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Non-Torsion Element Graph of a Module Over a Commutative Ring*

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ABSTRACT: Let R be a commutative ring with unity and M be a unitary R-module. Let $T(M)$ be the set of all torsion elements of M and $NT(M) = M - T(M)$ be the set of all non-torsion elements of M. The non-torsion element graph of M over R is an undirected simple graph $G_{NT}(M)$ with $NT(M)$ as vertex set and any two distinct vertices x and y are adjacent if and only if $x + y \in T(M)$. In this paper, we study the basic properties of the graph $G_{NT}(M)$. We also study the diameter and girth of $G_{NT}(M)$. Further, we determine the domination number and the bondage number of $G_{NT}(M)$. We establish a relation between diameter and domination number of $G_{NT}(M)$. We also establish a relation between girth and bondage number of $G_{NT}(M)$.

Key Words: Non-torsion element graph, Diameter, Girth, Domination number, Bondage number, Torsion elements, Non-torsion elements.

Contents

1. Introduction

The study of graphs by associating various algebraic structures has become an intriguing research topic since the last three decades. Recently, this area is growing rapidly leading to many interesting results and questions. It was Beck [7] who first introduced the concept of zero-divisor graph of commutative rings. He was mostly interested in coloring of commutative rings. Beck's introduction has been slightly modified by Anderson and Naseer in [2]. After that, many fundamental papers assigning graphs to rings and modules have been published, for instance, see [1,3,4,6,20].

In 2008, Anderson and Badawi [3] introduced the total graph of a commutative ring and later this notion was generalised to many algebraic structures, in particular to module over a commutative ring (see [6,10,11]). Atani and Habibi [6] have generalised the total graph as in [3] by introducing the total torsion element graph of a module over a commutative ring. They have studied the characteristics of this total graph and it's induced subgraphs by considering two cases, $T(M)$ is a submodule of M or is not a submodule of M.

The concepts of dominating sets and domination numbers play a vital role in graph theory. Dominating sets are the focus of many books of graph theory, for example see [13] and [14]. But not much research has been done on domination of graphs associated to algebraic structures such as groups, rings, modules in terms of algebraic properties. However, some works on domination of graphs associated to rings and modules have appeared recently, for instance, see [9,12,17,18,19,21].

In this paper, we define the non-torsion element graph $G_{NT}(M)$ of a module M over a commutative ring R as a subgraph of the Total torsion element graph [6]. We study the basic properties of the graph $G_{NT}(M)$. We also determine the diameter and girth of $G_{NT}(M)$. Further, we investigate the domination number and the bondage number of $G_{NT}(M)$. We establish a relationship between diameter

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and domination number of $G_{NT}(M)$. We also establish a relationship between girth and bondage number of $G_{NT}(M)$.

2. Preliminaries

In this section, we recall some basic definitions and results which are useful in the later sections.

Throughout this paper, R is a commutative ring with unity and M is an unitary R-module, unless otherwise specified. An element a of a commutative ring R is called a zero-divisor of R if $ab = 0$ for some non-zero element b of R. Let $Z(R)$ be the set of zero-divisors of R. Let $T(M) = \{m \in M | \; rm = 0,$ for some $r(\neq 0) \in R\}$ be the set of torsion elements of M. Let $NT(M) = M - T(M)$ be the set of non-torsion elements of M. A module M is called torsion module if $T(M) = M$. On the other hand, a module M is called torsion-free if $T(M) = \{0\}$. For any undefined terminology in rings and modules we refer to [5,15,16].

By a graph G, we mean a simple undirected graph without loops. For a graph G, we denote the set of all vertices by $V(G)$ and edges by $E(G)$. We call a graph is finite if both $V(G)$ and $E(G)$ are finite sets, and we use the symbol $|G|$ to denote the number of vertices in the graph G. We say that G is a null graph if both $V(G) = \phi$ and $E(G) = \phi$. Two vertices x and y of a graph G are connected if there is a path in G connecting them. Also, a graph G is connected if there is a path between any two distinct vertices. A graph G is disconnected if it is not connected. A graph G is said to be totally disconnected if $E(G) = \phi$. A graph G is complete if every pair of distinct vertices of the graph are adjacent. We denote a complete graph on n vertices by K_n. If the vertex set $V(G)$ of the graph G is partitioned into two non-empty disjoint sets X and Y of cardinality $|X| = m$ and $|Y| = n$, and two vertices are adjacent if and only if they are not in the same partite set, then G is called a bipartite graph. A graph G is called a complete bipartite graph if every vertex in X is adjacent to every vertex in Y. We denote the complete bipartite graph on m and n vertices by $K_{m,n}$.

For vertices $x, y \in G$ the distance $d(x, y)$ is defined as the length of the shortest path between x and y, if the vertices are connected in G. If they are not connected, $d(x, y) = \infty$. The diameter of the graph G is

$$diam(G) = sup\{d(x, y)| \; x, y \in G\}$$

A cycle is a closed path which begins and ends at the same vertex. The cycle of n vertices is denoted by C_n. The girth of the graph G, denoted by $gr(G)$ is the length of the shortest cycle in G and $gr(G) = \infty$ if G has no cycles.

A subset $S \subseteq V$ is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is defined to be the minimum cardinality of a dominating set in G and such a dominating set is called γ-set of G. If G is a trivial graph, then $\gamma(G) = 0$. The bondage number $b(G)$ is the minimum number of edges whose removal increases the domination number. For basic definitions and results in domination we refer to [13] and for any undefined graph-theoretic terminology we refer to [8].

