Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–21.
ISSN-0037-8712 in press
doi:10.5269/bspm.65454

On the Existence and Uniqueness Results for Fuzzy Fractional Boundary Value Problem Involving Caputo Fractional Derivative

AZIZ EL GHAZOUANI, FOUAD IBRAHIM ABDOU AMIR, M'HAMED ELOMARI and SAID MELLIANI

ABSTRACT: In this paper, we investigate the existence and uniqueness of solutions for fuzzy boundary value problems involving fuzzy Caputo fractional derivatives of order $q \in (2, 3)$. As a preliminary step, we construct a generic structure of the solution associated with our proposed model by utilizing the Green's function. We establish the existence of a unique solution of the proposed model paired with the given initial conditions by using Banach fixed point theorem. At last, as application, an illustrative example is given to show the applicability of our theoritical results.

Key Words: Fuzzy Metric Spaces, generalized Derivatives, generalized fractional Caputo derivatives, Green's function, fixed point theorem.

Contents

1 Introduction **1**

2 Preliminaries **2**
2.1 The metric space E^1 2
2.2 Zadeh extension Principe 4

3 Fuzzy derivative and integration **5**
3.1 Hukuhara generalized difference 5
3.2 Hukuhara's derivative 5
3.3 Fuzzy integration 6

4 Fuzzy fractional derivative **7**

5 Main Results **14**

6 Applications **19**

7 Conclusions **20**

1. Introduction

Fractional differential equations of non-integer order play an important role in describing physical phenomena more accurately than classical differential equations of integer order. In fact, fractional differential equations are real-order expansions of differential equations. One reason for the need for fractional differential equations is the fact that many phenomena cannot be modeled as integer differential equations. For this reason, in recent years, much attention has been paid to the consequences of the existence of solutions to differential equations of fractional order. This kind of fractional differential equation is applicable to many practical fields such as Physics of polymers, viscous materials, viscous damping and seismic analysis. you can see [23,17,10,18]. On the other hand, when analyzing real-world phenomena, uncertain factors must also be dealt with. Under these circumstances, Fuzzy set theory is one of the best non-statistical or non-random approaches leading to the theoretical study of fuzzy differential equations. Recently, the topic of existence and uniqueness of solutions of linear and nonlinear fuzzy fractional differential equations has been studied more extensively and discussed by many researchers in various aspects. For example, in [4], Arshad proved the existence and uniqueness of the solution

2010 *Mathematics Subject Classification:* 03E72, 34K36.
Submitted October 17, 2022. Published December 31, 2022

to Riemann-Liouville fuzzy fractional differential equation, and in [21] the existence and uniqueness of solutions, as well as approximate solutions of fractional fuzzy differential equations under Liouville-Caputo H differentiability, have been demonstrated by Salahshour and et al. Furthermore, the existence and uniqueness of solutions to fuzzy fractional differential equations under generalized Liouville-Caputo Hukuhara differentiability have been demonstrated by Allahviranloo et al. in [1]. In [8] Minhao Chen et al corrected the errors of [15] and presented the necessary conditions to demonstrate the existence and uniqueness of such problems.

Motivated by the above work and its approach, this article deals with the existence and uniqueness of solutions to the nonlinear fuzzy fractional differential equation of order $q \in (2,3)$ with fuzzy boundary conditions :

$$\begin{cases} D^q x(t) = f\left(t, x(t)\right), \quad 0 < t < a \\ x(a) = \tilde{0} \in E^1, x(a) = A \in E^1, \quad x(b) = B \in E^1 \quad 0 < b < a \end{cases} \tag{1.1}$$

Where $f : [0,a] \times E^1 \to E^1$ is a continuous map.

The main tools used are based on the Green function, Ascoli lemma and Banach fixed point theorem.

This article is organized as follows: After this introduction, we have presented some concepts related to fuzzy metric spaces. Fuzzy derivatives and integrals take place in part 3. In part 4, we'll learn about fuzzy fractional derivatives. The main results were then discussed in Section 5. We conclude our work with an example.

2. Preliminaries

In this section, we present some definitions and introduce essential symbols that will be used throughout the article.

2.1. The metric space E^1

Definition 2.1. *Consider E^1 as a function space defined as :*

$$E^1 = \{v : \mathbb{R} \to [0,1], \quad v \quad \textit{satisfies } (1-4) \textit{ below } \}$$

1. *v is normal, i.e. there is a $x_0 \in \mathbb{R}$ such that $v\left(x_0\right) = 1$;*
2. *v is a fuzzy convex set;*
3. *v is upper semi-continuous;*
4. *The closure of $\{x \in \mathbb{R}, \quad v(x) > 0\}$ is compact.*

For every $\alpha \in (0,1]$, the α-cut of the elements of E^1 is defined as

$$v^\alpha = \{x \in \mathbb{R}, v(x) \geq \alpha\}$$

Through the previous property, we can write

$$u^\alpha = [\underline{v}(\alpha), \bar{v}(\alpha)]$$

The distance between two elements of E^1 is (see [9])

$$D(u,v) = \sup_{\alpha \in (0,1]} \max\{|\bar{u}(\alpha) - \bar{v}(\alpha)|, |\underline{u}(\alpha) - \underline{v}(\alpha)|\}$$

Theorem 2.2. *The metric space $\left(E^1, d\right)$ is complete.*

Before proving the theorem 2.2, we announce the following theorem :

Theorem 2.3. (*Negoita-Ralescu theorem [6] page 56*)
Given a family of subsets $\{M_r : r \in [0,1]\}$ that satisfy conditions (i)-(iv).

(i) *For all* $r \in [0,1]$, M_r *is a non-empty closed interval;*

(ii) *If* $0 \leq r_1 \leq r_2 \leq 1$, *then* $M_{r_2} \subseteq M_{r_1}$;

(iii) *Any sequence* r_n *that converges from below to* $r \in (0,1]$ *we've got*

$$\bigcap_{n=1}^{\infty} M_{r_n} = M_r$$

(iv) *For every sequence* r_n *that converges to 0 from above, we have*

$$cl\left(\bigcup_{n=1}^{\infty} M_{r_n}\right) = M_0$$

Then for every $r \in [0,1]$ *there is a unique* $u \in \mathbb{R}_F$ *where* $u_r = M_r$.

Proof. (proof of 2.2)

To prove that $\left(E^1, d\right)$ is a complete metric space it must be shown that any Cauchy sequence of elements of E^1 is convergent to an element of E^1.

Let $(u_n)_{n\in\mathbb{N}}$ be the Cauchy sequence of the elements of E^1, we have :

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in N \text{ tell que } \forall p, q \in N, (p > q > N(\varepsilon)) \Longrightarrow d_\infty(u_p, u_q) < \varepsilon,$$

that is to say $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$ tell que $\forall p, q \in \mathbb{N}, (p > q > N(\varepsilon))$ we have

$$\sup_{r\in[0,1]} \max\left\{\left|(u_p)_r^- - (u_q)_r^-\right|, \left|(u_p)_r^+ - (u_q)_r^+\right|\right\} < \varepsilon,$$

which means that $\forall \varepsilon > 0, \exists N(\varepsilon) \in N$ such that $\forall p, q \in N, (p > q > N(\varepsilon))$,

$$\left|(u_p)_r^- - (u_q)_r^-\right| < \varepsilon \text{ and } \left|(u_p)_r^+ - (u_q)_r^+\right| < \varepsilon, \text{ for all } r \in [0,1].$$

Then the real sequences $\left((u_n)_r^-\right)_n$ and $\left((u_n)_r^+\right)_n$ are Cauchy for all $r \in [0,1]$, so they converge in $\mathbb{R}$ toward u_r^- and u_r^+ for all $r \in [0,1]$. Like $(u_r)_n^- < (u_r)_n^+$ for all $n \in \mathbb{N}$ and all $r \in [0,1]$, then by passing to the limit $u_r^- < u_r^+$, for all $r \in [0,1]$. As a result $M_r = [u_r^-, u_r^+]$ is an interval of $\mathbb{R}$. It remains for us to show that M_r is a r chopped off. For this, we show that M_r verifies the assumptions of Negoita-Ralescu characterization theorem. □

The following theorem is easily derived from Theorem 1 of [20] and Theorem 1.1 of [19].

Theorem 2.4. *Let* $u \in E^1$. $u^\alpha = [\underline{u}(\alpha), \bar{u}(\alpha)]$ *denotes the* α*-cut of* u. *For* $[0,1]$, *the following (1) to (3) hold.*

1. $\underline{u}(\alpha), \bar{u}(\alpha)$ *are in* $\mathcal{C}([0,1], \mathbb{R})$;
2. $\underline{u}(\alpha)$ *is monotonically increasing and* $\bar{u}(\alpha)$ *is monotonically decreasing.*
3. $\underline{u}(1) = \bar{u}(1)$

Conversely, if $i(\alpha), s(\alpha) : [0,1] \to \mathbb{R}$ satisfies the above conditions (1) to (3),

$$u(a) = \begin{cases} \sup\{\alpha \in [0,1] \mid i(\alpha) \leq a \leq s(\alpha)\}, & a \in [i(0), s(0)] \\ 0, \quad a \neq [i(0), a(0)] \end{cases} \tag{2.1}$$

Then there exists $u \in E^1$ such that $u^\alpha = [i(\alpha), s(\alpha)], \quad \forall \alpha \in [0,1]$.

