central multilinear polynomial over C. If R satisfies (2.1), then either a or b or c are scalar matrices.*

Proof. Since R and U satisfy the same generalized polynomial identities (see [4]), U satisfies

$$\begin{aligned} &a'f(\xi)cf(\xi)^2 + 2af(\xi)pf(\xi)^2 + f(\xi)p'f(\xi)^2 \\ &+ f(\xi)a'f(\xi)^2c + 2f(\xi)af(\xi)bf(\xi)c + f(\xi)^2b'f(\xi)c \\ &- a'f(\xi)^2cf(\xi) - 2af(\xi)bf(\xi)cf(\xi) - f(\xi)b'f(\xi)cf(\xi) \\ &- f(\xi)a'f(\xi)cf(\xi) - 2f(\xi)af(\xi)pf(\xi) - f(\xi)^2p'f(\xi) \end{aligned} \tag{2.4}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$. Suppose that this is a trivial GPI for U. So,

$$\begin{aligned} & a' f(\xi) c f(\xi)^2 + 2a f(\xi) p f(\xi)^2 + f(\xi) p' f(\xi)^2 \\ & + f(\xi) a' f(\xi)^2 c + 2 f(\xi) a f(\xi) b f(\xi) c + f(\xi)^2 b' f(\xi) c \\ & - a' f(\xi)^2 c f(\xi) - 2a f(\xi) b f(\xi) c f(\xi) - f(\xi) b' f(\xi) c f(\xi) \\ & - f(\xi) a' f(\xi) c f(\xi) - 2 f(\xi) a f(\xi) p f(\xi) - f(\xi)^2 p' f(\xi) \end{aligned} \tag{2.5}$$

is zero element in $T = U *_C C\{\xi_1, \ldots, \xi_n\}$, the free product of U and $C\{\xi_1, \ldots, \xi_n\}$, the free C-algebra in noncommuting indeterminates $\xi_1, \ldots, \xi_n$. This implies $\{1, c\}$ is linearly $C-$dependent, that is $c \in C$, as desired. Let us assume $c \notin C$, then by (2.5)

$$\{f(\xi) a' f(\xi) + 2 f(\xi) a f(\xi) b + f(\xi)^2 b'\} f(\xi) c = 0 \in T. \tag{2.6}$$

This again implies that $\{1, b, b'\}$ is linearly $C-$dependent. There exist $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 + \alpha_2 b + \alpha_3 b' = 0$. If $\alpha_3 = 0$, then $\alpha_2 \neq 0$ and hence $b \in C$, as desired. Thus we assume $\alpha_3 \neq 0$ and $b \notin C$. Then by (2.6)

$$\{f(\xi) a' f(\xi) + 2 f(\xi) a f(\xi) b + \alpha f(\xi)^2 b + \beta f(\xi)^2\} f(\xi) c = 0 \in T. \tag{2.7}$$

Assume $a \notin C$, then $2 f(\xi) a f(\xi) b f(\xi) c$ appears nontrivially in (2.7), which is a contradiction. So, either a or b or c is central, as desired.
Next suppose that (2.4) is a non-trivial GPI for Q. Let $\overline{C}$ be the algebraic closure of C. We know that U and $U \otimes_C \overline{C}$ satisfy the same GPIs. Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [9, Theorems 2.5 and 3.5], we may replace R by U or $U \otimes_C \overline{C}$ according to C finite or infinite and then applying Martindale's theorem [21], we can say that R is a primitive ring with nonzero socle $soc(R)$ and with C as its associated division ring. Then, by Jacobson's theorem [15, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(\xi_1, \ldots, \xi_n)$ is not central valued on R, R must be noncommutative and so $m \geq 2$. In this case, by Proposition 2.1, we get that either a or b or c are in C. If V is infinite dimensional over C, then for any $e^2 = e \in soc(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. Since a_2, a_3, a_5 are not in C, there exist $h_1, h_2, h_3 \in soc(R)$ such that $[a, h_1] \neq 0$ $[b, h_2] \neq 0$, $[c, h_3] \neq 0$. By Litoff's Theorem [13], there exists idempotent $e \in soc(R)$ such that $ah_1, h_1 a, bh_2, h_2 b, ch_3, h_3 c, h_1, h_2, h_3 \in eRe$. Since R satisfies generalized identity