Now we summarize some results on domination number and bondage number of a graph which will be useful for the later sections.

Lemma 2.1. *[14]*

(i) *If G is a graph of order n, then $1 \leq \gamma(G) \leq n$. A graph G of order n has domination number 1 if and only if G contains a vertex v of degree $n-1$; while $\gamma(G) = n$ if and only if $G \cong \overline{K_n}$.*

(ii) *$\gamma(K_n) = 1$ for a complete graph K_n, but the converse is not true, in general and $\gamma(\overline{K_n}) = n$ for a null graph $\overline{K_n}$.*

(iii) *Let G be a complete r-partite graph $(r \geq 2)$ with partite sets $V_1, V_2, ..., V_r$. If $|V_i| \geq 2$ for $1 \leq i \leq r$, then $\gamma(G) = 2$; because one vertex of V_1 and one vertex of V_2 dominate G. If $|V_i| = 1$ for some i, then $\gamma(G) = 1$.*

(iv) *$\gamma(K_{1,n}) = 1$ for a star graph $K_{1,n}$.*

(v) If G is a union of disjoint subgraphs $G_1, G_2, ..., G_k$, then $\gamma(G) = \gamma(G_1) + \gamma(G_2) + ... + \gamma(G_k)$.

(vi) Domination number of a bistar graph is 2, because the set consisting of two centres of the graph is a minimal dominating set.

(vii) Let C_n and P_n be an n-cycle and a path with n vertices, respectively. Then $\gamma(C_n) = \lceil \frac{n}{3} \rceil = \gamma(P_n)$.

Lemma 2.2. *[14]*

(i) If G is a simple graph of order n, then $1 \leq b(G) \leq n - 1$.

(ii) $b(K_n) = n - 1$ for a complete graph K_n, but the converse is not true, in general and $b(\overline{K_n}) = 0$ for a null graph $\overline{K_n}$.

(iii) Let G be a complete r-partite graph with partite sets $V_1, V_2, ..., V_r$. Then $b(G) = min\{|V_1|, |V_2|, ..., |V_r|\}$. In particular, $b(K_{m,n}) = min\{m, n\}$.

(iv) If G is a union of disjoint subgraphs $G_1, G_2, ..., G_k$, then $b(G) = min\{b(G_1), b(G_2), ..., b(G_k)\}$.

(v) Let C_n and P_n be an n-cycle and a path with n vertices, respectively. Then $b(P_n) = 1$ and $b(C_n) = 2$.

3. Non-torsion element Graph $G_{NT}(M)$ of a Module

In this section we introduce the Non-Torsion element Graph $G_{NT}(M)$ of a module M and study some of it's basic properties. We begin with the following definition.

Definition 3.1. Let R be a commutative ring and M be an R-module. The non-torsion element graph $G_{NT}(M)$ of a Module M is an undirected simple graph defined by taking $NT(M)$ as the vertex set and two distinct vertices x and y are adjacent if and only if $x + y \in T(M)$.

Now, we discuss some examples.

Example 3.2. Let $R = \mathbb{Z}_8$ and $M = R = \mathbb{Z}_8$ is a module over itself. $T(M) = \{0, 2, 4, 6\}$ is a submodule of M. Therefore $NT(M) = \{1, 3, 5, 7\}$. Now, we can observe that the graph $G_{NT}(M)$ is the complete graph K_4 as shown in the following figure.

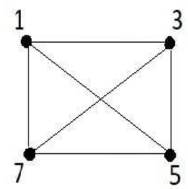

Figure 1: $G_{NT}(\mathbb{Z}_8)$

Example 3.3. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_4$. Then M is an R-module with the usual operations, and $T(M) = \{0, 1, 2, 3\}$. Now, we can see that the graph $G_{NT}(M)$ is the null graph.

Proposition 3.4. Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. If two distinct vertices of $G_{NT}(M)$ are connected, then there exists a path of length 2 or 1 between them. In particular, if $G_{NT}(M)$ is connected, then $diam(G_{NT}(M)) \leq 2$.

Proof. Let x, y be two distinct vertices of $G_{NT}(M)$ which are connected. If x, y are adjacent in $G_{NT}(M)$, then obviously we get a path of length 1 between them. If x, y are not adjacent in $G_{NT}(M)$, then there exists a path $x \sim x_1 \sim x_2 \sim ... \sim x_{n-1} \sim y$ of length n (> 1) in $G_{NT}(M)$.

Now if n is odd, then we have

$x + y = (x + x_1) - (x_1 + x_2) + ... - (x_{n-2} + x_{n-1}) + (x_{n-1} + y) \in T(M)$, a contradiction.

Therefore, n must be even and so we have

$x-y=(x+x_1)-(x_1+x_2)+...+(x_{n-2}+x_{n-1})-(x_{n-1}+y)\in T(M)$.
Hence, there exists a path $x\sim(-y)\sim y$ of length 2 between x and y . Similarly, we also get a path $x\sim(-x)\sim y$ of length 2 between x and y. Thus, if $G_{NT}(M)$ is connected, then $diam(G_{NT}(M))\leq 2$. □

Proposition 3.5. Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M. Then the following conditions are equivalent :
(1) $G_{NT}(M)$ is connected.
(2) Either $x+y\in T(M)$ or $x-y\in T(M)$ $\forall$ $x,y\in NT(M)$.
(3) Either $x+y\in T(M)$ or $x+2y\in T(M)$ $\forall$ $x,y\in NT(M)$. In particular, either $2x\in T(M)$ or $3x\in T(M)$ (not both) $\forall$ $x\in NT(M)$.