Lemma 2.5. *[12] You can embed* $\mathbb{R}$ *into* E^1 *using the following map*

$$g\begin{cases} \mathbb{R} \to E^1 \\ r \to g_r : g_r(x) = \begin{cases} 1, x = r \\ 0, x \neq r \end{cases} \end{cases}$$

2.2. Zadeh extension Principe

Definition 2.6. *This principle makes it possible to extend an application* $f:\mathbf{X}_1\times\mathbf{X}_2\longrightarrow\mathbf{Y}$ *(where* $\mathbf{X}_1,\mathbf{X}_2,\mathbf{Y}$ *classical sets) in*

$$\begin{aligned}\tilde{f}:\mathbb{F}(\mathbf{X}_1)\times\mathbb{F}(\mathbf{X}_2)&\longrightarrow\mathbb{F}(\mathbf{Y})\\(\nu_1,\nu_2)&\longmapsto\tilde{f}(\nu_1,\nu_2),\end{aligned}$$

for all $y\in\mathbb{Y}$

$$\tilde{f}(\nu_1,\nu_2)(y)=\begin{cases}\sup\limits_{(x_1,x_2)\in f^{-1}(\{y\})}\{\nu_1(x_1)\wedge\nu_2(x_2)\} & \text{if} \quad f^{-1}(\{y\})\neq\emptyset\\ \\ 0 & \text{if}\quad f^{-1}(\{y\})=\emptyset.\end{cases}$$

a) Addition extension

Let $\mathbf{X}$ *be a vector space,*

$$\begin{aligned}+:\mathbf{X}\times\mathbf{X}&\longrightarrow\mathbf{X}\\(x,y)&\longmapsto x+y.\end{aligned}$$

According to Zadeh's principle:

$$\begin{aligned}\oplus:\mathbb{F}(\mathbf{X})\times\mathbb{F}(\mathbf{X})&\longrightarrow\mathbb{F}(\mathbf{x})\\(\nu_1,\nu_2)&\longmapsto\nu_1\oplus\nu_2,\end{aligned}$$

And for every $x\in\mathbf{X}$

$$\begin{aligned}(\nu_1\oplus\nu_2)(x)&=\sup_{(x_1,x_2)\in+^{-1}(\{x\})}\nu_1(x_1)\wedge\nu_2(x_2)\\&=\sup\nu_1(x_1)\wedge\nu_2(x_2),\\&=\sup_{x_1\in\mathbf{X}}\nu_1(x_1)\wedge\nu_2(x-x_1).\end{aligned}$$

Similarly for all $x\in\mathbf{X}$

$$(\nu_1\odot\nu_2)(x)=\sup_{x_1-x_2=x}\nu_1(x_1)\wedge\nu_2(x_2).$$

b) Multiplication by a scalar

$$\begin{aligned}f:\quad\mathbf{X}&\longrightarrow\mathbf{X}\\x&\longmapsto\lambda x\quad(\lambda\in\mathbb{K}).\end{aligned}$$

By the principe of Zadeh :

$$\begin{aligned}\tilde{f}:\mathbb{F}(\mathbf{X})&\longrightarrow\mathbb{F}(\mathbf{X})\\\nu&\longmapsto\tilde{f}(\mu)\quad\forall y\in\mathbf{X},\end{aligned}$$

$$\tilde{f}(\nu)(y)=\sup_{x\in f^{-1}(\{y\})}\nu(x)=\sup_{\lambda x=y}\nu(x)=\nu\left(\frac{y}{\lambda}\right).$$

Then, for all $\lambda\in\mathbb{K}$, $\forall\nu\in\mathbb{F}(\mathbf{X}),\forall x\in\mathbf{X}$,

$$(\lambda\odot\nu)(x)=\begin{cases}\nu\left(\frac{x}{\lambda}\right) & \text{if}\quad\lambda\neq0\\0 & \text{if}\quad\lambda=0.\end{cases}$$

By Zadeh's extension principle,

$$\begin{aligned}(\mu+\nu)^\alpha&=\mu^\alpha+\nu^\alpha\\(\lambda\mu)^\alpha&=\lambda\mu^\alpha\end{aligned}$$

For all $\mu,\nu\in E^1$ and $\lambda\in\mathbb{R}$.

Remark 2.7. $(E^1;\oplus;\odot)$ *is not a vector space.*

3. Fuzzy derivative and integration

This section presents some preliminary definitions and theorems about fuzzy set-valued functions.

3.1. Hukuhara generalized difference

Definition 3.1. *[5] The generalized Hukuhara difference of two fuzzy numbers $\mu, \nu \in E^1$ is defined as*

$$\mu \ominus_g \nu = \varpi \Leftrightarrow \begin{cases} \mu = \nu + \varpi \\ or \quad \nu = \mu + (-1)\varpi \end{cases}$$

For the α-levels,

$$(\mu \ominus_g \nu)^\alpha = [\min\{\underline{\mu}(\alpha) - \underline{\nu}(\alpha), \bar{\mu}(\alpha) - \bar{\nu}(\alpha)\}, \max\{\underline{\mu}(\alpha) - \underline{\nu}(\alpha), \bar{\mu}(\alpha) - \bar{\nu}(\alpha)\}]$$

and the existence condition of $\varpi = \mu \ominus_g \nu \in E^1$ are

$$\text{case (i)} \quad \begin{cases} \underline{\varpi}(\alpha) = \underline{\mu}(\alpha) - \underline{\nu}(\alpha) \text{ and } \overline{\varpi}(\alpha) = \overline{\mu}(\alpha) - \overline{\nu}(\alpha) \\ \text{with } \ \underline{\varpi}(\alpha) \text{ increasing, } \overline{\varpi}(\alpha) \text{ decreasing, } \underline{\varpi}(\alpha) \le \overline{\varpi}(\alpha) \end{cases} \tag{3.1}$$

$$\text{case (ii)} \quad \begin{cases} \underline{\varpi}(\alpha) = \overline{\mu}(\alpha) - \overline{\nu}(\alpha) \text{ and } \overline{\varpi}(\alpha) = \underline{\mu}(\alpha) - \underline{\nu}(\alpha) \\ \text{with } \ \underline{\varpi}(\alpha) \text{ increasing, } \overline{\varpi}(\alpha) \text{ decreasing, } \underline{\varpi}(\alpha) \le \overline{\varpi}(\alpha) \end{cases} \tag{3.2}$$

for every $\alpha \in [0, 1]$.

The following properties were obtained in [13]

Proposition 3.2. *(Stefanini [13])*
Let $\mu, \nu \in E^1$ be two fuzzy numbers, after that

(i) If there is a gH difference, it is unique.

(ii) $\mu \ominus_g \nu = \mu \ominus \nu$ or $\mu \ominus_g \nu = -(\nu \ominus \mu)$ whenever the expressions on the right exist; in particular, $\mu \odot_{gH} \mu = \mu \odot_H \mu = 0$,

(iii) if $\mu \ominus_g \nu$ exists in the sense (i), then $\nu \ominus_g \mu$ exists in the sense (ii) and vice versa,

(iv) $(\mu + \nu) \ominus_g \nu = \mu$,

(v) $0 \ominus_g (\mu \ominus_g \nu) = \nu \ominus_g \mu$,

(vi) $\mu \ominus_g \nu = \nu \ominus_g \mu = \varpi$ if and only if $\varpi = -\varpi$; furthermore, $\varpi = 0$ if and only if $\mu = \nu$.

Proposition 3.3. *[14]*

$$\|\mu \ominus_g \nu\| = d(\mu, \nu)$$

Theorem 3.4. *[16] The space $\left(E^1, \|.\|\right)$ is a linear normed space.*

For the rest of this work, we will assume $u \ominus_g v \in E^1$. Denotes $\|x\|_1 = d(x, \widehat{0})$ for every $x \in E^1$.

3.2. Hukuhara's derivative

Let $f : [a, b] \subset \mathbb{R} \to E^1$ be a fuzzy function. The α level of f is

$$f(x, \alpha) = \left[\underline{f}(x, \alpha), \overline{f}(x, \alpha)\right], \ \forall x \in [a, b], \ \forall \alpha \in [0, 1].$$

Definition 3.5. *[5] Let $x_0 \in (a, b)$ and h be such that $x_0 + h \in (a, b)$, then the generalized Hukuhara derivation of the fuzzy value function $f : (a, b) \to E^1$ at x_0 is*

$$\lim_{h \to 0} \left\| \frac{f(x_0 + h) -_g f(x_0)}{h} -_g f'_{gH}(x_0) \right\|_1 = 0 \tag{3.3}$$

If $f_{gH}(x_0) \in E^1$ satisfies (3.3) then we say that f is generalized Hukuhara differentiable at x_0.

Definition 3.6. *[5] Let $f:[a,b]\to E^1$ and $x_0\in(a,b)$, with $\underline{f}(x,\alpha)$ and $\bar{f}(x,\alpha)$ both are differentiable with respect to x_0.*

1. *f is $[(i)-gH]$-differentiable at x_0 if*

$$f'_{i,gH}(x_0)=\left[\underline{f}'(x,\alpha),\bar{f}'(x,\alpha)\right] \tag{3.4}$$

2. *f is $[(ii)-gH]$-differentiable at x_0 if*

$$f'_{ii,gH}(x_0)=\left[\bar{f}'(x,\alpha),\underline{f}'(x,\alpha)\right] \tag{3.5}$$

Theorem 3.7. *[2]Let $f:J\subset\mathbb{R}\to E^1$ and $\phi:J\to\mathbb{R}$ and $x\in J$. Suppose $\phi(x)$ is a function differentiable on x and the fuzzy-valued function $f(x)$ is gH differentiable on x. So*

$$(f\phi)'_g(x)=(f'\phi)_g(x)+(f\phi')_g(x)$$

Definition 3.8. *[22] Let $f:[a,b]\to E^1$ and $f'_gH(x)$ Also, (a,b) has no switch point, and $\underline{f}(x,\alpha)$ and $\overline{f}(x,\alpha)$ are both at x_0. Differentiable.*

- *f' is $[(i)-gH]$-differentiable at x_0 if*

$$f''_{i,gH}(x_0)=\left[\underline{f}''(x,\alpha),\overline{f}''(x,\alpha)\right]$$

- *f' is $[(ii)-gH]$-differentiable at x_0 if*

$$f''_{ii,gH}(x_0)=\left[\overline{f}''(x,\alpha),\underline{f}''(x,\alpha)\right]$$

3.3. Fuzzy integration

Definition 3.9. *[24] Let $f:[a,b]\to E^1$. $f(x)$ is fuzzy Riemann integrable on $I\in E^1$ for every $\epsilon>0$ there exists $\delta>0$ such that for any division $P=\{[u,v];\xi\}$ with the norms $\Delta(P)<\delta$,we have*

$$d\left(\sum_p^*(v-u)f(\xi),I\right)<\epsilon$$

Where $\sum_p^*$ stands for fuzzy sum.