$$\begin{aligned} e\Big\{ & a' f(e\xi_1 e, \ldots, e\xi_n e) c f(e\xi_1 e, \ldots, e\xi_n e)^2 + 2a f(e\xi_1 e, \ldots, e\xi_n e) p f(e\xi_1 e, \ldots, e\xi_n e)^2 \\ & + f(e\xi_1 e, \ldots, e\xi_n e) p' f(e\xi_1 e, \ldots, e\xi_n e)^2 + f(e\xi_1 e, \ldots, e\xi_n e) a' f(e\xi_1 e, \ldots, e\xi_n e)^2 c \\ & + 2 f(e\xi_1 e, \ldots, e\xi_n e) a f(e\xi_1 e, \ldots, e\xi_n e) b f(e\xi_1 e, \ldots, e\xi_n e) c \\ & + f(e\xi_1 e, \ldots, e\xi_n e)^2 b' f(e\xi_1 e, \ldots, e\xi_n e) c - a' f(e\xi_1 e, \ldots, e\xi_n e)^2 c f(e\xi_1 e, \ldots, e\xi_n e) \\ & - 2a f(e\xi_1 e, \ldots, e\xi_n e) b f(e\xi_1 e, \ldots, e\xi_n e) c f(e\xi_1 e, \ldots, e\xi_n e) \\ & - f(e\xi_1 e, \ldots, e\xi_n e) b' f(e\xi_1 e, \ldots, e\xi_n e) c f(e\xi_1 e, \ldots, e\xi_n e) \\ & - f(e\xi_1 e, \ldots, e\xi_n e) a' f(e\xi_1 e, \ldots, e\xi_n e) c f(e\xi_1 e, \ldots, e\xi_n e) \\ & - 2 f(e\xi_1 e, \ldots, e\xi_n e) a f(e\xi_1 e, \ldots, e\xi_n e) p f(e\xi_1 e, \ldots, e\xi_n e) \\ & - f(e\xi_1 e, \ldots, e\xi_n e)^2 p' f(e\xi_1 e, \ldots, e\xi_n e) \Big\} e = 0, \end{aligned}$$

then the subring eRe satisfies

$$\begin{aligned}&ea'ef(\xi_1,\dots,\xi_n)ecef(\xi_1,\dots,\xi_n)^2+2eaef(\xi_1,\dots,\xi_n)epef(\xi_1,\dots,\xi_n)^2\\&+f(\xi_1,\dots,\xi_n)ep'ef(\xi_1,\dots,\xi_n)^2+f(\xi_1,\dots,\xi_n)ea'ef(\xi_1,\dots,\xi_n)^2ece\\&+2f(\xi_1,\dots,\xi_n)eaef(\xi_1,\dots,\xi_n)ebef(\xi_1,\dots,\xi_n)ece\\&+f(\xi_1,\dots,\xi_n)^2eb'ef(\xi_1,\dots,\xi_n)ece-ea'ef(\xi_1,\dots,\xi_n)^2ecef(\xi_1,\dots,\xi_n)\\&-2eaef(\xi_1,\dots,\xi_n)ebef(\xi_1,\dots,\xi_n)ecef(\xi_1,\dots,\xi_n)\\&-f(\xi_1,\dots,\xi_n)eb'ef(\xi_1,\dots,\xi_n)ecef(\xi_1,\dots,\xi_n)\\&-f(\xi_1,\dots,\xi_n)ea'ef(\xi_1,\dots,\xi_n)ecef(\xi_1,\dots,\xi_n)\\&-2f(\xi_1,\dots,\xi_n)eaef(\xi_1,\dots,\xi_n)epef(\xi_1,\dots,\xi_n)\\&-f(\xi_1,\dots,\xi_n)^2ep'ef(\xi_1,\dots,\xi_n)=0.\end{aligned}$$

Then by the above finite dimensional case, either eae or ebe or ece are central elements of eRe. Thus either $ah_1=(eae)h_1=h_1(eae)=h_1a$ or $bh_2=(ebe)h_2=h_2(ebe)=h_2b$ or $ch_3=(ece)h_3=h_3(ece)=h_3c$, in any case we get a contradiction.

