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Some Properties of Weak $\star$ Dunford-Pettis Operators on Banach Lattices

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ABSTRACT: We study relationships between the class of weak* Dunford-Pettis operators and other classes of operators like Dunford-Pettis, weak Dunford-Pettis and unbounded Dunford-Pettis operators.

Key Words: Banach lattices, Dunford-Pettis operator, weak Dunford-Pettis operator, unbounded Dunford-Pettis operator, unbounded absolutely weakly convergence.

Contents

1. Introduction

The class of weak* Dunford-Pettis operators appeared for the first time in [12]. In [2] the authors studied its relationship between M-weakly compact and almost Dunford-Pettis operator, the class of unbounded absolutely weakly Dunford-Pettis operators was introduced in [17]. Here, we study the relationship between the class of weak* Dunford-Pettis operator and other classes of operators such that the well known Dunford-Pettis operators, weak Dunford-Pettis operators and σ-unbounded absolutely weakly Dunford-Pettis operators.
By Theorem 3.1 we show necessary and sufficient conditions under which a weak* Dunford-Pettis operator is σ-unbounded Dunford-Pettis operator. Clearly a Dunford-Pettis operator is a weak* Dunford-Pettis, but the converse is not true in general, by Theorem 3.5 we study when the converse is true. Since the weak convergence implies the weak* convergence, every weak* Dunford-Pettis operator is weak Dunford-Pettis, but the converse is not true in general, by theorem 3.7, we give properties on Banach lattices whenever every weak Dunford-Pettis is weak* Dunford-Pettis operator. We start this article by recalling few definitions in the first section.

2. Preliminaries

Throughout this paper X and Y denote two Banach lattices, X and Y two Banach spaces. The positive cone of E is denoted by E^+. B_X is the closed unit ball of X. The term operator between two Banach spaces means a bounded linear mapping.
A norm bounded subset A of X is said to be

- Dunford-Pettis set if every weak null sequence (f_n) of X' converges uniformly on A, that is, $\lim\limits_{n\to\infty} \sup_{x\in A} |\langle f_n, x\rangle| = 0$.
- Limited set if every weak* null sequence (f_n) of X' converges uniformly on A, that is, $\lim\limits_{n\to\infty} \sup_{x\in A} |\langle f_n, x\rangle| = 0$.

Let us recall that an operator $T : X \longrightarrow Y$ is said to be:

- Dunford-Pettis if T carries weakly convergent sequences to norm convergent sequences [1].

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- Weak Dunford-Pettis whenever $x_n \to 0$ for $\sigma(X, X')$-topology (shortly, $x_n \xrightarrow{w} 0$) and $f_n \to 0$ for $\sigma(Y', Y'')$-topology imply that $\lim_{n\to\infty} f_n(T(x_n)) = 0$ [1].
- Weak* Dunford-Pettis whenever $x_n \to 0$ for $\sigma(X, X')$-topology and $f_n \to 0$ for $\sigma(Y', Y)$-topology (shortly, $f_n \xrightarrow{w^\star} 0$) imply that $\lim_{n\to\infty} f_n(T(x_n)) = 0$ [12].
- Limited if $T(B_X)$ is a limited set of Y, alternatively, T is limited if, and only if, $||T'(f_n)|| \to 0$ for every weak* null sequence (f_n) in Y' [4].
- M-weakly compact whenever $||T(x_n)|| \to 0$ holds for every norm bounded disjoint sequence $(x_n) \subset X$ [1].

A Banach space X has

- The Dunford-Pettis* property (shortly, DP* property) if every relatively compact subset of X is limited, which is equivalent to say that X has the DP * property if, and only if, $\lim_{n\to\infty} f_n(x_n) = 0$ for every weak null sequence (x_n) in X and every weak* null sequence (f_n) of X', by Borwein in [5]
- The Grothendieck property if for every sequence (f_n) in X' such that $f_n \xrightarrow{w^\star} 0$ then $f_n \xrightarrow{w} 0$ in X' [19].
- The Schur property if each weakly null sequence in E converges to zero in the norm. For example, the Banach lattice ℓ^1 has the Schur property but the Banach lattice $L^1([0,1])$ does not have the Schur property.