Proof. (1) $\Rightarrow$ (2) Directly follows from Proposition 3.4.
(2) $\Rightarrow$ (3) Since $(x+y)-y=x\notin T(M)$. So by our assumption $(x+y)+y=x+2y\in T(M)$. If $x=y$, either $2x$ or $3x\in T(M)$. Now, $3x-2x=x\notin T(M)$, so $2x$ and $3x$ both can not be in $T(M)$.
(3) $\Rightarrow$ (1) Let $x,y\in NT(M)$ and $x+y\notin T(M)$. Then by our assumption, $x+2y\in T(M)$. Now $2y\notin T(M)$. So, $3y=y+2y\in T(M)$. Now, $x\neq 2y$. So there exist a path $x\sim 2y\sim y$ from x to y. Therefore, $G_{NT}(M)$ is connected. □

Proposition 3.6. Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)|=\lambda$ and $|\frac{M}{T(M)}|=\mu$. Then
(1) If $2=1_R+1_R\in Z(R)$, then $G_{NT}(M)$ is the union of $\mu-1$ disjoint K_λ's.
(2) If $2=1_R+1_R\notin Z(R)$, then $G_{NT}(M)$ is the union of $\frac{\mu-1}{2}$ disjoint $K_{\lambda,\lambda}$'s.

Proof. (1) For any $x\in NT(M)$, $x+T(M)\subseteq NT(M)$. Now, if $2\in Z(R)$, then $2x\in T(M)$ for any x. For any x_1, $x_2\in T(M)$, $(x+x_1)+(x+x_2)=2x+x_1+x_2\in T(M)$. Thus, $x+T(M)$ is a complete graph with λ elements.

Now, if $x+T(M)$ and $y+T(M)$ are distinct cosets and $x+x_1$, $y+x_2$ are adjacent. $\Rightarrow x+y\in T(M)$. So, we have $x-y\in T(M)$, as $2y\in T(M)$. Therefore, $x+T(M)=y+T(M)$. Hence, $G_{NT}(M)$ is the union of $\mu-1$ disjoint K_λ's.
(2) Let $x\in NT(M)$ and $x+x_1$, $x+x_2$ are adjacent in $x+T(M)$. Now, $2x\in T(M)$. So, $\exists$ $r(\neq 0)\in R$ such that $r(2x)=0$. But $x\notin T(M)$. We get $2r=0$, a contradiction. Hence, no pair of elements of $x+T(M)$ are adjacent.

Now, $x+T(M)$ and $-x+T(M)$ are disjoint cosets as $2x\notin T(M)$. For any two elements $x+x_1$ and $-x+x_2$ in $x+T(M)$ and $-x+T(M)$ respectively, $(x+x_1)+(-x+x_2)=x_1+x_2\in T(M)$. Thus $(x+T(M))\cup(-x+T(M))$ forms a complete bipartite subgraph $K_{\lambda,\lambda}$ of $G_{NT}(M)$. If $x+x_1$ and $y+x_2$ are adjacent vertices of two distinct cosets of $T(M)$, then $x+x_1+y+x_2\in T(M)$. So, $x+y\in T(M)$. Therefore, $x+T(M)=-y+T(M)$. Thus, $G_{NT}(M)$ is the union of $\frac{\mu-1}{2}$ disjoint $K_{\lambda,\lambda}$'s. □

Example 3.7. Let $R=\mathbb{Z}_4$ and $M=\mathbb{Z}_4\times\mathbb{Z}_4$. Then M is an R-module with the usual operations, and $T(M)=\{(0,0),(0,2),(2,0),(2,2)\}$ is a submodule of M. Therefore $NT(M)=\{(0,1),(0,3),(1,0),(1,1),(1,2),(1,3),(2,1),(2,3),(3,0),(3,1),(3,2),(3,3)\}$. Clearly, $|T(M)|=4$ and so $|\frac{M}{T(M)}|=4$. Also, it is clear that $2\in Z(R)$. We see that, the graph $G_{NT}(M)$ is union of 3 disjoint K_4's in the following figure.

Example 3.8. Let $R=\mathbb{Z}_5$. Then $R=M$ is a module over itself and $T(M)=\{0\}$ is a submodule of M. Therefore, $NT(M)=\{1,2,3,4\}$. Clearly, $|T(M)|=1$ and so $|\frac{M}{T(M)}|=5$. Also, it is clear that $2\notin Z(R)$. We see that, the graph $G_{NT}(M)$ is union of two disjoint $K_{1,1}$'s as shown in the following figure.

Proposition 3.9. Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M with $M-T(M)\neq\phi$. Then

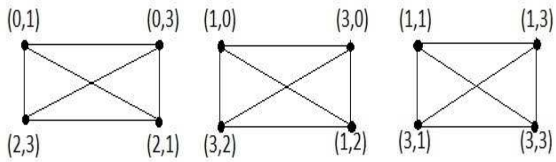

Figure 2: $G_{NT}(\mathbb{Z}_4 \times \mathbb{Z}_4)$

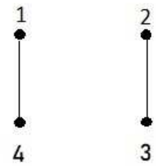

Figure 3: $G_{NT}(\mathbb{Z}_5)$

(1) $G_{NT}(M)$ is complete if and only if either $|\dfrac{M}{T(M)}| = 2$ or $|\dfrac{M}{T(M)}| = |M| = 3$.
(2) $G_{NT}(M)$ is connected if and only if either $|\dfrac{M}{T(M)}| = 2$ or $|\dfrac{M}{T(M)}| = 3$.
(3) $G_{NT}(M)$ is totally disconnected if and only if $T(M) = \{0\}$ and $2 \in Z(R)$.