Theorem 3.10. *[5] If f is gH differentiable and there are no switching points on the interval $[a,b]$, then*

$$\int_a^b f'(t)dt=f(b)\ominus_g f(a)$$

Theorem 3.11. *Let $f:E^1\to\mathbb{R}$ be a H-derivative function. $f'=0$ if only f is a constant function.*

Proof. By 3.10 it remain to prove that $\int_a^b f'(t)dt\neq\widetilde{0}$ when $a\neq b$. If $a\neq b$ then by Banach theorem there is $\varphi\in(E^1)^*$ such that $|||\varphi|||=1$ and $\varphi(f(b)\ominus_g f(a))=\|f(b)\ominus_g f(a)\|$. we consider the function $g(t)=\varphi(f(t)\ominus_g f(a))$, $\forall t\in[a,b]$. We have $g'(t)=\varphi(f'(t))=0$ this function is at real values and derivative which implies that $g=0$, we conclude that $g(b)=0$ which completes the proof. □

Theorem 3.12. *[11] Let $f(x)$ be a fuzzy function on $(-\infty,\infty)$, expressed as $f(x,\alpha)=\left[\underline{f}(x,\alpha),\overline{f}(x,\alpha)\right]$ for any fixed $\alpha\in[0,1]$. Assume that $|\underline{f}(x,\alpha)|$ and $|\overline{f}(x,\alpha)|$ are Riemann integrable on $(-\infty,\infty)$ for all $\alpha\in[0,1]$. Then $f(x)$ is improperly fuzzy Riemann integrable over $(-\infty,\infty)$ and the improperly fuzzy Riemann integral is a fuzzy number. we also have*

$$\int_{-\infty}^{\infty}f(x)dx=\left[\int_{-\infty}^{\infty}\underline{f}(x,\alpha)dx,\int_{-\infty}^{\infty}\overline{f}(x,\alpha)dx\right]$$

From this theorem, we can discuss improper integration of fuzzy Riemann.

Lemma 3.13. *Let $f : \mathbb{R} \times \mathbb{R}^+ \to E^1$, $f(x,t;\alpha) = [\underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$, and let $a \in \mathbb{R}^+$.*
If $\int_a^\infty \underline{f}(x,t;\alpha)dt$ and $\int_a^\infty \overline{f}(x,t;\alpha)dt$ are converges, then

$$\int_a^\infty f(x,t;\alpha)dt \in E^1$$

Proof. Just use the theorem 2.4. □

Theorem 3.14. *[7] Let $f : \mathbb{R} \times \mathbb{R}^+ \to E^1$ be fuzzy value function such that $f(x,t;\alpha) = [\underline{f}(x,t;\alpha), \overline{f}(x,t;\alpha)]$. For each $x \in [a,\infty)$, the fuzzy integral $\int_c^\infty f(x,t)dt$ is convergent and also $\int_a^\infty f(x,t)dx$ as a function of t is convergent on $[c,\infty)$. Then*

$$\int_c^\infty \int_a^\infty f(x,t)dxdt = \int_a^\infty \int_c^\infty f(x,t)dtdx$$

Theorem 3.15. *Suppose both, $f(x,t)$ and $\partial_{x_{gH}} f(x,t)$, are fuzzy continuous in $[a,b] \times [c,\infty)$. Moreover, the integral converges for $x \in \mathbb{R}$, and the integral $\int_c^\infty f(x,t)dt$ uniformly converges on $[a,b]$. Then F is gH-differentiable on $[a,b]$ and*

$$F'_{gH}(x) = \int_c^\infty \partial_{x_{gH}} f(x,t)dt$$

Proof. By the convergence domaine theorem of $\underline{f}(x,t;\alpha)$ and $\overline{f}(x,t;\alpha)$, the continuity of $\partial_{x_{gH}} f(x,t)$ on $[a,b]$ and use the condition (3.1). □

Theorem 3.16. *[2] Let $g : [a,b] \to E^1$ and $f : [a,b] \to \mathbb{R}$ are two differentiable functions, then*

$$\int_a^b g'_{gH}(x)f(x)dx = ((g(b) \odot f(b)) \ominus_g (g(a) \odot f(a))) \ominus_g \int_a^b g(x)f'(x)dx$$

Remark 3.17. *[2] If $f, g \in A^{E^1}$ with $\lim\limits_{|y|\to\infty} f(y) = 0$, $\lim_{|y|\to\infty} g(y) = 0$ then*

$$\int_{-\infty}^\infty f'_{gH}(y)g(y)dx = \int_{-\infty}^\infty f(y)g'(y)dx$$

4. Fuzzy fractional derivative

We present the derivation of generalized fuzzy fractions and their properties.

Definition 4.1. *[3] Let $f \in L^{E^1}([a,b])$. The fuzzy Riemann-Liouville integral of the fuzzy function f is defined as*

$$I^q_{RL} f(t) = \frac{1}{\Gamma(q)} \odot \int_a^t (t-s)^{q-1} \odot f(s)ds, \quad a < s < t, \quad 0 < q < 1 \tag{4.1}$$

Definition 4.2. *[2] **[Riemann-Liouville fractional derivative RL]***
Let $f \in L^{E^1}([a,b])$ be a fuzzy value function,

$$D^q_{RL_{gH}} f(s) = \begin{cases} \dfrac{1}{\Gamma(n-q)} \odot \left(\dfrac{d}{ds}\right)^n \displaystyle\int_a^s (s-t)^{n-q-1} \odot f(t)dt, & n-1 < q < n \\ \left(\dfrac{d}{ds}\right)^{n-1} f(s) & , \quad q = n-1 \end{cases} \tag{4.2}$$

In other words forall $q \in (\mathbf{n-1}, \mathbf{n})$

$$D^q_{RL_{gH}} f(t) = \left(\boldsymbol{I}^{n-q}_{RL} f(t)\right)^{(n)}, t \in [a,b]$$

The second derivative in the sense of gH-differentiability exists. So these limits exist.

$$\left(\boldsymbol{I}^{n-q}_{RL} f(t)\right)^{(n)} = \lim_{h\to 0} \frac{\left(I^{n-q}_{RL} f\right)^{(n-1)} (t+h) \ominus_{gH} \left(I^{2-q}_{RL} f\right)^{(n-1)} (t)}{h}$$

$$\left(\boldsymbol{I}^{n-q}_{RL} f(t)\right)^{(n-1)} = \lim_{h\to 0} \frac{\left(I^{n-q}_{RL} f\right)^{(n-2)} (t+h) \ominus_{gH} \left(I^{2-q}_{RL} f\right)^{(n-2)} (t)}{h}$$

Analogiquly

$$\left(\boldsymbol{I}^{n-q}_{RL} f(t)\right)'' = \lim_{h\to 0} \frac{\left(I^{n-q}_{RL} f\right)' (t+h) \ominus_{gH} \left(I^{2-q}_{RL} f\right)' (t)}{h}$$

$$\left(\boldsymbol{I}^{n-q}_{RL} f(t)\right)' = \lim_{h\to 0} \frac{I^{n-q}_{RL} f(t+h) \ominus_{gH} I^{2-q}_{RL} f(t)}{h}$$

Where

$$I^{n-q}_{RL} f(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} \odot f(s) ds, t \in [t_0, T]$$

Proposition 4.3. *Let $f \in L^{E^1}([a,b])$ be a fuzzy value function, the following equality hold*

$$D^q_{RL_{gH}} I^q_{RL} f(t) = f(t)$$

$$D^q_{RL_{gH}} I^p_{RL} f(t) = I^{p-q}_{RL} f(t), \quad p > q$$

Proof. For the first one, by using the definition,

$$\boldsymbol{D}^q_{RL_{gH}} \boldsymbol{I}^q_{RL} f(t) = \boldsymbol{D}^q_{RL_{gH}} \left(\boldsymbol{I}^q_{RL} f(t)\right) = \left(\boldsymbol{I}^{n-q}_{RL} \boldsymbol{I}^q_{RL} f(t)\right)^{(n)} = \left(\boldsymbol{I}^{(n)}_{RL} f(t)\right)^{(n)}$$

And

$$\left(\boldsymbol{I}^{(n)}_{RL} x(t)\right)^{(n)} = \left(\int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} f(u) du dt_{n-1} \cdots dt_1\right)^{(n)}$$

Finally,

$$\boldsymbol{D}^q_{RL_{gH}} \boldsymbol{I}^q_{RL} f(t) = \left(\int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} f(u) du dt_{n-1} \cdots dt_1\right)^{(n)} = f(t)$$

For the second equality, we have

$$\begin{aligned}
\boldsymbol{D}^q_{RL_{gH}} \boldsymbol{I}^p_{RL} f(t) &= \left(\boldsymbol{I}^{n-q}_{RL} \boldsymbol{I}^p_{RL} x(t)\right)^{(n)} \\
&= \frac{1}{\Gamma(n-q)} \odot \left(\int_a^t (t-s)^{n-q-1} \odot \boldsymbol{I}^p_{RL} f(s) ds\right)^{(n)} \\
&= \frac{1}{\Gamma(n-q)\Gamma(p)} \odot \left(\int_a^t (t-s)^{n-q-1} \int_a^s (s-u)^{p-1} \odot f(u) du ds\right)^{(n)}
\end{aligned}$$

By the Dirichlet's formula, we have

$$\boldsymbol{D}^q_{RL_{gH}}\boldsymbol{I}^p_{RL}f(t) = \frac{1}{\Gamma(n-q)\Gamma(p)} \odot \left(\int_a^t f(u) \odot \int_u^t (t-s)^{n-q-1} \odot (s-u)^{p-1} ds du\right)^{(n)}_{gH}$$

$$= \frac{1}{\Gamma(n-q)\Gamma(p)} \odot \left(\int_a^t f(s) \odot \int_s^t (t-\tau)^{n-q-1} \odot (\tau-s)^{p-1} d\tau ds\right)^{(n)}_{gH}$$