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.
Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice.
The unbounded absolute weak convergence (uaw-convergence) was introduced first by Zabeti in [18]. A sequence $(x_n)_n$ is unbounded absolutely weakly convergent (shortly, uaw-convergent) to a vector x in E if $(|x_n - x| \wedge u)$ is weakly convergent to zero for every $u \in E^+$; we write $x_n \xrightarrow{uaw} x$.
The unbounded norm convergence (un-convergence) in a Banach lattices was first introduced by V. Troitsky in [8]. A sequence is said to be unbounded norm convergent (shortly un-convergence) to a vector x in E if $||\,|x_n - x| \wedge u||$ converges to zero for every $u \in E^+$; we write $x_n \xrightarrow{un} x$.
The lattice operations in E' are called weak* sequentially continuous if the sequence $(|f_n|)_n$ converges to 0 by the weak* topology whenever the sequence (f_n) converges weak* to 0 in E'.
An operator $T : X \to Y$ between two Banach spaces is said to be an embedding whenever there exist two positive constants K and M satisfying:

$$K||x|| \leq ||T(x)|| \leq M||x|| \qquad \text{for all} \qquad x \in X$$

For all unexplained terminology and standard facts on vector and Banach lattices, we refer the reader to the monographs [1] and the paper [18].

3. Main results

Recall that an operator $T : E \to F$ is σ-unbounded Dunford-Pettis if for every norm bounded sequence (x_n), $x_n \xrightarrow{uaw} 0$ in E implies that $T(x_n) \xrightarrow{un} 0$ [15]. Note that there exists a weak* Dunford-Pettis operators which is not σ-unbounded Dunford-Pettis. In fact, the identity operator $Id_{\ell^\infty} : \ell^\infty \to \ell^\infty$ is weak* Dunford-Pettis (bcause ℓ^∞ has the Dunford-Pettis* property) but fails to be σ-unbounded Dunford-Pettis. In fact, the standard basis $(e_n)_n$ is uaw-null in ℓ^∞. By considering $(1,1,1,,..........)\in\ell^\infty$ where (1,1,1,.........) is the constant sequence with all rang value is 1. It is clear that $\|e_n\wedge(1,1,1,1,1,..........)\| = 1$. Hence, $e_n \overset{un}{\nrightarrow} 0$.

In the following result, we show necessary and sufficient conditions under which a weak* Dunford-Pettis operator is σ-unbounded Dunford-Pettis.

Theorem 3.1. *Let E and F be two Banach lattices such that F is a Dedekind σ-complete. Then the following assertions are equivalent:*

1. *Every positive weak* Dunford-Pettis operator $T : E \to F$ is M-weakly compact.*
2. *Every positive weak* Dunford-Pettis operator $T : E \to F$ is σ-unbounded Dunford-Pettis.*
3. *One of the following is valid:*
   - *(a) The norms of E' and F are order continuous.*
   - *(b) E is finite-dimensional.*
   - *(c) $F = \{0\}$.*

Proof. (1) $\Longrightarrow$ (2) Let $T : E \to F$ be an M-weakly compact operator, by [21, Theorem 2.7], $\|Tx_n\| \longrightarrow 0$ for every uaw-null sequence $x_n \subset B_E$. Since norm convergence implies un-convergence then, T is σ-unbounded Dunford-Pettis.

(2) $\Longrightarrow$ (3) It suffices to prove separately the two following assertions:
(α) If the norm of F is not order continuous, then E is finite-dimensional (β) If the norm of E' is not order continuous, then $F = \{0\}$.

Assume that (α) is false, i.e. the norm of F is not order continuous and E is infinite-dimensional. Then, it results from [1, Theorem 4.51] that ℓ^∞ is a lattice embedding in F. Let $i : \ell^\infty \to F$ be a lattice embedded, then there exist two positive constants m and M satisfying

$$m\|x\|_\infty \le \|i(x)\| \le M\|x\|_\infty \qquad \text{for all} \qquad x\in\ell^\infty(*)$$

On the other hand, since E is an infinite-dimensional Banach lattice, it follows from [3, Lemma 2.3] and [3, Lemma 2.5] the existence of a positive disjoint sequence (x_n) in E^+ with $\|x_n\| = 1$ for all $n\in\mathbb{N}$ and there exists a positive disjoint sequence (g_n) of E' with $\|g_n\|\le 1$ for each n, such that $g_n(x_n) = 1$ and $g_n(x_m) = 0$ for $n \neq m$.