Proof. **(1)** If $G_{NT}(M)$ is complete, then by Proposition 3.6, $G_{NT}(M)$ contains one K_λ or $K_{1,1}$. For K_λ, $\mu - 1 = 1$, so $|\dfrac{M}{T(M)}| = 2$ and for $K_{1,1}$, $\lambda = 1$, $\dfrac{\mu-1}{2} = 1$, so $|\dfrac{M}{T(M)}| = |M| = 3$.

Conversely, if $|\dfrac{M}{T(M)}| = 2$. $\dfrac{M}{T(M)} = \{T(M), x + T(M)\}$, $x \in NT(M)$. Then $2x \in T(M)$. So, $2 \in Z(R)$. By Proposition 3.6, $G_{NT}(M)$ contains one K_λ, hence complete.
If $|\dfrac{M}{T(M)}| = |M| = 3$, then $T(M) = \{0\}$. Now if $2 \in Z(R)$, then $2a \in T(M)$ $\forall a \in M$. i.e $2a = 0$ $\forall\, a \in M$, a contradiction as M is a cyclic group of order 3. So $2 \notin$ Z(R). By Proposition 3.6(2) $G_{NT}(M)$ is $K_{1,1}$, a complete graph.
(2) If $G_{NT}(M)$ is connected, then by Proposition 3.6, $G_{NT}(M)$ is either K_λ or $K_{\lambda,\lambda}$. Thus $\mu - 1 = 1$ or $\dfrac{\mu-1}{2} = 1$. So, $|\dfrac{M}{T(M)}| = 2$ or $|\dfrac{M}{T(M)}| = 3$.

Now if $|\dfrac{M}{T(M)}| = 2$, then $G_{NT}(M)$ is complete by Part 1, so it is connected.
If $|\dfrac{M}{T(M)}| = 3$, and if $2 \in Z(R)$, then $2x \in T(M)$ for all x. $\dfrac{M}{T(M)} = \{T(M), x + T(M), y + T(M)\}$, $x, y \in NT(M)$. So, $x + y \in T(M)$ and $2y \in T(M)$ as $2 \in Z(R)$. Now, $x + y - 2y = x - y \in T(M)$. Which gives $x + T(M) = y + T(M)$, a contradiction. Therefore $2 \notin Z(R)$. So $G_{NT}(M)$ is $K_{\lambda,\lambda}$, which is connected.
(3) $G_{NT}(M)$ is totally disconnected if and only if it is a disjoint union of K_1's. Therefore, by Proposition 3.6(1) we have $2 \in Z(R)$ and $|T(M)| = 1$ which gives $T(M) = \{0\}$. This completes the proof. $\square$

Proposition 3.10. Let $\prod_{i \in I} R_i = R$ and $\prod_{i \in I} M_i = M$, where M_i is R_i-module for each $i \in I$. Then the following holds:
(1) If $G_{NT}(M_i)$ is a complete graph and $2 \in Z(R_i)$ for some $i \in I$, then $G_{NT}(M)$ is a complete graph.
(2) If $G_{NT}(M)$ is complete and $2 \notin Z(R)$, then $G_{NT}(M_i)$ is complete for each $i \in I$.

Proof. (1) If $G_{NT}(M_i)$ is complete, then for any $x_i,\ y_i \in NT(M_i)$, $x_i + y_i \in T(M_i)$.
Let $x = (x_1, x_2, ..., x_i, ...)$, $y = (y_1, y_2, ..., y_i, ...) \in NT(M)$. Then, $x+y = (x_1+y_1, x_2+y_2, ..., x_i+y_i, ...)$.
Since $x_i + y_i \in T(M_i)$, so $\exists\ r_i(\neq 0) \in R_i$ such that $r_i.(x_i + y_i) = 0$.

Now, $(0, 0, ..., 0, r_i, 0, ...).(x_1 + y_1, x_2 + y_2, ..., x_i + y_i, ...) = 0$, i.e. $r.(x + y) = 0$ where $r = (0, 0, ...0, r_i, 0, ...)(\neq 0) \in R$. So $x + y \in T(M)$.

Again if $x_i = y_i = w$, then $x + y = (x_1 + y_1, x_2 + y_2, ..., x_{i-1} + y_{i-1}, 2w, x_{i+1} + y_{i+1}, ...)$. Since $2 \in Z(R_i)$, so $\exists\ r_i(\neq 0) \in R_i$ such that $2r_i = 0$, Let $r = (0, 0, ..., 0, r_i, 0, ...)$, then $r.(x + y) = 0$, So $x + y \in T(M)$.
Hence, $G_{NT}(M)$ is complete.
(2) Let us assume that $G_{NT}(M)$ is complete. Let $x_i,\ y_i \in NT(M_i)$ for some $i \in I$. Then $(x_1, x_2, ..., x_{i-1}, x_i, x_{i+1}, ...)$, $(x_1, x_2, ..., x_{i-1}, y_i, x_{i+1}, ...) \in NT(M)$.