Hence, we say that either a or b or c are in C. □

By the same way as above we can prove the following prepositions.

Proposition 2.3. *Let R be a prime ring of char $(R)\neq 2$, C the extended centroid of R and $f(x_1,\dots,x_n)$ a non-central multilinear polynomial over C. If c and $k\in R$ such that*

$$f(\xi)kcf(\xi)^2-f(\xi)kf(\xi)cf(\xi)-f(\xi)^2kcf(\xi)+f(\xi)^2kf(\xi)c=0$$

for all $\xi=(\xi_1,\dots,\xi_n)\in R^n$, then either $k\in C$ or $c\in C$.

Proposition 2.4. *Let R be a prime ring of char $(R)\neq 2$, C the extended centroid of R and $f(x_1,\dots,x_n)$ a non-central multilinear polynomial over C. If c and $k\in R$ such that*

$$kf(\xi)cf(\xi)^2-kf(\xi)^2cf(\xi)-f(\xi)kf(\xi)cf(\xi)+f(\xi)kf(\xi)^2c=0$$

for all $\xi=(\xi_1,\dots,\xi_n)\in R^n$, then either $k\in C$ or $c\in C$.

Lemma 2.5. *Let R be a noncommutative prime ring of char$(R)\neq 2$, U be its Utumi quotient ring, C be its extended centroid and $f(x_1,\dots,x_n)$ be a noncentral multilinear polynomial over C. Suppose for some $a,b,c\in U$, $G(x)=ax+xb$, and $d(x)=[c,x]$ for all $x\in R$ with $c\notin C$. If*

$$G^2(f(\xi))d(f(\xi))\in C$$

for all $\xi=(\xi_1,\dots,\xi_n)\in R^n$, then one of the following holds:

1. *$G(x)=(a+b)x$ for all $x\in R$ with $(a+b)^2=0$;*
2. *$G(x)=x(a+b)$ for all $x\in R$ with $(a+b)^2=0$.*

Proof. By the hypothesis, we have

$$\big(a^2f(\xi)+2af(\xi)b+f(\xi)b^2\big)\big(cf(\xi)-f(\xi)c\big)\in C \tag{2.8}$$

that is

$$\big[\big(a^2f(\xi)+2af(\xi)b+f(\xi)b^2\big)\big(cf(\xi)-f(\xi)c\big),f(\xi)\big]=0 \tag{2.9}$$

for all $\xi=(\xi_1,\dots,\xi_n)\in R^n$. Then by Proposition 2.2, either $a\in C$ or $b\in C$ or $c\in C$. Since $c\notin C$, so either $a\in C$ or $b\in C$.

If $a\in C$, it follows hypothesis as

$$f(\xi)(a+b)^2\big(cf(\xi)-f(\xi)c\big)\in C$$

that is

$$f(\xi)(a+b)^2cf(\xi)^2 - f(\xi)(a+b)^2f(\xi)cf(\xi) - f(\xi)^2(a+b)^2cf(\xi) + f(\xi)^2(a+b)^2f(\xi)c = 0$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. Then by Proposition 2.3, $(a+b)^2 \in C$.

If $b \in C$, it follows hypothesis as

$$(a+b)^2 f(\xi)\big(cf(\xi) - f(\xi)c\big) \in C$$

that is

$$(a+b)^2f(\xi)cf(\xi)^2 - (a+b)^2f(\xi)^2cf(\xi) - f(\xi)(a+b)^2f(\xi)cf(\xi) + f(\xi)(a+b)^2f(\xi)^2c = 0$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. Then by Proposition 2.4, $(a+b)^2 \in C$. Thus in both the above cases we have $(a+b)^2 \in C$ and hence we can write $(a+b)^2x = G^2(x) = x(a+b)^2$ for all $x \in f(R)$.

Considering $G^2(f(\xi)) = f(\xi)(a+b)^2$, our hypothesis $G^2(f(\xi))d(f(\xi)) \in C$ gives $f(\xi)\big[(a+b)^2c, f(\xi)\big] \in C$ for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. Then by [19] we have $(a+b)^2c \in C$. This implies $(a+b)^2 = 0$ as $c \notin C$.