To finish the proof, we have to construct a positive weak* Dunford-Pettis operator $T : E \to F$ which is not σ-unbounded Dunford-Pettis. Now, we define the operator $P : E \longrightarrow \ell^\infty$ defined by $P(x) = (g_n(x))_n$. Clearly P is well defined and is positive. Let

$$T = i\circ P : E \longrightarrow \ell^\infty \longrightarrow F$$

then T is a positive weak* Dunford-Pettis operator (because ℓ^∞ has the Dunford-Pettis* property). But T is not σ-unbounded Dunford-Pettis operator. Indeed, by [Lemma 2, [18]], $x_n \overset{uaw}{\longrightarrow} 0$, we will prove the existence of an element $u\in F^+$ such that $\|T(x_n)\wedge u\| \nrightarrow 0$. In fact, let consider $(1,1,1,.........)\in\ell^\infty$ where $(1,1,1.......)$ is the constant sequence from ℓ^∞ with all rang value is 1. Since i is a positive operator defined on $\ell^\infty \longrightarrow F$, then $i(1,1,1,1.........)\in F^+$, as the operator i is a lattice homomorphism then, by [[1], Theorem 2.14] we have the following

$$\begin{aligned}
\|T(x_n)\wedge i(1,1,1,.........)\| &= \|i\circ P(x_n)\wedge i(1,1,1,.........)\| \\
&= \|i[(g_k(x_n))_k \wedge (1,1,1,.........)]\| \\
&= \|i(g_k(x_n))_k\| \\
&\ge m\|g_k(x_n))_k\|_\infty \qquad \text{from}(*) \\
&\ge m = m|g_n(x_n)|
\end{aligned}$$

for all $n\in\mathbb{N}$. Then, $\|T(x_n)\wedge i(1,1,1,1.........)\| \nrightarrow 0$ and hence $T(x_n) \overset{un}{\nrightarrow} 0$, so T is not σ-unbounded Dunford-Pettis.

(β) Assume that (1) holds. We will prove that if the norm on E' is not order continuous, then $F = \{0\}$. If not, there exists some $y > 0$ in F and since the norm of E' is not order continuous then, il follows from Theorem 2.4.14 and Proposition 2.3.11 [14] that ℓ^1 is a closed sublattice of E and there exists a positive projection P: $E \longrightarrow \ell^1$.
Consider the positive operator S defined by

$$S : \ell^1 \longrightarrow F, (\alpha_n)_n \mapsto \sum_{n=1}^{\infty} \alpha_n y$$

The composed operator $T = S \circ P$ is compact (because S is compact as its rank is one), hence T is weak* Dunford-Pettis but T is not σ-unbounded Dunford-Pettis. Indeed, let (e_n) be the canonical basis of ℓ^1 we have $e_n \xrightarrow{uaw} 0$ but $\|T(e_n) \wedge y\| \nrightarrow 0$ ($T(e_n) \wedge y = y \wedge y = y$ for each $n \in \mathbb{N}$). Thus, T is not a σ-unbounded Dunford-Pettis operator.

$(3.a) \Longrightarrow (1)$ We will adapt here the implication $(1) \Longrightarrow (2)$ in the proof of [13, Theorem 2.1].
We will prove that $\|Tx_n\| \longrightarrow 0$ for every disjoint sequence $x_n \subset B_E$.
Let $T : E \longrightarrow F$ be a positive weak* Dunford-Pettis operator and (x_n) be a disjoint sequence in B_E. By [[7], Corollary 2.6], it suffices to prove that $|Tx_n| \longrightarrow 0$ in the $\sigma(F, F')$-topology of F and $f_n(Tx_n) \longrightarrow 0$ for every disjoint and norm bounded sequence $(f_n) \subset (F')^+$. In fact,
- Let $f \in (F')^+$. As the norm of E' is order continuous then, $x_n \longrightarrow 0$ and $|x_n| \longrightarrow 0$ in the $\sigma(E, E')$-topology of E (because (x_n) is a disjoint sequence). It follows from [1, Theorem 1.23] that for each n there exists some $g_n \in [-f, f]$ with $f|T(x_n)| = g_n(T(x_n))$. Now, since $|x_n| \longrightarrow 0$ in the $\sigma(E, E')$-topology of E and T is positive then,