So, $(2x_1, 2x_2, ..., 2x_{i-1}, x_i + y_i, 2x_{i+1}, ...) \in T(M)$. Since $2 \notin Z(R)$, so $2x_j \in NT(M_j)$ for $j \neq i$. Therefore, $x_i + y_i \in T(M_i)$. Hence $G_{NT}(M_i)$ is complete. □

Proposition 3.11. Let $\prod_{i\in I} R_i = R$ and $\prod_{i\in I} M_i = M$, where M_i is R_i-module for each $i \in I$. If $M - T(M) \neq \phi$, then $|\frac{M}{T(M)}| = |\frac{M_i}{T(M_i)}|$ for some i.

Proof. If $|\frac{M}{T(M)}| = \alpha$, then there exists at least $(\alpha - 1)$ elements $x_1, x_2, ..., x_{\alpha-1} \in NT(M)$. So there exists $x_{1i}, x_{2i}, \ .\ .\ .\ x_{(\alpha-1)i} \in NT(M_i)$.

Therefore, $|\frac{M_i}{T(M_i)}| \geq \alpha = |\frac{M}{T(M)}|$

Now, if $|\frac{M_i}{T(M_i)}| = \beta$, we can find at least $(\beta - 1)$ elements $y_1, y_2, ..., y_{\beta-1} \in NT(M_i)$. Since $M - T(M) \neq \phi$, so there exists an element $x = (x_1, x_2, ...) \in NT(M)$. Using $y_1, y_2, ..., y_{\beta-1}$ we can get $(\beta - 1)$ non-torsion elements of M such as $(x_1, x_2, ..., x_{i-1}, y_1, x_{i+1}, ...)$, $(x_1, x_2, ..., x_{i-1}, y_2, x_{i+1}, ...)$, . . .
So, we have $|\frac{M}{T(M}| \geq \beta = |\frac{M_i}{T(M_i)}|$

Which gives $|\frac{M}{T(M}| = |\frac{M_i}{T(M_i)}|$. □

Proposition 3.12. Let $\prod_{i\in I} R_i = R$ and $\prod_{i\in I} M_i = M$, where M_i is R_i-module for each $i \in I$. When $M - T(M) \neq \phi$
(1) If $2 \in Z(R)$, then $G_{NT}(M)$ is complete if and only if $|\frac{M_i}{T(M_i)}| = 2$ for some $i \in I$.
(2) If $2 \notin Z(R)$, then $G_{NT}(M)$ is complete if and only if $|\frac{M_i}{T(M_i)}| = |M_i| = 3$ for each $i \in I$.

Proof. (1) Suppose $2 \in Z(R)$ and $G_{NT}(M)$ is complete. Then, $|\frac{M}{T(M)}| = 2$ by Proposition 3.9(1). So, $|\frac{M_i}{T(M_i)}| = 2$, for some i by Proposition 3.11.

Conversely, If $2 \in Z(R)$ and $|\frac{M_i}{T(M_i)}| = 2$. Then by Proposition 3.9(1) we have, $G_{NT}(M_i)$ is complete. Also, $2 \in Z(R_i)$ as $|\frac{M_i}{T(M_i)}| = 2$. Hence by Proposition 3.9(1), $G_{NT}(M)$ is complete.
(2) If $G_{NT}(M)$ is complete and $2 \notin Z(R)$, then by Proposition 3.10(2) we have, $G_{NT}(M_i)$ is complete for each i. So, $|\frac{M_i}{T(M_i)}| = 2$ or $|\frac{M_i}{T(M_i)}| = |M_i| = 3$ for each i by Proposition 3.9(1).
Since $2 \notin Z(R_i)$ for each i, So $|\frac{M_i}{T(M_i)}| \neq 2, \Rightarrow |\frac{M_i}{T(M_i)}| = |M_i| = 3$ for each i.

Conversely, If $|\dfrac{M_i}{T(M_i)}| = |M_i| = 3$ for each $i \in I$, then each $G_{NT}(M_i)$ is complete by Proposition 3.9(1). So for any two distinct elements $x, y \in NT(M)$, there exists $x_i \neq y_i$ and $x_i + y_i \in T(M_i)$. So $x + y \in T(M)$. Hence, $G_{NT}(M)$ is complete. □

4. Diameter and Girth of $G_{NT}(M)$

In this section, we have discussed the diameter and the girth of the Non-Torsion Graph $G_{NT}(M)$. We begin with the following proposition.

Proposition 4.1. Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M and $M - T(M) \neq \phi$. Then

(1) $diam(G_{NT}(M)) = 0$ if and only if $T(M) = \{0\}$ and $|M| = 2$.

(2) $diam(G_{NT}(M)) = 1$ if and only if either $T(M) \neq \{0\}$ and $|\dfrac{M}{T(M)}| = 2$ or $T(M) = \{0\}$ and $|M| = 3$.

(3) $diam(G_{NT}(M)) = 2$ if and only if $T(M) \neq \{0\}$ and $|\dfrac{M}{T(M)}| = 3$.

(4) Otherwise, $diam(G_{NT}(M))) = \infty$.