Thus we arrive either $G(x) = (a+b)x$ or $x(a+b)$, with $(a+b)^2 = 0$. These are our required conclusions. □

3. Proof of the main theorem

In light of the notion in [17, Theorem 3], generalized derivation G has its form $G(x) = ax + \delta(x)$ for some $a \in U$ and δ is a derivation on U.

Now if we consider $f(\xi_1, \ldots, \xi_n)$ be a noncentral multilinear polynomial over the field C and d is a derivation on R.

We shall use the notation

$$f(\xi_1, \ldots, \xi_n) = \xi_1\xi_2\cdots\xi_n + \sum_{\sigma \in S_n, \sigma \neq id} \alpha_\sigma \xi_{\sigma(1)}\xi_{\sigma(2)}\cdots\xi_{\sigma(n)}$$

for some $\alpha_\sigma \in C$, and S_n denotes the symmetric group of degree n.

Then we have

$$d(f(\xi_1, \ldots, \xi_n)) = f^d(\xi_1, \ldots, \xi_n) + \sum_i f(\xi_1, \ldots, d(\xi_i), \ldots, \xi_n),$$

where $f^d(\xi_1, \ldots, \xi_n)$ be the polynomials obtained from $f(\xi_1, \ldots, \xi_n)$ replacing each coefficients α_σ with $d(\alpha_\sigma)$. Similarly, by calculation, we have

$$\begin{aligned} d^2(f(\xi_1, \ldots, \xi_n)) &= f^{d^2}(\xi_1, \ldots, \xi_n) + 2\sum_i f^d(\xi_1, \ldots, d(\xi_i), \ldots, \xi_n) \\ &+ \sum_i f(\xi_1, \ldots, d^2(\xi_i), \ldots, \xi_n) \\ &+ \sum_{i \neq j} f(\xi_1, \ldots, d(\xi_i), \ldots, d(\xi_j), \ldots, \xi_n). \end{aligned}$$

By hypothesis, we have

$$\big(G(a)f(\xi) + 2a\delta(f(\xi)) + \delta^2(f(\xi))\big)d(f(\xi)) \in C$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$. By [18], we have

$$\big(G(a)f(\xi) + 2a\delta(f(\xi)) + \delta^2(f(\xi))\big)d(f(\xi)) \in C \tag{3.1}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in U^n$.

If d and δ both are inner derivations, then by Proposition 2.2, we have our conclusions of Main Theorem. Thus, to prove our Main Theorem, we need to consider the case when not both d and δ are inner. Indeed we have to consider the two following embedded cases.

- d and δ are linearly C-independent modulo inner derivations of U.
- d and δ are linearly C-dependent modulo inner derivations of U.

Case-1: *When d and δ are linearly C-independent modulo inner derivations of U.*

By (3.1) U satisfies

$$\Big\{F(a)f(\xi_1,\ldots,\xi_n)+2a\{f^{\delta}(\xi_1,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,\delta(\xi_i),\ldots,\xi_n)\}$$
$$+\{f^{\delta^2}(\xi_1,\ldots,\xi_n)+2\sum_i f^{\delta}(\xi_1,\ldots,\delta(\xi_i),\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,\delta^2(\xi_i),\ldots,\xi_n)$$
$$+\sum_{i\neq j} f(\xi_1,\ldots,\delta(\xi_i),\ldots,\delta(\xi_j),\ldots,\xi_n)\}\Big\}\{f^{d}(\xi_1,\ldots,\xi_n)$$
$$+\sum_i f(\xi_1,\ldots,d(\xi_i),\ldots,\xi_n)\}\in C.$$

for all $\xi_1,\ldots,\xi_n\in U$. Since d and δ are not inner, by Kharchenko's theorem [16], U satisfies

$$\Big\{F(a)f(\xi_1,\ldots,\xi_n)+2a\{f^{\delta}(\xi_1,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,x_i,\ldots,\xi_n)\}$$
$$+\{f^{\delta^2}(\xi_1,\ldots,\xi_n)+2\sum_i f^{\delta}(\xi_1,\ldots,x_i,\ldots,\xi_n)+\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)$$
$$+\sum_{i\neq j} f(\xi_1,\ldots,x_i,\ldots,x_j,\ldots,\xi_n)\}\Big\}\{f^{d}(\xi_1,\ldots,\xi_n)$$
$$+\sum_i f(\xi_1,\ldots,z_i,\ldots,\xi_n)\}\in C.$$