$$0 \leq f|Tx_n| = g_n(Tx_n) = T'(g_n)(x_n) \leq |T'(g_n)||x_n| \leq T'(f)|x_n| \longrightarrow 0$$

and hence $|T(x_n)| \longrightarrow 0$ in the $\sigma(F, F')$-topology of F.
- Let $(f_n) \subset (F')^+$ be a disjoint and norm bounded sequence. As the norm of F is order continuous, then by [14, Corollary 2.4.3], $f_n \longrightarrow 0$ in the $\sigma(F', F)$-topology of F'. Now, since T is weak* Dunford-Pettis then, $f_n(T(x_n)) \longrightarrow 0$. This finishes the proof.
$(3.b) \Longrightarrow (1)$ It follows from [13, Theorem 2.4].
$(3.c) \Longrightarrow (1)$ Obvious. □

Remark 3.2. *If we consider only the condition of order continuity of E', it will not be sufficient for a weak* Dunford-Pettis operator to be σ-unbounded Dunford-Pettis, we consider again the example of the identity from ℓ^∞ to ℓ^∞, which is weak* Dunford-Pettis. The dual of ℓ^∞ is order continuous, but this identity is not σ-unbounded Dunford-Pettis.*

To establish our next results, we need the following Lemma,

Lemma 3.3. *Let A be a bounded subset of a Banach space X. if for each $\varepsilon > 0$ there exists a limited set A_ε in X such that $A \subseteq A_\varepsilon + \varepsilon B_X$, then A is a limited set.*

Proof. Let (f_n) be a weak* null sequence in X', and ε an arbitrary element of $\mathbb{R}$ such that $\varepsilon > 0$. Pick some $M > 0$ with $\|f_n\| \leq M$ for all n. By hypothesis, there exits some limited subset A_ε of X such that $A \subseteq A_\varepsilon + \frac{\varepsilon}{2M} B_X$, since ε is arbitrary, then

$$\sup_{x \in A} |f_n(x)| \leq \sup_{x \in A_\varepsilon} |f_n(x)| + \frac{\varepsilon}{2}.$$

As A_ε is limited, there exists some n_0 with $\sup_{x \in A} |f_n(x)| \leq \frac{\varepsilon}{2}$ for all $n \geq n_0$. Thus, $\sup_{x \in A} |f_n(x)| \leq \varepsilon$ for all $n \geq n_0$. This implies $\sup_{x \in A} |f_n(x)| \longrightarrow 0$, and then A is limited. □

Recall that an operator from a Banach lattice E into a Banach space Y is almost Dunford-Pettis if $\|T(x_n)\| \to 0$ for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E [22].

Theorem 3.4. *Let E and F be two Banach lattices. If F' has weak* sequentially continuous lattice operations, then every positive almost Dunford-Pettis operator from E into F is weak* Dunford-Pettis.*

Proof. Let $T : E \longrightarrow F$ be a positive almost Dunford-Pettis operator, and let W be a relatively weakly compact set in E. We want to show that $T(W)$ is a limited set in F.
Let $\varepsilon > 0$, and let $A = Sol(W)$. By [1, Theorem 4.36], there exists $u \geq 0$ lying in the ideal generated be A satisfying $T(W) \subset [-T(u), T(u)] + \varepsilon B_X$. Or F' has weak* sequentially continuous lattice operations then, il follows from [11, Proposition 3.1] that $[-T(u), T(u)]$ is a limited set in F. Then, by Lemma 3.3 the set $T(W)$ is limited in F and hence T is weak* Dunford-Pettis. □

Clearly, a Dunford-Pettis operator is a weak* Dunford-Pettis but a weak* Dunford-Pettis operator is not necessary Dunford-Pettis. For example, the identity operator $Id_{\ell^\infty} : \ell^\infty \longrightarrow \ell^\infty$ is weak* Dunford-Pettis (because ℓ^∞ has the DP* property) but is not Dunford-Pettis (because ℓ^∞ does not have the Schur property).