Proof. (1) If $T(M) = \{0\}$ and $|M| = 2$, then $|NT(M)| = 1$, so $diam(G_{NT}(M)) = 0$.
Conversely, if $diam(G_{NT}(M)) = 0$, $G_{NT}(M)$ contains one K_1. Which gives $T(M) = \{0\}$ and $|M| = 2$.
(2) If $diam(G_{NT}(M)) = 1$ then $G_{NT}(M)$ is complete and the proof follows from Proposition 3.9(1).
If T(M) $\neq \{0\}$ and $|\dfrac{M}{T(M)}| = 2$, then $G_{NT}(M))$ contains one K_λ, $\lambda > 1$, so $G_{NT}(M)$ is complete.
If $T(M) = \{0\}$ and $|M| = 3$, then $G_{NT}(M)$ is the complete bipartite graph $K_{1,1}$ by Proposition 3.9(1) and Proposition 3.6. Hence the proof.
(3) If $diam(G_{NT}(M)) = 2$, $G_{NT}(M)$ is connected. So, by Proposition 3.9(2), $|\dfrac{M}{T(M)}| = 3$ and $T(M) \neq \{0\}$.

Conversely, if $T(M) \neq \{0\}$ and $|\dfrac{M}{T(M)}| = 3$, $G_{NT}(M)$ is connected by Proposition 3.9(2) and $diam(G_{NT}(M)) \leq 2$ by Proposition 3.4. Therefore by part (1) and part (2) above, we have $diam(G_{NT}(M)) \neq 0, 1$ which yields $diam(G_{NT}(M)) = 2$.
(4) If $G_{NT}(M)$ is connected, then $diam(G_{NT}(M)) \leq 2$. So If $G_{NT}(M)$ is disconnected, $diam(G_{NT}(M)) = \infty$ □

Example 4.2. Let $R = \mathbb{Z}_9$, then $R = M$ is a module over itself. Then $T(M) = \{0, 3, 6\}$ is a submodule of M. Therefore $NT(M) = \{1, 2, 4, 5, 7, 8\}$. Clearly, $T(M) \neq \{0\}$ and $|\dfrac{M}{T(M)}| = 3$. In the following figure, We see the graph $G_{NT}(M)$ has diameter 2.

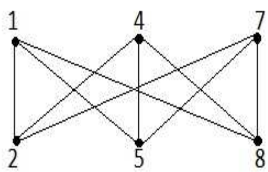

Figure 4: $G_{NT}(\mathbb{Z}_9)$

Proposition 4.3. Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M and $M - T(M) \neq \phi$. Then the following holds :

(1) $gr(G_{NT}(M)) = 3$ if and only if $2 \in Z(R)$ and $|T(M)| \geq 3$.

(2) $gr(G_{NT}(M)) = 4$ if and only if $2 \notin Z(R)$ and $|T(M)| \geq 2$.

(3) Otherwise, $gr(G_{NT}(M)) = \infty$.

Proof. (1) If $2 \in Z(R)$, $|T(M)| \geq 3$, $G_{NT}(M)$ is disjoint union of K_λ, $\lambda \geq 3$ by Proposition 3.6(1). Therefore K_λ will contain a 3-cycle which yields $gr(G_{NT}(M)) = 3$.

If $gr(G_{NT}(M)) = 3$, $G_{NT}(M)$ can not be union of bipartite graph, so $2 \in Z(R)$ by Proposition 3.6(2) and $|T(M)| \geq 3$.
(2) If $2 \notin Z(R)$ and $|T(M)| \geq 2$, then $G_{NT}(M)$ is union of complete bipartite graph $K_{\lambda,\lambda}$, $\lambda \geq 2$ by Proposition 3.6(2). So $gr(G_{NT}(M)) = 4$.

Now, if $gr(G_{NT}(M)) = 4$, then $G_{NT}(M)$ contains complete bipartite graph with $|T(M)| \geq 2$ and so $2 \notin Z(R)$ by Proposition 3.6(2).
(3) Since $G_{NT}(M)$ contains either complete graphs or complete bipartite graphs by Proposition 3.6 . So $gr(G_{NT}(M))$ is either 3 or 4 if $G_{NT}(M)$ contains a cycle. Otherwise $gr(G_{NT}(M)) = \infty$. □

Example 4.4. Let $R = \mathbb{Z}_8$, then $R = M$ is a module over itself. Then $T(M) = \{0, 2, 4, 6\}$ is a submodule of M. Therefore $NT(M) = \{1, 3, 5, 7\}$. Clearly, $|T(M)| = 4$ and $2 \in Z(R)$. In the following figure, We see the graph $G_{NT}(M)$ has girth 3.

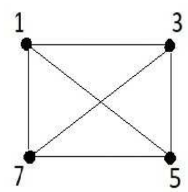

Figure 5: $G_{NT}(\mathbb{Z}_8)$

5. Domination Number and Bondage number of $G_{NT}(M)$

In this section we determine the domination number and the bondage number of the the Non-torsion element Graph $G_{NT}(M)$. We establish a relationship between diameter and domination number of $G_{NT}(M)$. We also establish a relationship between girth and bondage number of $G_{NT}(M)$. We begin with the following proposition.

Proposition 5.1. Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \lambda$, $\lambda \geq 2$ and $|\frac{M}{T(M)}| = \mu$, then $\gamma(G_{NT}(M)) = \mu - 1$.