If we suppose $x = \dfrac{\tau - s}{t-s}$ then

$$\begin{cases} \text{if} \quad \tau = s \quad \Rightarrow \quad x = 0 \\ \text{if} \quad \tau = t \quad \Rightarrow \quad x = 1 \end{cases} \quad \text{and} \quad d\tau = (t-s)dx$$

Also

$$\begin{cases} (\tau - t) = (t-s)(1-x) \\ (\tau - s) = x(t-s) \end{cases}$$

we obtain

$$\boldsymbol{D}^q_{RL_{gH}}\boldsymbol{I}^p_{RL}f(t) = \frac{1}{\Gamma(n-q)\Gamma(p)} \odot \left(\int_a^t f(s) \odot (t-s)^{n-q+p-1} ds \int_0^1 x^{p-1}(1-x)^{n-q-1} dx\right)^{(n)}_{gH}$$

On the other hand,

$$\int_0^1 x^{p-1}(1-x)^{n-q-1} dx = \frac{\Gamma(n-q)\Gamma(p)}{\Gamma(p-q+n)}$$

Then,

$$\boldsymbol{D}^q_{RL_{gH}}\boldsymbol{I}^p_{RL}f(t) = \frac{1}{\Gamma(p-q+n)} \odot \left(\int_a^t f(s) \odot (t-s)^{n-q+p-1} ds\right)^{(n)}_{gH}$$

Now we are going to find the, $\left(\int_a^t f(s) \odot (t-s)^{n-q+p-1} ds\right)^{(n)}_{gH}$.

We have, first derivative

$$\left(\int_a^t f(s) \odot (t-s)^{n-q+p-1} ds\right)' = \int_a^t (n-q+p-1) f(s) \odot (t-s)^{n-q+p-2} ds$$

Second derivative,

$$\left(\int_a^t (n-q+p-1) f(s) \odot (t-s)^{n-q+p-2} ds\right)' = \int_a^t (n-q+p-1)(n-q+p-2) f(s) \odot (t-s)^{n-q+p-3} ds.$$

nth derivative,

$$\left(\int_a^t (n-q+p-1)\cdots(p-q+1) f(s) \odot (t-s)^{p-q} ds\right)'$$
$$= \int_a^t (n-q+p-1)\cdots(p-q+1)(p-q) f(s) \odot (t-s)^{p-q-1} ds,$$

By substituting in,

$$\boldsymbol{D}^q_{RL_{gH}}\boldsymbol{I}^p_{RL}f(t) = \frac{1}{\Gamma(p-q+n)} \odot \left(\int_a^t f(s) \odot (t-s)^{n-q+p-1}ds\right)^{(n)}_{gH}$$

$$= \frac{(p-q+n-1)\cdots(p-q+1)(p-q)}{\Gamma(p-q+n)} \int_a^t f(s) \odot (t-s)^{p-q-1}ds.$$

Since,

$$\frac{(p-q+n-1)\cdots(p-q+1)(p-q)}{\Gamma(p-q+n)} = \frac{1}{\Gamma(p-q)}$$

$$\boldsymbol{D}^q_{\boldsymbol{RL}_{gH}}\boldsymbol{I}^p_{\boldsymbol{RL}}f(t) = \frac{1}{\Gamma(p-q)} \odot \int_a^t (t-s)^{p-q-1} \odot f(s)ds = \boldsymbol{I}^{p-q}_{RL}f(t).$$

Definition 4.4. *[2] The definition of the RL fractional derivative assumes that the integer order of the derivative is the operator in the integral and in the operand functions* $f(t) \in E^1, t \in [a,b]$.

$${}^C_{gH}D^q f(s) = \begin{cases} \dfrac{1}{\Gamma(n-q)} \odot \displaystyle\int_a^s (s-t)^{n-q-1} \odot f^{(n)}_{gH}(t)dt, & n-1 < q < n \\ \left(\dfrac{d}{ds}\right)^{n-1} f(s) & , \quad q = n-1 \end{cases} \tag{4.3}$$

In other words for all $q \in (n-1,n)$ and $t \in [a,b]$, we have

$${}^C_{gH}D^q f(t) = \boldsymbol{D}^q_{\boldsymbol{RL}_{gH}}\left(f(t) \ominus_{gH} f(a) \ominus_{gH} (t-a) \odot f'(a) \ominus_{gH} \cdots \ominus_{gH} \frac{(t-a)^{(n-1)}}{(n-1)!} \odot f^{(n-1)}(a)\right),$$

We get the following relations :

$$\int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} f^{(n)}(u)du dt_{n-1} \cdots dt_1 = f(t) \ominus_{gH} f(a) \ominus_{gH} (t-a) \odot f'(a) \ominus_{gH} \cdots \ominus_{gH} \frac{(t-a)^{(n-1)}}{(n-1)!} \odot f^{(n-1)}(a).$$

By substituting,

$$\boldsymbol{D}^q_{\boldsymbol{RL}_{gH}}\left(\int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} f^{(n)}(u)du dt_{n-1} \cdots dt_1\right) = \boldsymbol{D}^q_{\boldsymbol{RL}_{gH}}\Big(f(t) \ominus_{gH} f(a) \ominus_{gH} (t-a) \odot f'(a) \ominus_{gH} \cdots \ominus_{gH} \frac{(t-a)^{(n-1)}}{(n-1)!} \odot f^{(n-1)}(a)\Big).$$

Indeed,

$${}^C_{gH}D^q f(t) = \boldsymbol{D}^q_{\boldsymbol{RL}_{gH}}\left(\int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} f^{(n)}(u)du dt_{n-1} \cdots dt_1\right)$$

$$= \boldsymbol{D}^q_{\boldsymbol{RL}_{gH}}\left(\boldsymbol{I}^n_{\boldsymbol{RL}} f^{(n)}(t)\right)$$

$$= \boldsymbol{I}^{n-q}_{\boldsymbol{RL}} f^{(n)}(t).$$

Finally,

$$ {}_{gH}^{C}D^q f(t) = I_{RL}^{n-q} f^{(n)}(t) = \frac{1}{\Gamma(n-q)} \int_a^t (t-s)^{n-q-1} \odot f^{(n)}(s)ds. $$

□

Lemma 4.5. Let $f \in A^{E^1}$ and $q \in (0,1)$ thus $f = [\underline{f}, \overline{f}]$ then

1. If f is $[(i)-gH]$-differentiable with respect to t_0 then $D^q f$ is $[(i)-gH]$ differentiable with respect to t_0 and $f' = [\underline{f'}, \overline{f'}]$ and $D^q f = [D^q \underline{f}, D^q \overline{f}]$.
2. If f is $[(ii)-gH]$-differentiable with respect to t_0 then $D^q f$ is $[(ii)-gH]$ differentiable with respect to t_0 and $f' = [\overline{f'}, \underline{f'}]$ and $D^q f = [D^q \overline{f}, D^q \underline{f}]$.

Proof. 1. Suppose that f is $[(i)-gH]$-differentiable with respect to t_0, Note that

$$ {}_{gH}D^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f'_{gH}(s)ds. $$

Since $\frac{1}{\Gamma(1-q)}(t-s)^{-q}$ s always a non-negative quantity for $0 < t < s$ then ${}_{gH}D^q f(t)$ is $[(i)-gH]$ differentiable with respect to t_0.

Moreover, we have

$$ f' = [\underline{f'}, \overline{f'}] $$
$$ \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f'_{gH}(s)ds = [\frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \underline{f'}(s)ds, \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \overline{f'}(s)ds] $$
$$ D^q f = [D^q \underline{f}, D^q \overline{f}]. $$

we do the same for point 2.

Theorem 4.6. Let $f \in A^{E^1}$ and $q \in (1,2)$, then

$$ {}_{gH}D^q f(t) = {}_{gH}D^{q-1} f'_{gH}(t). $$

Proof. Set $f(t) = [\underline{f}(t;\alpha), \bar{f}(t;\alpha)]$ and use lemma (4.5)

case 1. If f is [(i)]-differentiable and f' is [(i)]-differentiable then

$$ f(t)' = \left[\underline{f}'(t;\alpha), \bar{f}'(t;\alpha)\right] \text{ and } f(t)'' = \left[\underline{f}''(t;\alpha), \bar{f}''(t;\alpha)\right]. $$

Note that

$$ {}_{gH}D^q f(t) = \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} f''_{gH}(s)ds, $$

we have

$$ f(t)'' = \left[\underline{f}''(t;\alpha), \bar{f}''(t;\alpha)\right] $$
$$ \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} f''_{gH}(s)ds = \left[\frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} \underline{f}''(s;\alpha)ds, \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} \bar{f}''(s;\alpha)ds\right]. $$
$$ {}_{gH}D^q f(t) = \left[{}_{gH}D^q \underline{f}(t;\alpha), {}_{gH}D^q \overline{f}(t;\alpha)\right]. $$

Moreover, since f' is [(i)-differentiable] and $0 \le q-1 \le 1$ then

$$ f(t)' = \left[\underline{f}'(t;\alpha), \bar{f}'(t;\alpha)\right] $$
$$ {}_{gH}D^{q-1} f(t)' = \left[{}_{gH}D^{q-1} \underline{f}'(t;\alpha), {}_{gH}D^{q-1} \overline{f}'(t;\alpha)\right]. $$

$$ {}_{gH}D^{q-1}f(t)' = \left[{}_{gH}D^{q}\underline{f}(t;\alpha), {}_{gH}D^{q}\overline{f}(t;\alpha)\right], \tag{4.4} $$

thus, by (4) and (4.4) we obtain ${}_{gH}D^{q-1}f(t)' = {}_{gH}D^{q}f(t)$

case 2. If f is [(i)]-differentiable and f' is [(ii)]-differentiable then

$$ f(t)' = \left[\underline{f}'(t;\alpha), \bar{f}'(t;\alpha)\right] \text{ and } f(t)'' = \left[\overline{f}''(t;\alpha), \underline{f}''(t;\alpha)\right]. $$