In the following result, we show properties on Banach lattices E and F whenever every weak* Dunford-Pettis operator is Dunford-Pettis,

Theorem 3.5. *Let E and F be two Banach lattices. If every weak* Dunford-Pettis operator $T : E \longrightarrow F$ is Dunford-Pettis, then one of the following is valid:*

1. *E has the Schur property,*

2. *The norm of F is order continuous.*

Proof. Let suppose that (1) and (2) doesn't hold, i.e., the norm of F is not order continuous and E does not have the Schur property. We will construct an operator $T : E \longrightarrow F$ which is weak* Dunford-Pettis but is not Dunford-Pettis. Indeed, suppose that E does not have the Schur property. Then there exists a weakly null sequence $(x_n) \subset E$ and $\varepsilon > 0$ and a sequence $(f_n) \subset B_{E'}$ such that $|f_n(x_n)| > \varepsilon$.
Now, consider the operator $P : E \longrightarrow \ell^\infty$ defined by

$$P(x) = (f_n(x))_n$$

Since the norm of F is not order continuous, it follows from [1, Theorem 4.51] that ℓ^∞ is lattice embeddable in F, i.e., there exists a lattice homomorphism $S : \ell^\infty \longrightarrow F$ and there exist two positive constants M and m satisfying

$$m \left\|(\lambda_k)_k\right\|_\infty \leq \left\|S\left((\lambda_k)_k\right)\right\| \leq M \left\|(\lambda_k)_k\right\|_\infty$$

for all $(\lambda_k)_k \in \ell^\infty$.
Let consider the operator $T = S \circ P : E \longrightarrow \ell^\infty \longrightarrow F$, and note that T is a weak* Dunford-Pettis operator (because ℓ^∞ has the DP* property), but is not Dunford-Pettis. In fact, since (x_n) is weakly null sequence in E and as

$$\begin{aligned} \|T(x_n)\| &= \|S \circ P(x_n)\| \\ &= \|S((f_k(x_n))_k)\| \\ &\geq m\|(f_k(x_n))\|_\infty \\ &\geq m|f_n(x_n)| \\ &> m\varepsilon \end{aligned}$$

for every n. Then, T is not Dunford-Pettis, and this finishes the proof.
□

As an immediate consequence, we have the following result,

Corollary 3.6. *Let E be a Banach lattice. If every weak* Dunford-Pettis operator $T : E \longrightarrow E$ is Dunford-Pettis, then the norm of E is order continuous.*

Clearly, a weak* Dunford-Pettis operator is weak Dunford-Pettis, but the converse doens't hold in general. In fact, the identity operator $Id_{c_0} : c_0 \longrightarrow c_0$ is weak Dunford-Pettis (because c_0 has the DP property) but it is not weak* Dunford-Pettis (because c_0 fails the DP* property).

Although, by virtue of following result, we show properties on Banach lattices E and F whenever a weak Dunford-Pettis operator is weak* Dunford-Pettis operator.

Theorem 3.7. *Let E and F be two Banach lattices such that F is Dedekind σ-complete and the norm on E' is order continuous. If each weak Dunford-Pettis operator $T : E \longrightarrow F$ is weak* Dunford-Pettis, then one of the following assertions is valid:*

1. *E has the wDP* property.*
2. *F is a KB-space.*

Proof. Suppose that F is not a KB-space, we will prove that E has the wDP* property. Since F is not a KB-space, then by [[1],theorem 4.60] c_0 is lattice embeddable in F, let consider an arbitrary positif operator $T : E \longrightarrow c_0$.
The Banach lattice c_0 has the Dunford-Pettis property, then T is a weak Dunford-Pettis operator, by assumption, T is weak* Dunford-Pettis, since the norm on c_0 and E' are order continuous then by theorem 3.1, T is M-weakly compact, by [21, Theorem 2.7], T is uaw-Dunford-Pettis. Since c_0 is discrete and order continuous then by virtue of theorem 3 of [9], T is Dunford-Pettis. Theorem 3.5 of [6] shows that E has the wDP* property. □