Proof. Let us consider the following two cases of $Z(R)$.
Case 1: Suppose that $2 = 1_R + 1_R \in Z(R)$. Then we have from Proposition 3.6(1) that the graph $G_{NT}(M)$ is the union of $\mu - 1$ disjoint K_λ's and we know that $\gamma(K_\lambda) = 1$. Thus $\gamma(G_{NT}(M)) = \mu - 1$.
Case 2: Suppose that $2 = 1_R + 1_R \notin Z(R)$. Then again we have from Proposition 3.6(2) that the graph $G_{NT}(M)$ is the union of $\frac{\mu - 1}{2}$ disjoint $K_{\lambda,\lambda}$'s and we know that $\gamma(K_{\lambda,\lambda}) = 2$. Thus $\gamma(G_{NT}(M)) = \frac{\mu - 1}{2} \times 2 = \mu - 1$.
Hence, $\gamma(G_{NT}(M)) = \mu - 1$. □

Example 5.2. Let $R = \mathbb{Z}_4$ and $M = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then M is an R-module with the usual operations, and $T(M) = \{(0, 0), (0, 2), (2, 0), (2, 2)\}$ is a submodule of M. Therefore $NT(M) = \{(0, 1), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 0), (3, 1), (3, 2), (3, 3)\}$. Clearly, $\lambda = |T(M)| = 4$ and so $\mu = |\frac{M}{T(M)}| = 4$.

Also, it is clear that $2 \in Z(R)$. From Example 3.7 we can observe that, the graph $G_{NT}(M)$ is union of 3 disjoint K_4's. Therefore, $\gamma(G_{NT}(M)) = \gamma(K_4 \cup K_4 \cup K_4) = \gamma(K_4)+\gamma(K_4)+\gamma(K_4) = 1+1+1 = 3 = \mu-1$.

Proposition 5.3. Let M be a non-zero torsion-free module over a commutative ring R such that $|\frac{M}{T(M)}| = \mu$, then $\gamma(G_{NT}(M)) = \frac{\mu-1}{2}$.

Proof. Since M is torsion-free, so we have $T(M) = \{0\}$. Therefore, $|\frac{M}{T(M)}| = |M| = \mu$. Now, we show that $Z(R) = \{0\}$. Let $x\ (\neq 0) \in Z(R)$, then there exists $y\ (\neq 0) \in R$ such that $xy = 0$. Let us consider an element $m\ (\neq 0) \in M$, and we have $(xy)m = 0$ which yields $x(ym) = 0$. Then $ym = 0$ as $x \neq 0$ which implies either $y = 0$ or $m = 0$, a contradiction. Therefore, $Z(R) = 0$. So, $2 = 1_R + 1_R \notin Z(R)$ and from Proposition 3.6(2) we have the graph $G_{NT}(M)$ is the union of $\frac{\mu-1}{2}$ disjoint $K_{1,1}$'s.
Thus $\gamma(G_{NT}(M)) = \frac{\mu-1}{2} \times 1 = \frac{\mu-1}{2}$. □

Example 5.4. Let $R = \mathbb{Z}_5$. Then $R = M$ is a module over itself and $T(M) = \{0\}$ is a submodule of M. Therefore, $NT(M) = \{1, 2, 3, 4\}$. Clearly, $\lambda = |T(M)| = 1$ and so $\mu = |\frac{M}{T(M)}| = 5$. From Example 3.8 we can see that, the graph $G_{NT}(M)$ is union of two disjoint $K_{1,1}$'s. Therefore, $\gamma(G_{NT}(M)) = \gamma(K_{1,1} \cup K_{1,1}) = \gamma(K_{1,1}) + \gamma(K_{1,1}) = 1 + 1 = 2 = \frac{\mu-1}{2}$.

Proposition 5.5. Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M with $M - T(M) \neq \phi$. Then $\gamma(G_{NT}(M)) = 1$ if and only if $|\frac{M}{T(M)}| = 2$ or $|\frac{M}{T(M)}| = |M| = 3$.

Proof. Let us assume that $\gamma(G_{NT}(M)) = 1$. Then clearly $G_{NT}(M)$ is connected. If $2 \in Z(R)$, then $\mu - 1 = 1$ and hence $\mu = 2$, where $\mu = |\frac{M}{T(M)}|$, by Proposition 3.6(1) . Thus $|\frac{M}{T(M)}| = 2$.
If $2 \notin Z(R)$, then $\frac{\mu-1}{2} = 1$ and so $\mu = |\frac{M}{T(M)}| = 3$, by Proposition 3.6(2) . Also by assumption, we have $\lambda = |T(M)| = 1$ and hence $T(M) = \{0\}$. Thus $|\frac{M}{T(M)}| = |M| = 3$.
Conversely, let us assume that $|\frac{M}{T(M)}| = 2$ or $|\frac{M}{T(M)}| = |M| = 3$. Then by Proposition 3.9(1), $G_{NT}(M)$ is complete and hence $\gamma(G_{NT}(M)) = 1$.

In the following corollary, a relationship between diameter and domination number of $G_{NT}(M)$ has been established. □

Proposition 5.6. Let M be a module over commutative ring R such that $T(M)$ is a submodule of M. Then

(1) $diam(G_{NT}(M)) = 1$ if and only if $\gamma(G_{NT}(M)) = 1$.

(2) $diam(G_{NT}(M)) = 2$ if and only if $\gamma(G_{NT}(M)) = 2$.