In particular U satisfies the blended component

$$\Big[\sum_i f(\xi_1,\ldots,y_i,\ldots,\xi_n)\sum_i f(\xi_1,\ldots,z_i,\ldots,\xi_n),f(\xi_1,\ldots,\xi_n)\Big]=0. \tag{3.2}$$

Putting $y_i=[q',\xi_i]$ for each $i\in\{1,\ldots,n\}$, for some $q'\notin C$ and $z_1=\xi_1$, $z_2=\ldots,z_n=0$, we get

$$\big[[q'f(\xi_1,\ldots,\xi_n)]f(\xi_1,\ldots,\xi_n),f(\xi_1,\ldots,\xi_n)\big]=0$$

that is

$$[q',f(\xi_1,\ldots,\xi_n)]_2 f(\xi_1,\ldots,\xi_n)=0$$

this implies

$$[q',f(\xi_1,\ldots,\xi_n)]_2=0 \quad\text{as}\quad f(\xi_1,\ldots,\xi_n)\notin C$$

for all $\xi_1,\ldots,\xi_n\in U$. Then by [22] we get $q'\in C$, which is a contradiction.

Case-2: *When d and δ are linearly C-dependent modulo inner derivations of U.*

In this case we get $\alpha,\beta\in C$ and $q\in U$ such that $\alpha d+\beta\delta=ad_q$. It is clear from the context that $(\alpha,\beta)\neq(0,0)$. So with out loss of generality we arrive the following two subcases:

<u>Sub-case-i:</u> *When $\alpha=0$.*
Then we get $\delta(x)=[p,x]$, where $p=\beta^{-1}q$. It is obvious that d is not inner, otherwise we get contradiction. Now from (3.1) we have

$$(a'^2f(\xi_1,\ldots,\xi_n)+2a'f(\xi_1,\ldots,\xi_n)b'+f(\xi_1,\ldots,\xi_n)b'^2)df\big(d(f(\xi_1,\ldots,\xi_n))\big)\in C$$

for all $\xi_1, \dots, \xi_n \in U$ and $a' = a + p, b' = -p \in U$. Now from above we can write

$$\begin{aligned} &\big(a'^2 f(\xi_1, \dots, \xi_n) + 2a' f(\xi_1, \dots, \xi_n)b' + f(\xi_1, \dots, \xi_n)b'^2\big). \\ &\big(f^d(\xi_1, \dots, \xi_n) + \sum_i f(\xi_1, \dots, d(\xi_i), \dots, \xi_n)\big) \in C. \end{aligned} \tag{3.3}$$

Since d is not inner, by Kharchenko's theorem [16]

$$\begin{aligned} &\big(a'^2 f(\xi_1, \dots, \xi_n) + 2a' f(\xi_1, \dots, \xi_n)b' + f(\xi_1, \dots, \xi_n)b'^2\big). \\ &\big(f^d(\xi_1, \dots, \xi_n) + \sum_i f(\xi_1, \dots, y_i, \dots, \xi_n)\big) \in C. \end{aligned}$$

In particular U satisfies the blended component

$$\begin{aligned} &\Big[\big(a'^2 f(\xi_1, \dots, \xi_n) + 2a' f(\xi_1, \dots, \xi_n)b' + f(\xi_1, \dots, \xi_n)b'^2\big). \\ &\sum_i f(\xi_1, \dots, y_i, \dots, \xi_n), f(\xi_1, \dots, \xi_n)\Big] = 0. \end{aligned} \tag{3.4}$$

Replacing y_i by $[q, \xi_i]$, for some $q \notin C$ in (3.4) we have

$$\begin{aligned} &\Big[\big(a'^2 f(\xi_1, \dots, \xi_n) + 2a' f(\xi_1, \dots, \xi_n)b' + f(\xi_1, \dots, \xi_n)b'^2\big) \\ &[q, f(\xi_1, \dots, \xi_n)], f(\xi_1, \dots, \xi_n)\Big] = 0. \end{aligned} \tag{3.5}$$

Which is similar as (2.9) of Lemma 2.5, so from there we get our conclusions (1) and (2) of main theorem.