Note that

$$ {}_{gH}D^{q}f(t) = \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} f''_{gH}(s)ds, $$

We have

$$ f(t)'' = \left[\overline{f}''(t;\alpha), \underline{f}''(t;\alpha)\right]. $$

Which implies

$$ \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} f''_{gH}(s)ds = \left[\frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q}\overline{f}''(s;\alpha)ds, \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q}\underline{f}''(s;\alpha)ds\right]. $$

So

$$ {}_{gH}D^{q}f(t) = \left[{}_{gH}D^{q}\overline{f}(t;\alpha), {}_{gH}D^{q}\underline{f}(t;\alpha)\right]. \tag{4.5} $$

Moreover, since f' is [(ii)-differentiable] and $0 \le q-1 \le 1$ then

$$ f(t)' = \left[\underline{f}'(t;\alpha), \overline{f}'(t;\alpha)\right] $$

$$ {}_{gH}D^{q-1}f(t)' = \left[{}_{gH}D^{q-1}\overline{f}'(t;\alpha), {}_{gH}D^{q-1}\underline{f}'(t;\alpha)\right]. $$

$$ {}_{gH}D^{q-1}f(t)' = \left[{}_{gH}D^{q}\overline{f}(t;\alpha), {}_{gH}D^{q}\underline{f}(t;\alpha)\right]. \tag{4.6} $$

Thus, by (4.5) and (4.6) we obtain ${}_{gH}D^{q-1}f(t)' = {}_{gH}D^{q}f(t)$

Case 3. If f is [(ii)-differentiable] and f' is [(i)]-differentiable then

$$ f(t)' = \left[\overline{f}'(t;\alpha), \underline{f}'(t;\alpha)\right] \text{ and } f(t)'' = \left[\overline{f}''(t;\alpha), \underline{f}''(t;\alpha)\right]. $$

Note that

$$ {}_{gH}D^{q}f(t) = \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} f''_{gH}(s)ds. $$

We have

$$ f(t)'' = \left[\overline{f}''(t;\alpha), \underline{f}''(t;\alpha)\right] $$

$$ \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} f''_{gH}(s)ds = \left[\frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q}\overline{f}''(s;\alpha)ds, \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q}\underline{f}''(s;\alpha)ds\right]. $$

$$ {}_{gH}D^{q}f(t) = \left[{}_{gH}D^{q}\overline{f}(t;\alpha), {}_{gH}D^{q}\underline{f}(t;\alpha)\right]. \tag{4.7} $$

Moreover, since f' is [(i)-differentiable] and $0 \le q-1 \le 1$, then

$$ f(t)' = \left[\overline{f}'(t;\alpha), \underline{f}'(t;\alpha)\right] $$

$$ {}_{gH}D^{q-1}f(t)' = \left[{}_{gH}D^{q-1}\overline{f}'(t;\alpha), {}_{gH}D^{q-1}\underline{f}'(t;\alpha)\right]. $$

$$_{gH}D^{q-1}f(t)' = \left[_{gH}D^{q}\overline{f}(t;\alpha), {}_{gH}D^{q}\underline{f}(t;\alpha)\right], \tag{4.8}$$

thus, by (4.7) and (4.8) we obtain $_{gH}D^{q-1}f(t)' = {}_{gH}D^{q}f(t)$

case 4. If f is [(ii)]-differentiable and f' is [(ii)]-differentiable, then

$$f(t)' = \left[\overline{f}'(t;\alpha), \underline{f}'(t;\alpha)\right] \text{ and } f(t)'' = \left[\underline{f}''(t;\alpha), \overline{f}''(t;\alpha)\right].$$

Note that

$$_{gH}D^{q}f(t) = \frac{1}{\Gamma(2-q)}\int_0^t (t-s)^{1-q} f''_{gH}(s)ds.$$

We have

$$f(t)'' = \left[\underline{f}''(t;\alpha), \bar{f}''(t;\alpha)\right].$$

Which implies

$$\frac{1}{\Gamma(2-q)}\int_0^t (t-s)^{1-q} f''_{gH}(s)ds = \left[\frac{1}{\Gamma(2-q)}\int_0^t (t-s)^{1-q}\underline{f}''(s;\alpha)ds, \frac{1}{\Gamma(2-q)}\int_0^t (t-s)^{1-q}\bar{f}''(s;\alpha)ds\right],$$

$$_{gH}D^{q}f(t) = \left[_{gH}D^{q}\underline{f}(t;\alpha), {}_{gH}D^{q}\overline{f}(t;\alpha)\right]. \tag{4.9}$$

Moreover, since f' is [(ii)-differentiable] and $0 \le q-1 \le 1$ then

$$f(t)' = \left[\overline{f}'(t;\alpha), \underline{f}'(t;\alpha)\right]$$
$$_{gH}D^{q-1}f(t)' = \left[_{gH}D^{q-1}\underline{f}'(t;\alpha), {}_{gH}D^{q-1}\overline{f}'(t;\alpha)\right]$$

$$_{gH}D^{q-1}f(t)' = \left[_{gH}D^{q}\underline{f}(t;\alpha), {}_{gH}D^{q}\overline{f}(t;\alpha)\right] \tag{4.10}$$

thus, by (4.9) and (4.10) we obtain $_{gH}D^{q-1}f(t)' = {}_{gH}D^{q}f(t)$

from where, forall $q \in (1,2)$ we have $_{gH}D^{q}f(t) = {}_{gH}D^{q-1}f'_{gH}(t)$

Lemma 4.7. Let $f \in L^{E^1}$, $\forall t \in [0,b]$ and $\forall q \in (n-1,n)$ we have

(i) $^{C}D^{q}I^{q}f(t) = f(t)$

(ii) $I^{q\,C}D^{q}f(t) = f(t) \ominus_{gH} f(0) \ominus_{gH} (t) \odot f'(0) \ominus_{gH} \cdots \ominus_{gH} \frac{(t)^{(n-1)}}{(n-1)!} \odot f^{(n-1)}(0).$

Proof. . For the first one, by using the definition, we have

$$^{C}_{gH}D^{q}f(t) = \boldsymbol{D}^{q}_{R\boldsymbol{L}_{gH}}\left(f(t) \ominus_{gH} f(0) \ominus_{gH} (t) \odot f'(0) \ominus_{gH} \cdots \ominus_{gH} \frac{(t)^{(n-1)}}{(n-1)!} \odot f^{(n-1)}(0)\right),$$

Which implies that

$$\begin{aligned} ^{C}_{gH}D^{q}I^{q}f(t) &= \boldsymbol{D}^{q}_{R\boldsymbol{L}_{gH}}\left(I^{q}f(t) \ominus_{gH} I^{q}f(0) \ominus_{gH} (t) \odot I^{q}f'(0) \ominus_{gH} \cdots \ominus_{gH} \frac{(t)^{(n-1)}}{(n-1)!} \odot I^{q}f^{(n-1)}(a)\right) \\ &= \boldsymbol{D}^{q}_{R\boldsymbol{L}_{gH}}\left(I^{q}f(t)\right) = f(t) \end{aligned}$$

Indeed, forall $f \in L^{E^1}$,it exists a constant K such that

$$K = \sup_{t\in[a,b]} D[f^{(n)}(t), \hat{0}]$$

then

$$D_0\left[I^q f^{(n)}(t),\hat{0}\right] \leq K\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds = K\frac{(t)^q}{\Gamma(1+q)}$$

Which implies that, $I^q f(t) = I^q f'(t) = \cdots = I^q f^{(n-1)}(t) = 0$ in $t=0$.

□

The second property, forall $q \in (n-1,n)$ we have

$${}_{gH}^{C}D^q f(t) = \boldsymbol{D}^q_{R\boldsymbol{L}_{gH}}\left(f(t)\ominus_{gH} f(0)\ominus_{gH}(t)\odot f'(0)\ominus_{gH}\cdots\ominus_{gH}\frac{(t)^{(n-1)}}{(n-1)!}\odot f^{(n-1)}(0)\right),$$

Using the fractional integration of Riemann–Liouville in both sides, we have

$$\begin{aligned} I^q{}_{gH}^{C}D^q f(t) &= I^q\boldsymbol{D}^q_{R\boldsymbol{L}_{gH}}\left(f(t)\ominus_{gH} f(0)\ominus_{gH}(t)\odot f'(0)\ominus_{gH}\cdots\ominus_{gH}\frac{(t)^{(n-1)}}{(n-1)!}\odot f^{(n-1)}(0)\right)\\ &= f(t)\ominus_{gH} f(0)\ominus_{gH}(t)\odot f'(0)\ominus_{gH}\cdots\ominus_{gH}\frac{(t)^{(n-1)}}{(n-1)!}\odot f^{(n-1)}(0)\end{aligned}$$

□

5. Main Results

First, we prove the following lemma for explicit solution formulas for linear fractional problems subject to fractional-integer boundary conditions.

Lemma 5.1. *We consider the following initial-type problem of Caputo type fractional fuzzy differential equation with the non-integer order* $q\in]2,3[$

$$\begin{cases} D^q u(t) = f(t,u(t)), \quad 0<t<a \\ u(a)=0\in E^1, u(a)=A\in E^1, \quad u(b)=B\in E^1 \quad 0<b<a \end{cases} \tag{5.1}$$

Where $f:[0,a]\times E^1\to E^1$ *is a continuous and* $q\in]2,3[$.