Proof. (1) It is clear by Proposition 4.1(2) and Proposition 5.5 .
(2) If $diam(G_{NT}(M)) = 2$, then $T(M) \neq \{0\}$ and $|\frac{M}{T(M)}| = 3$, by Proposition 4.1(3) . Hence $G_{NT}(M)$ is connected, by Proposition 3.9(2) . Therefore $G_{NT}(M)$ is a complete bipartite graph $K_{\lambda,\lambda}$ with $\lambda \geq 2$. So $\gamma(G_{NT}(M)) = 2$.
Conversely, if $\gamma(G_{NT}(M)) = 2$, then $G_{NT}(M)$ is the union of two K_λ's or is a complete bipartite graph $K_{\lambda,\lambda}$ with $\lambda \geq 2$, by Proposition 3.6(1) and Proposition 3.6(2) . So $\mu - 1 = 2$ or $\frac{\mu-1}{2} = 1$. In either

case, $|\dfrac{M}{T(M)}| = 3$ and $|T(M)| \geq 2$. Thus $T(M) \neq \{0\}$ and $diam(G_{NT}(M)) = 2$, by Proposition 4.1(3) .
$\square$

Proposition 5.7. Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \lambda$ and $|\dfrac{M}{T(M)}| = \mu$. Then

$$b(G_{NT}(M)) = \begin{cases} \lambda - 1, & \text{if } 2 = 1_R + 1_R \in Z(R) \\ \lambda \quad , & \text{if } 2 = 1_R + 1_R \notin Z(R) \end{cases}$$

Proof. Let us assume that $2 = 1_R + 1_R \in Z(R)$. Then by Proposition 3.6(1), the graph $G_{NT}(M)$ is the union of $\mu - 1$ disjoint K_λ's and we know that $b(K_\lambda) = \lambda - 1$. Thus, $b(G_{NT}(M)) = \lambda - 1$.
Again, let us suppose that $2 = 1_R + 1_R \notin Z(R)$. Then by Proposition 3.6(2), $G_{NT}(M)$ is the union of $\dfrac{\mu - 1}{2}$ disjoint $K_{\lambda,\lambda}$'s and we know that $b(K_{\lambda,\lambda}) = \lambda$. Hence, $b(G_{NT}(M)) = \lambda$. $\square$

Example 5.8. Let $R = \mathbb{Z}_4$ and $M = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then M is an R-module with the usual operations, and $T(M) = \{(0,0), (0,2), (2,0), (2,2)\}$ is a submodule of M. Therefore $NT(M) = \{(0,1), (0,3), (1,0), (1,1), (1,2), (1,3), (2,1), (2,3), (3,0), (3,1), (3,2), (3,3)\}$. Clearly, $\lambda = |T(M)| = 4$ and so $\mu = |\dfrac{M}{T(M)}| = 4$. Also, it is clear that $2 \in Z(R)$. From Example 3.7 we can observe that, the graph $G_{NT}(M)$ is union of 3 disjoint K_4's. Therefore, $b(G_{NT}(M)) = b(K_4 \cup K_4 \cup K_4) = min\{b(K_4), b(K_4), b(K_4)\} = min\{3,3,3\} = 3 = \lambda - 1$.

Example 5.9. Let $R = \mathbb{Z}_5$. Then $R = M$ is a module over itself and $T(M) = \{0\}$ is a submodule of M. Therefore, $NT(M) = \{1,2,3,4\}$. Clearly, $|T(M)| = 1$ and so $|\dfrac{M}{T(M)}| = 5$. Also, it is clear that $2 \notin Z(R)$. From Example 3.8, we see that the graph $G_{NT}(M)$ is union of two disjoint $K_{1,1}$'s.
Thus, we have $b(G_{NT}(M)) = b(K_{1,1} \cup K_{1,1}) = min\{b(K_{1,1}), b(K_{1,1})\} = min\{1,1\} = 1$.

Proposition 5.10. Let M be a module over a commutative ring R such that $T(M)$ is a submodule of M, $|T(M)| = \lambda$ and $|\dfrac{M}{T(M)}| = \mu$. Then

(1) $gr(G_{NT}(M)) = 3$ if and only if $b(G_{NT}(M)) = \lambda - 1$ and $|T(M)| \geq 3$.

(2) $gr(G_{NT}(M)) = 4$ if and only if $b(G_{NT}(M)) = \lambda$ and $|T(M)| \geq 2$.

Proof. (1) If $gr(G_{NT}(M)) = 3$, then $2 \in Z(R)$ and $|T(M)| \geq 3$, by Proposition 4.3(1) . Since $2 \in Z(R)$ so $b(G_{NT}(M)) = \lambda - 1$, by Proposition 5.7 .

Conversely, let us assume that $b(G_{NT}(M)) = \lambda - 1$ and $|T(M)| \geq 3$. If $2 \notin Z(R)$, then $G_{NT}(M)$ is the union of $\dfrac{\mu - 1}{2}$ disjoint $K_{\lambda,\lambda}$'s, by Proposition 3.6(2) and hence $b(G_{NT}(M)) = \lambda$, a contradiction to the assumption. Therefore $2 \in Z(R)$, and then $gr(G_{NT}(M)) = 3$, by Proposition 4.3(1) .

(2) If $gr(G_{NT}(M)) = 4$, then $2 \notin Z(R)$ and $|T(M)| \geq 2$, by Proposition 4.3(2) . Since $2 \notin Z(R)$ so $b(G_{NT}(M)) = \lambda$, by Proposition 5.7 .

Conversely, let us suppose that $b(G_{NT}(M)) = \lambda$ and $|T(M)| \geq 2$. If $2 \in Z(R)$, then $b(G_{NT}(M)) = \lambda - 1$ by Proposition 3.6(1), a contradiction. So $2 \notin Z(R)$. Therefore, $G_{NT}(M)$ is the union of $K_{\lambda,\lambda}$'s, where $\lambda \geq 2$. Thus $gr(K_{\lambda,\lambda})$ and hence $gr(G_{NT}(M))$ is equal to 4. $\square$

Conflict of Interest:

On behalf of all authors, the corresponding author declares that there is no conflict of interest.

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