<u>Sub-case-ii:</u> *When $\alpha \neq 0$.*
Then we have $d = \mu\delta + ad_c$, for some $\mu \in C$ and $c \in U$. Here δ never be an inner derivation, otherwise both d and δ will be inner, a contradiction. Then from (3.1) we have

$$\begin{aligned} &\big(G(a) f(\xi_1, \dots, \xi_n) + 2a\delta(f(\xi_1, \dots, \xi_n)) + \delta^2(f(\xi_1, \dots, \xi_n))\big) \\ &\big(\mu\delta(f(\xi_1, \dots, \xi_n)) + [c, f(\xi_1, \dots, \xi_n)]\big) \in C \end{aligned} \tag{3.6}$$

for all $\xi = (\xi_1, \dots, \xi_n) \in U^n$. This is a differential identity containing the terms of the type δ and δ^2. As, δ and δ^2 are outer, by Kharchenko's theorem [16] $\delta(\xi_i)$ and $\delta^2(\xi_i)$ can be replaced by x_i and y_i respectively in (3.6). And hence U satisfies the blended component

$$\big(\sum_i f(\xi_1, \dots, y_i, \dots, \xi_n)\big)\big(\mu \sum_i f(\xi_1, \dots, x_i, \dots, \xi_n)\big) \in C,$$

that is

$$\Big[\mu \sum_i f(\xi_1, \dots, y_i, \dots, \xi_n) \sum_i f(\xi_1, \dots, x_i, \dots, \xi_n), f(\xi_1, \dots, \xi_n)\Big] = 0. \tag{3.7}$$

Replacing y_i by $[q, \xi_i]$, where $q \notin C$ and $x_1 = \xi_1$, $x_2 = \ldots = x_n = 0$ in (3.7) we get

$$\Big[\mu[q, f(\xi_1, \dots, \xi_n)] f(\xi_1, \dots, \xi_n), f(\xi_1, \dots, \xi_n)\Big] = 0,$$

that is

$$\Big[[\mu q, f(\xi_1, \dots, \xi_n)], f(\xi_1, \dots, \xi_n)\Big] f(\xi_1, \dots, \xi_n) = 0,$$

that is

$$[\mu q, f(\xi_1, \dots, \xi_n)]_2 = 0,$$

for all $\xi_1, \ldots, \xi_n \in R$, as $f(\xi_1, \ldots, \xi_n)$ is noncentral. So by [22] we get $\mu q \in C$, this says $\mu = 0$. Then from (3.6) we get

$$\Big(G(a)f(\xi_1, \ldots, \xi_n) + 2a\delta(f(\xi_1, \ldots, \xi_n)) + \delta^2(f(\xi_1, \ldots, \xi_n))\Big)[c, f(\xi_1, \ldots, \xi_n)] \in C, \tag{3.8}$$

for all $\xi_1, \ldots, \xi_n \in U$. Again from above putting the expressions of $\delta(f(\xi_1, \ldots, \xi_n))$ and $\delta^2(f(\xi_1, \ldots, \xi_n))$ we will find a blended component satisfied by U as follows:

$$\sum_i f(\xi_1, \ldots, \delta^2(\xi_i), \ldots, \xi_n)[c, f(\xi_1, \ldots, \xi_n)] \in C. \tag{3.9}$$

As it is mentioned earlier that δ is outer, then by applying Kharchenko's theorem [16], we replace $\delta^2(\xi_i)$ by y_i in (3.9) we get the following:

$$\sum_i f(\xi_1, \ldots, y_i, \ldots, \xi_n)[c, f(\xi_1, \ldots, \xi_n)] \in C.$$

In particular $y_1 = \xi_1$ and $y_2 = \cdots y_n = 0$ we get

$$f(\xi_1, \ldots, \xi_n)[c, f(\xi_1, \ldots, \xi_n)] \in C$$

for all $\xi_1, \ldots, \xi_n \in R$. Then from [19] we get $c \in C$. Finally we get $\mu = 0$ and $c \in C$, which implies $d = 0$, a contradiction.

This completes the proof.

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