Consider the equation

$$u(t) = \frac{a^2\odot B\ominus b^2\odot A}{a^2b-ab^2}\odot t\oplus\frac{a\odot B\ominus b\odot A}{ab^2-ba^2}\odot t^2\oplus\int_0^a G(t,s)\odot f(s,u(s))ds \tag{5.2}$$

where

$$G(t,s)=\begin{cases} -\dfrac{1}{\Gamma(q)}\left(-(t-s)^{q-1}-\dfrac{bt}{a(b-a)}(a-s)^{q-1}+\dfrac{at}{b(b-a)}(b-s)^{q-1}-\dfrac{t^2}{a(a-b)}(a-s)^{q-1}\right.\\ \left.+\dfrac{t^2}{b(a-b)}(b-s)^{q-1}\right), \quad 0<s<t<b<a. \\ -\dfrac{1}{\Gamma(q)}\left(-\dfrac{bt}{a(b-a)}(a-s)^{q-1}+\dfrac{at}{b(b-a)}(b-s)^{q-1}-\dfrac{t^2}{a(a-b)}(a-s)^{q-1}\right.\\ \left.+\dfrac{t^2}{b(a-b)}(b-s)^{q-1}\right), \quad 0<t<s<b<a. \\ -\dfrac{1}{\Gamma(q)}\left(\dfrac{bt}{a(b-a)}(a-s)^{q-1}+\dfrac{t^2}{a(a-b)}(a-s)^{q-1}\right), \quad 0<t<b<s<a. \end{cases}$$

Here $G(t,s)$ *is called the Green's function associated with the boundary value problem (5.1).*

Proof. Consider the following problem :

$$\begin{cases} D^q u(t) = f(t, u(t)), & 0 < t < a \\ u(0) = 0, \quad u(a) = A \in E^1; \quad u(b) = B \in E^1 \quad 0 < b < a \quad 2 < q < 3. \end{cases} \tag{5.3}$$

The first equation in this problem is equivalent to the following integral equation

$$I^q D^q u(t) = u(t) \ominus_{gh} u(0) \ominus_{gh} t \odot u'(0) \ominus_{gh} \frac{t^2}{2} \odot u''(0)$$

$$= I^q f(t, u(t)).$$

Which give

$$u(t) = u(0) + t \odot u'(0) + \frac{t^2}{2} \odot u''(0) + I^q f(t, u(t)). \tag{5.4}$$

Looking for $u'(0)$ and $u''(0)$

For $t = 0$, we have : $u(0) = 0$.

For $t = b$, we have

$$\begin{aligned} u(b) &= u(0) + b \odot u'(0) + \frac{b^2}{2} \odot u''(0) + I^q f(b, u(b)) \\ &= u(0) + b \odot u'(0) + \frac{b^2}{2} \odot u''(0) + \frac{1}{\Gamma(q)} \odot \int_0^b (b - s)^{q-1} \odot f(s, u(s))ds, \end{aligned}$$

which implies

$$b \odot u'(0) + \frac{b^2}{2} \odot u''(0) = B \ominus \frac{1}{\Gamma(q)} \odot \int_0^b (b - s)^{q-1} \odot f(s, u(s))ds. \tag{5.5}$$

For $t = a$, we obtain

$$u(a) = u(0) + a \odot u'(0) + \frac{a^2}{2} \odot u''(0) + I^q f(a, u(a))$$

$$= u(0) + a \odot u'(0) + \frac{a^2}{2} \odot u''(0) + \frac{1}{\Gamma(q)} \odot \int_0^a (a - s)^{q-1} \odot f(s, u(s))ds,$$

which implies

$$a \odot u'(0) + \frac{a^2}{2} \odot u''(0) = A \ominus \frac{1}{\Gamma(q)} \odot \int_0^a (a - s)^{q-1} \odot f(s, u(s))ds. \tag{5.6}$$

Doing $a \odot$ (5.5) $\ominus\, b \odot$ (5.6), we find

$$\begin{aligned} &\left(\frac{ab^2}{2} - \frac{ba^2}{2}\right) \odot u''(0) = a \odot B \ominus b \odot A \\ &\quad \ominus \frac{1}{\Gamma(q)} \odot \left[\int_0^b \left(a(b-s)^{q-1} \ominus b(a-s)^{q-1}\right) \odot f(s, u(s))ds \ominus \int_b^a b(a-s)^{q-1} \odot f(s, u(s))ds\right]. \end{aligned}$$

Thus

$$\begin{aligned} u''(0) = &\frac{2}{ab^2 - ba^2} (a \odot B \ominus b \odot A \\ &\ominus \frac{1}{\Gamma(q)} \odot \left[\int_0^b \left(a(b-s)^{q-1} \ominus b(a-s)^{q-1}\right) \odot f(s, u(s))ds \ominus \int_b^a b(a-s)^{q-1} \odot f(s, u(s))ds\right]\Bigg). \end{aligned} \tag{5.7}$$

And $a^2 \odot (5.5) \ominus b^2 \odot (5.6)$ given

$$\begin{aligned}\left(a^2b - ab^2\right) \odot u'(0) &= a^2 \odot B \ominus b^2 \odot A \\ &\ominus \frac{1}{\Gamma(q)} \odot \left[\int_0^b \left(a^2(b-s)^{q-1} \ominus b^2(a-s)^{q-1}\right) \odot f(s,u(s))ds \ominus \int_b^a b^2(a-s)^{q-1} \odot f(s,u(s))ds\right].\end{aligned}$$

Thus

$$\begin{aligned}u'(0) = &\frac{1}{a^2b - ab^2}\Big(a^2 \odot B \ominus b^2 \odot A \\ &\ominus \frac{1}{\Gamma(q)} \odot \left[\int_0^b \left(a^2(b-s)^{q-1} \ominus b^2(a-s)^{q-1}\right) \odot f(s,u(s))ds \ominus \int_b^a b^2(a-s)^{q-1} \odot f(s,u(s))ds\right]\Big).\end{aligned} \tag{5.8}$$

We substitute 5.7 and 5.8 in 5.4 we get

$$\begin{aligned}u(t) = &\frac{t}{a^2b - ab^2} \odot \Big(a^2 \odot B \ominus b^2 \odot A \\ &\ominus \frac{1}{\Gamma(q)} \odot \left[\int_0^b \left(a^2(b-s)^{q-1} \ominus b^2(a-s)^{q-1}\right) \odot f(s,u(s))ds \ominus \int_b^a b^2(a-s)^{q-1} \odot f(s,u(s))ds\right]\Big) \\ &+ \frac{t^2}{ab^2 - ba^2} \odot \Big(a \odot B \ominus b \odot A \\ &\ominus \frac{1}{\Gamma(q)} \odot \left[\int_0^b \left(a(b-s)^{q-1} \ominus b(a-s)^{q-1}\right) \odot f(s,u(s))ds \ominus \int_b^a b(a-s)^{q-1} \odot f(s,u(s))ds\right]\Big) \\ &\oplus I^q f(t,u(t)) \\ &= \frac{a^2 \odot B \ominus b^2 \odot A}{a^2b - ab^2} \odot t + \frac{a \odot B \ominus b \odot A}{ab^2 - ba^2} \odot t^2 \\ &\ominus \frac{1}{\Gamma(q)} \odot \int_0^b \Big(\frac{a^2t}{a^2b - ab^2} \odot (b-s)^{q-1} \ominus \frac{b^2t}{a^2b - ab^2} \odot (a-s)^{q-1} \\ &\oplus \frac{at^2}{ab^2 - ba^2} \odot (b-s)^{q-1} \ominus \frac{bt^2}{ab^2 - ba^2} \odot (a-s)^{q-1}\Big) \odot f(s,u(s))ds \\ &\oplus \frac{1}{\Gamma(q)} \int_b^a \left(\frac{b^2t}{a^2b - ab^2} \odot (a-s)^{q-1} \oplus \frac{bt^2}{ab^2 - ba^2} \odot (a-s)^{q-1}\right) \odot f(s,u(s))ds \\ &\oplus \frac{1}{\Gamma(q)} \odot \int_0^t (t-s)^{q-1} \odot f(s,u(s))ds\end{aligned}$$

So we obtained

$$\begin{aligned}u(t) = &\frac{a^2 \odot B \ominus b^2 \odot A}{a^2b - ab^2} \odot t + \frac{a \odot B \ominus b \odot A}{ab^2 - ba^2} \odot t^2 \\ &\ominus \frac{1}{\Gamma(q)} \odot \int_0^b \Big(\frac{at}{b(a-b)} \odot (b-s)^{q-1} \ominus \frac{bt}{a(a-b)} \odot (a-s)^{q-1} \oplus \frac{t^2}{b(b-a)} \odot (b-s)^{q-1} \\ &\ominus \frac{t^2}{a(b-a)} \odot (a-s)^{q-1}\Big) \odot f(s,u(s))ds \\ &\oplus \frac{1}{\Gamma(q)} \int_b^a \left(\frac{bt}{a(a-b)} \odot (a-s)^{q-1} \oplus \frac{t^2}{a(b-a)} \odot (a-s)^{q-1}\right) \odot f(s,u(s))ds \\ &\oplus \frac{1}{\Gamma(q)} \odot \int_0^t (t-s)^{q-1} \odot f(s,u(s))ds.\end{aligned}$$

Which implies

$$u(t) = \frac{a^2 \odot B \ominus b^2 \odot A}{a^2b - ab^2} \odot t \oplus \frac{a \odot B \ominus b \odot A}{ab^2 - ba^2} \odot t^2 \oplus \int_0^a G(t,s) \odot f(s,u(s))ds.$$

With $G(t,s)$ is given by

$$G(t,s)=\begin{cases} -\dfrac{1}{\Gamma(q)}\left(-(t-s)^{q-1}-\dfrac{bt}{a(b-a)}(a-s)^{q-1}+\dfrac{at}{b(b-a)}(b-s)^{q-1}-\dfrac{t^2}{a(a-b)}(a-s)^{q-1}\right. \\ \left.+\dfrac{t^2}{b(a-b)}(b-s)^{q-1}\right), \quad 0<s<t<b<a. \\ -\dfrac{1}{\Gamma(q)}\left(-\dfrac{bt}{a(b-a)}(a-s)^{q-1}+\dfrac{at}{b(b-a)}(b-s)^{q-1}-\dfrac{t^2}{a(a-b)}(a-s)^{q-1}\right. \\ \left.+\dfrac{t^2}{b(a-b)}(b-s)^{q-1}\right), \quad 0<t<s<b<a. \\ -\dfrac{1}{\Gamma(q)}\left(\dfrac{bt}{a(b-a)}(a-s)^{q-1}+\dfrac{t^2}{a(a-b)}(a-s)^{q-1}\right), \quad 0<t<b<s<a. \end{cases}$$