Remark 3.8.
1. *If the norm on E' is not order continuous, then the first condition is not sufficient, for instance, let consider $E = F = L^1[0,1]$. The Banach lattice $L^1[0,1]$ has the positive Schur property, then by [[6], proposition 3.3], $L^1[0,1]$ has the wDP* property.*
The identity operator $Id : L^1[0,1] \longrightarrow L^1[0,1]$ is weak Dunford-Pettis but not weak* Dunford-Pettis because $L^1[0,1]$ has the Dunford-Pettis property without the DP* property. (Indeed, a separable Banach space with the DP* property must have the Schur property, while $L^1[0,1]$ has the positive Schur property without the Schur property).
2. *The second condition of Theorem 3.7 is not necessary, indeed, we consider $E = F = L^\infty[0,1]$, by virtue of Corollary 4.45 of [1], $L^\infty[0,1]$ has the Grothendieck property, then every weak Dunford-Pettis operator $T : L^\infty[0,1] \longrightarrow L^\infty[0,1]$ is weak* Dunford-Pettis. Although $(L^\infty[0,1])'$ is order continuous and $L^\infty[0,1]$ is not KB-space.*

Remark 3.9. *$L^1[0,1]$ is KB-space, but it's not a discrete Banach lattice. As we saw above (remark 3.8), there exists a weak Dunford-Pettis ($Id : L^1[0,1] \longrightarrow L^1[0,1]$) which is not weak* Dunford-Pettis. we will see by theorem bellow that the statement holds for an operator $T : E \longrightarrow F$, if F is a discrete KB-space.*

Furthermore, we give bellow sufficient conditions on Banach lattices under which a weak Dunford-Pettis operator from E into F is weak* Dunford-Pettis.

Theorem 3.10. *Let E and F be two Banach lattices. Then each positive weak Dunford-Pettis operator from E into F is weak* Dunford-Pettis if one of the following assertions is valid:*

1. *E has the DP* property.*
2. *F has the Grothendieck property.*
3. *F is reflexive.*
4. *F is a dual KB-space and the lattice operations in E are weakly sequentially continuous.*
5. *F is a discrete KB-space.*
6. *The norm of the topological bi-dual F'' is order continuous and the lattice operations in E are weakly sequentially continuous.*

Proof. 1. Let $T : E \longrightarrow F$ a positive operator, $(f_n) \subset F'$, $x_n \subset E$ such that $x_n \xrightarrow{w} 0$ and $f_n \xrightarrow{w^\star} 0$. The composed operator $f_n \circ T \subset E'$ is a linear function, with $f_n \circ T \xrightarrow{w^\star} 0$, since E has the DP* property, then $f_n(T(x_n)) \longrightarrow 0$, this proves that T is weak* Dunford-Pettis operator.

2. Obvious.

3. In this case F has Grothendieck property.(Reflexive spaces have Grothendieck property.)

4. , (5) and (6) Follow from theorem 2.4 of [16] (since every Dunford-Pettis operator is weak* Dunford-Pettis).

□

Bellow we study weak compactness of weak* Dunford-Pettis operators.

Theorem 3.11. *Let E and F be two Banach lattices. If each weak* Dunford-Pettis operator $T : E \longrightarrow F$ is weakly compact then one of the following assertions is valid:*

1. E' is KB-space.

2. F is a reflexive.

Proof. Let assume that E' is not KB-space, we will prove that F is reflexive.
It follows from [[1], Theorem 4.59] that E' is not order continuous, then by [[14], Proposition 2.3.11 and Theorem 2.4.14], E contains a sublattice isomorphic to l^1 and there exists a positive projection $P : E \longrightarrow l^1$.
Let consider an arbitrary positif operator $T : l^1 \longrightarrow F$.
l^1 has the Dunford-Pettis* property, then T is a weak* Dunford-Pettis operator, by assumption, T is weakly compact. Theorem 5.29 of [1] shows that F is reflexive.

□

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