□

Theorem 5.2. *We set that $f\in\mathcal{C}\left([0,a]\times E^1,E^1\right)$ such that $\underline{f}(\alpha)$ and $\bar{f}(\alpha)$ are continuous functions with respect α and there exists $M:[0,1]\to\mathbb{R}_+$ such that $\forall(t,u)\in[0,a]\times E^1$*

$$\left|\frac{\partial}{\partial\alpha}\underline{f}(t,u,\alpha)\right|,\left|\frac{\partial}{\partial\alpha}\overline{f}(t,u,\alpha)\right|\le M(\alpha)$$

and

$$\left|\frac{d}{d\alpha}\underline{A}(\alpha)\right|,\left|\frac{d}{d\alpha}\bar{A}(\alpha)\right|\ge\frac{a^q}{\Gamma(q-1)}M(\alpha)\quad and\quad \left|\frac{d}{d\alpha}\underline{B}(\alpha)\right|,\left|\frac{d}{d\alpha}\overline{B}(\alpha)\right|\ge\frac{ba^{q-1}}{\Gamma(q-1)}M(\alpha).$$

u is a solution to 5.1 iff u is a solution to 5.2.

Proof. Let $u(t)$ be a solution of (5.1) then

$$D^q u(t)=f(t,u(t))\quad\text{and}\quad u(0)=0,u(b)=B,u(a)=A$$

Since, $1<q-1<2$, then by Lemme(4.7), we get

$$I^{q-1}D^q u(t)=I^{q-1}D^{q-1}u'(t)=u'(t)\ominus u'(0)\ominus u''(0)\odot t=I^{q-1}f(t,u(t)).$$

which implies

$$u'(t)=u'(0)+u''(0)\odot t+I^{q-1}f(t,u(t)).$$

by integrating over $[0,b]$, respectively over $[0,a]$, we obtain

$$\begin{cases} bu'(0)+\dfrac{b^2}{2}u''(0)=B\ominus\dfrac{1}{\Gamma(q-1)}\displaystyle\int_0^b\int_0^t(t-s)^{q-2}f(s,u(s))dsdt \\ au'(0)+\dfrac{a^2}{2}u''(0)=A\ominus\dfrac{1}{\Gamma(q-1)}\displaystyle\int_0^a\int_0^t(t-s)^{q-2}f(s,u(s))dsdt, \end{cases}$$

thus

$$\begin{cases} \left(a^2b-b^2a\right)u'(0)=\left(Ba^2\ominus\dfrac{a^2}{\Gamma(q-1)}\displaystyle\int_0^b\int_0^t(t-s)^{q-2}f(s,u(s))dsdt\right)\ominus\left(Ab^2\ominus\dfrac{b^2}{\Gamma(q-1)}\displaystyle\int_0^a\int_0^t(t-s)^{q-2}f(s,u(s))dsdt\right) \\ \left(\dfrac{ab^2}{2}-\dfrac{ba^2}{2}\right)u''(0)=\left(Ba\ominus\dfrac{a}{\Gamma(q-1)}\displaystyle\int_0^b\int_0^t(t-s)^{q-2}f(s,u(s))dsdt\right)\ominus\left(Ab\ominus\dfrac{b}{\Gamma(q-1)}\displaystyle\int_0^a\int_0^t(t-s)^{q-2}f(s,u(s))dsdt\right). \end{cases}$$

Either again

$$\begin{cases} \left(a^2b-b^2a\right)u'(0)=a\left(\displaystyle\int_0^b\int_0^t\frac{Ba}{bt}\ominus\frac{a}{\Gamma(q-1)}(t-s)^{q-2}f(s,u(s))dsdt\right)\ominus b\left(\displaystyle\int_0^a\int_0^t\frac{Ab}{at}\ominus\frac{b}{\Gamma(q-1)}(t-s)^{q-2}f(s,u(s))dsdt\right) \\ \left(\dfrac{ab^2}{2}-\dfrac{ba^2}{2}\right)u''(0)=\left(\displaystyle\int_0^b\int_0^t\frac{Ba}{bt}\ominus\frac{a}{\Gamma(q-1)}(t-s)^{q-2}f(s,u(s))dsdt\right)\ominus\left(\displaystyle\int_0^a\int_0^t\frac{Ab}{at}\ominus\frac{b}{\Gamma(q-1)}(t-s)^{q-2}f(s,u(s))dsdt\right). \end{cases}$$

In other hand

$$\frac{d}{d\alpha}\left(\frac{\underline{B}a}{bt}-\frac{a}{\Gamma(q-1)}(t-s)^{q-2}f(s,u(s),\alpha)\right)\geq\frac{d\underline{B}}{d\alpha}\frac{a}{ba}-\frac{a^{q-1}}{\Gamma(q-1)}\frac{d}{d\alpha}f(s,u(s),\alpha)\geq 0$$

and

$$\frac{d}{d\alpha}\left(\frac{\overline{B}a}{bt}-\frac{a}{\Gamma(q-1)}(t-s)^{q-2}f(s,u(s),\alpha)\right)\leq\frac{d\underline{B}}{d\alpha}\frac{a}{ba}-\frac{a^{q-1}}{\Gamma(q-1)}\frac{d}{d\alpha}f(s,u(s),\alpha)\leq 0,$$

as far as

$$\frac{d}{d\alpha}\left(\frac{\underline{A}b}{at}-\frac{b}{\Gamma(q-1)}(t-s)^{q-2}f(s,u(s),\alpha)\right)\geq\frac{d\underline{A}}{d\alpha}\frac{b}{a^2}-\frac{a^{q-1}}{\Gamma(q-1)}\frac{d}{d\alpha}f(s,u(s),\alpha)\geq 0,$$

and

$$\frac{d}{d\alpha}\left(\frac{\overline{A}b}{at}-\frac{b}{\Gamma(q-1)}(t-s)^{q-2}f(s,u(s),\alpha)\right)\leq\frac{d\overline{A}}{d\alpha}\frac{b}{a^2}-\frac{a^{q-1}}{\Gamma(q-1)}\frac{d}{d\alpha}f(s,u(s),\alpha)\leq 0.$$

By theorem (2.4) and lemme (4.7)

$$u(t)=\int_0^t u'(s)ds\in E^1,\quad \forall t\in[0,a].$$

This means that $u(t)$ is continuously derivable according to Lemma (4.7)), we have

$$\begin{aligned}
&D^q u(t)=f\left(t,u(t)\right)\\
&\qquad\Leftrightarrow u(t)=u(0)+tu'(0)+\frac{t^2}{2}u''+I^q f\left(t,u(t)\right)\\
&\qquad\Leftrightarrow u(t)=tu'(0)+\frac{t^2}{2}u''+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f\left(s,u(s)\right)ds
\end{aligned}$$

where

$$\begin{cases}
u'(0)=\dfrac{1}{a^2b-ab^2}\left(a^2\odot B\ominus b^2\odot A\ominus\dfrac{1}{\Gamma(q)}\odot\left[\displaystyle\int_0^b\left(a^2(b-s)^{q-1}\ominus b^2(a-s)^{q-1}\right)\odot f(s,u(s))ds\ominus\int_b^a b^2(a-s)^{q-1}\odot f(s,u(s))ds\right]\right)\\
u''(0)=\dfrac{2}{ab^2-ba^2}\left(a\odot B\ominus b\odot A\ominus\dfrac{1}{\Gamma(q)}\odot\left[\displaystyle\int_0^b\left(a(b-s)^{q-1}\ominus b(a-s)^{q-1}\right)\odot f(s,u(s))ds\ominus\int_b^a b(a-s)^{q-1}\odot f(s,u(s))ds\right]\right).
\end{cases}$$

Substitute into the previous expression obtained, we get

$$u(t)=\frac{a^2\odot B\ominus b^2\odot A}{a^2b-ab^2}\odot t\oplus\frac{a\odot B\ominus b\odot A}{ab^2-ba^2}\odot t^2\oplus\int_0^a G(t,s)\odot f(s,u(s))ds,$$

that is $u(t)$ is a solution of (5.2).

Suppose $u(t)$ is the solution of (5.2).

First, $u(0)=\tilde{0}$, $u(b)=B$ and $u(a)=A$.

Since,

$$\begin{aligned}
u(t)&=\frac{a^2\odot B\ominus b^2\odot A}{a^2b-ab^2}\odot t\oplus\frac{a\odot B\ominus b\odot A}{ab^2-ba^2}\odot t^2\oplus\int_0^a G(t,s)\odot f(s,u(s))ds\\
&=\frac{a\odot B}{ab^2-ba^2}t^2\ominus\frac{a^2\odot B}{ab^2-ba^2}t+\frac{b^2\odot A}{ab^2-ba^2}t\ominus\frac{b\odot A}{ab^2-ba^2}t^2\oplus\int_0^a G(t,s)\odot f(s,u(s))ds\\
&=\int_0^a\left(\frac{t^2}{ab^2-ba^2}\ominus\frac{at}{ab^2-ba^2}\right)\odot B+\left(\frac{b^2\odot t}{a^2b^2-ba^3}\ominus\frac{b\odot t^2}{a^2b^2-ba^3}\right)\odot A\oplus G(t,s)\odot f(s,u(s))ds\\
&=\int_0^t\left(\frac{t^2}{ab^2-ba^2}\ominus\frac{at}{ab^2-ba^2}\right)\odot B+\left(\frac{b^2\odot t}{a^2b^2-ba^3}\ominus\frac{b\odot t^2}{a^2b^2-ba^3}\right)\odot A\oplus G(t,s)\odot f(s,u(s))ds\\
&\quad+\int_t^a\left(\frac{t^2}{ab^2-ba^2}\ominus\frac{at}{ab^2-ba^2}\right)\odot B+\left(\frac{b^2\odot t}{a^2b^2-ba^3}\ominus\frac{b\odot t^2}{a^2b^2-ba^3}\right)\odot A\oplus G(t,s)\odot f(s,u(s))ds,
\end{aligned}$$

it is very easy to verify that $u(t) \in E^1$. In this case, Using the property 4.3 we get

$$\begin{aligned} D^q u(t) &= D^q \int_0^a G(t,s) f\left(s, u(s)\right) ds \\ &= D^q \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\left(s, u(s)\right) ds \right) \\ &= f\left(s, u(s)\right) \end{aligned}$$

Thus $u(t)$ is a solution of (5.1). □

Theorem 5.3. *$f \in \mathcal{C}\left([0,a] \times E^1, E^1\right)$ and $(A,B) \in E^1 \times E^1$ satisfies the conditions of the theorem (5.2),and $\forall (x,y) \in E^1 \times E^1$; $d(f(t,x), f(t,y)) \leq K d(x,y), \quad \forall t \in [0,a]$ where $\frac{2K}{\Gamma(q+1)} \left[\dfrac{a^{q+1} + a^2 b^{q-1}}{(a-b)} \right] < 1$ Then the problem (5.1) has a unique solution in $\mathcal{C}^1\left([0,a], E^1\right)$.*

Proof. Consider the following map

$$F : \begin{cases} \mathcal{C}\left([0,a], E^1\right) \to \mathcal{C}\left([0,a], E^1\right) \\ x \to Fx : Fx(t) = w(t) + \displaystyle\int_0^a G(t,s) f\left(s, x(s)\right) \end{cases}$$

First F is well defined :

Since $G(.,s)$ and f are continuous then F has its value in $\mathcal{C}\left([0,a], E^1\right)$, so F is well defined.

$\left(E^1, d\right)$ is a complete metric space and $[0,a]$ is a compact of $\mathbb{R}$ then $\left(\mathcal{C}\left([0,a], E^1\right), d_\infty\right)$, where $d_\infty(x,y) = \sup_{t\in[0,a]} d(x(t), y(t))$, is a complete metric space.

For all $x, y \in \mathcal{C}\left([0,a], E^1\right)$, we have

$$\begin{aligned} d_\infty(Fx, Fy) &\leq \int_0^a |G(t,s)| d\left(f\left(s, x(s)\right), f\left(s, y(s)\right)\right) ds \\ &\leq K \sup_{t\in[0,a]} \int_0^a |G(t,s)|\, ds d_\infty(x,y) \\ &\leq K \sup_{t\in[0,a]} \left[\int_0^t |G(t,s)|\, ds + \int_t^b |G(t,s)|\, ds + \int_b^a |G(t,s)|\, ds \right] d_\infty(x,y) \\ &\leq \frac{K}{\Gamma(q+1)} \sup_{t\in[0,a]} \left[t^q + \frac{bt}{(a-b)} a^{q-1} + \frac{at}{(a-b)} b^{q-1} + \frac{t^2}{(a-b)} a^{q-1} + \frac{t^2}{(a-b)} b^{q-1} \right] d_\infty(x,y) \\ &\leq \frac{K}{\Gamma(q+1)} \left[a^q + \frac{ba}{(a-b)} a^{q-1} + \frac{a^2}{(a-b)} b^{q-1} + \frac{a^2}{(a-b)} a^{q-1} + \frac{a^2}{(a-b)} b^{q-1} \right] d_\infty(x,y) \\ &\leq \frac{2K}{\Gamma(q+1)} \left[\frac{a^{q+1} + a^2 b^{q-1}}{(a-b)} \right] d_\infty(x,y) \end{aligned}$$

Since $K \sup_{t\in[0,a]} \int_0^a |G(t,s)|\, ds < 1$. In this case, by Banach's fixed point theorem, F has a unique fixed point that is the solution to (5.1). □

6. Applications

In this section we satisfy the conditions of the theorem (5.3) and as an explanation of the theorem (5.3) $t \in [0,2]$, we have :

$$\begin{cases} D^q u(t) = p \odot \phi(u(t)), & 0 < t < 2 \\ u(0) = \tilde{0} \in E^1; \quad u(1) = (0.1 \quad 0.2 \quad 0.3); \quad u(2) = (0.2 \quad 0.5 \quad 1) & 2 < q < 3, \end{cases} \tag{6.1}$$

where p is a singleton fuzzy number and $\phi \in \mathcal{C}\left([0,2], E^1\right)$.

Since ϕ is continuous on [0;2] then there is $A > 0$ such that $|\phi| < A$ which implies by the Mean Value Theorem we know there exists $\xi \in]0;2[$ such that :

$$d_\infty\left(p \odot \phi(u(t)); p \odot \phi(v(t))\right) < |p|\xi d_\infty(u(t), v(t))$$

This implies the existence of a solution. Then, according to the theorem (5.3) to have uniqueness it is necessary that :

$$\frac{2|p|\xi}{\Gamma(q+1)}\left[2^{q+1}+2^2\right] < 1 \quad \text{or} \quad |p| < \frac{\Gamma(q+1)}{2^4\left(2^{q-1}+1\right)}$$

7. Conclusions

In this study, an attempt was made to give a solution to a fuzzy differential equation under the Caputo derivation using the Green's function and Banach's fixed point theorem. An example to support the results is presented.

References

[1] A. Tofigh, A. Armand, and Z. Gouyandeh. "Fuzzy fractional differential equations under generalized fuzzy Caputo derivative." Journal of Intelligent and Fuzzy Systems 26.3 (2014): 1481-1490.

[2] A. Tofigh. "Fuzzy fractional differential operators and equations." Studies in fuzziness and soft computing 397 (2021).

[3] A. Armand, Z. Gouyandeh, Fuzzy fractional integro-differential equations under generalized Caputo differentiability. Annals of Fuzzy Mathematics and Informatics Volume 10, No. 5, (November 2015), pp. 789-798.

[4] A. sadia. "On existence and uniqueness of solution of fuzzy fractional differential equations." (2013): 137-151.

[5] B. Bede, L. Stefanini, Generalized differentiability of fuzzy valued functions, Fuzzy Sets Syst. 230(0) (2013) 119-141.

[6] B. Bede ,Mathematics of Fuzzy Sets and Fuzzy Logic,Studies in Fuzziness and Soft Computing , Springer 2013.

[7] H. Brezis. Sobolev Spaces and Partial Differential Equations, Functional Analysis, Springer. (2010)

[8] C. Minghao, et al. "On fuzzy boundary value problems." Information Sciences 178.7 (2008): 1877-1892.

[9] G.A. Anastassiou, Fuzzy Mathematics: Approximation Theory. Studies in Fuzziness and Soft Computing, Springer, Berlin Heidelberg, 2010.

[10] R. Hilfer. "Applications of Fractional Calculus in Physics. Orlando." (1999).

[11] H.C. Wu, The improper fuzzy Riemann integral and its numerical integration, Inf. Sci. 111(14) (1998) 109-137.

[12] O. Kaleva. "Fuzzy differential equations." Fuzzy sets and systems 24.3 (1987): 301-317.

[13] L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, Fuzzy Sets Syst. 161 (2010) 1564?1584.

[14] L. Stefanini, B. Bede, Generalized hukuhara differentiability of interval-valued functions and interval differential equations, Nonlinear Anal. 71(8) (2009) 1311-1328.

[15] V. Lakshmikantham, K. N. Murty, and J. Turner. "Two-point boundary value problems associated with non-linear fuzzy differential equations." Mathematical inequalities and applications 4 (2001): 527-534.

[16] S. Melliani, et al. "Fuzzy fractional differential wave equation." International Journal On Optimization and Applications (2021): 42.

[17] F. C. Meral, T. J. Royston, and R. Magin. "Fractional calculus in viscoelasticity: an experimental study." Communications in nonlinear science and numerical simulation 15.4 (2010): 939-945.

[18] I. Podlubny. "Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications." (1999).

[19] R. Goetschel Jr., W. Voxman, Elementary fuzzy calculus, Fuzzy Sets Syst. 18 (1986) 3134.

[20] S. Saito, Qualitative approaches to boundary value problems of fuzzy differential equations by theory of ODE, J. Nonlinear Convex Anal. 5 (2004) 121130.

[21] S. Salahshour, et al. "Existence and uniqueness results for fractional differential equations with uncertainty." Advances in Difference Equations 2012.1 (2012): 1-12.

[22] T. Allahviranloo, Z. Gouyandeh, A. Armand, A. Hasanoglu, On fuzzy solutions for heat equation based on generalized hukuhara differentiability, Fuzzy Sets Syst. 265 (2015) 1-23.

[23] J. J. Trujillo, et al. Fractional calculus: models and numerical methods. Vol. 5. World Scientific, 2016.

[24] W. Congxin, M. Ming, Embedding problem of fuzzy number space: Part III, Fuzzy Sets Syst. 46(2) (1992) 281-286.

Aziz EL Ghazouani,
Laboratory of Applied Mathematics Scientific Calculus,
Sultan Moulay Slimane University,
BP 523, 23000, Beni Mellal, Morocco.
E-mail address: aziz.elghazouani@usms.ac.ma

and

Fouad Ibrahim Abdou Amir,
Laboratory of Applied Mathematics Scientific Calculus,
Sultan Moulay Slimane University,
BP 523, 23000, Beni Mellal, Morocco.
E-mail address: fouadibramih@gmail.com

and

M'hamed Elomari,
Laboratory of Applied Mathematics Scientific Calculus,
Sultan Moulay Slimane University,
BP 523, 23000, Beni Mellal, Morocco.
E-mail address: m.elomari@usms.ma

and

Said Melliani,
Laboratory of Applied Mathematics Scientific Calculus,
Sultan Moulay Slimane University,
BP 523, 23000, Beni Mellal, Morocco.
E-mail address: s.melliani@usms.ma