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Analysis and Qualitative Behaviour of a Tenth-Order Rational Difference Equation

E. M. Elsayed[1,2] and M. T. Alharthi[1,3]

ABSTRACT: In this article, we examine the qualitative behavior of the solutions of the following difference equation

$$z_{n+1} = az_{n-4} + \frac{bz_{n-4}}{cz_{n-4} - dz_{n-9}}, \quad n = 0, 1, ...,$$

where the initial conditions z_{-9}, z_{-8}, z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}, z_{-1}, z_0 are arbitrary non-zero real numbers and a, b, c, d are positive constants.

Key Words: Difference equation, recursive sequences, system of difference equations.

Contents

1. Introduction

In recent years, many researchers has studied the theory of difference equations due to the importance of this field in modeling a large number of real- life problems. To study modeling some natural phenomena that appear in biology, physics, economy, engineering, etc we use difference equations. Recent researches have been a great deal of interest in studying boundedness character, the global attractivity and the periodic nature of nonlinear difference equations.

Many researchers have established to study the behavior of the solution of difference equations for example:

Cinar [6] studied the solution of the difference equation

$$z_{n+1} = \frac{az_{n-1}}{1 + bz_n z_{n-1}}.$$

Elsayed [15] examined the periodic solution and investigated the global stability of the following difference equation

$$z_{n+1} = az_{n-l} + \frac{bz_{n-l}}{cz_{n-l} + dz_{n-k}}.$$

El-Moneam et al. [14] presented results on the dynamic behaviour of the following difference equation

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$$z_{n+1} = Az_n + Bz_{n-k} + Cz_{n-l} + Dz_{n-\sigma} + \frac{bz_{n-k} + hz_{n-l}}{dz_{n-k} + ez_{n-l}}.$$

Khaliq and Hassan [27] investigated the qualitative behaviour of the recursive equation given on the form

$$z_{n+1} = az_n + \frac{\alpha + \beta z_{n-k}}{A + Bz_{n-k}}.$$

Elabbasy et al. [8] analyzed the global stability, periodicity character, and obtained the solution of some special cases of the difference equation

$$z_{n+1} = \frac{dz_{n-l}z_{n-k}}{cz_{n-s} - b} + a.$$

Aloqeili [4] gave the form of the solutions of the recursive equation

$$z_{n+1} = \frac{z_{n-1}}{a - +z_n z_{n-1}}.$$

Ibrahim [29] introduced some relevant results of the recursive equation

$$z_{n+1} = \frac{z_n z_{n-2}}{z_{n-1}(a + bz_n z_{n-2})}.$$

For more researches about the qualitative behavior of difference equations see refs. [1]- [40].

In this paper we study some properties of the solutions of the of the following recursive equation:

$$z_{n+1} = az_{n-4} + \frac{bz_{n-4}}{cz_{n-4} - dz_{n-9}}, n - 0, 1, ..., \tag{1.1}$$

where the initial conditions $z_{-9},\ z_{-8},\ z_{-7},\ z_{-6},\ z_{-5},\ z_{-4},\ z_{-3},\ z_{-2},\ z_{-1},\ z_0$ are arbitrary non-zero real numbers and a, b, c, d are positive constants.

2. Basic Properties and Definitions

In this section we introduce some basic definitions and theorems that we need in the sequel.

Defintion 1. Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be continuously differentiable function. Then for every set of initial condition $x_{-k},\ x_{-k+1},\ ..., x_0 \in I$,the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, x_{n-2}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 2. A point $x^* \in I$ is called an equilibrium point of Eq.(2.1) if

$$x^* = F(x^*, x^*, ..., x^*),$$

that is, $x_n = x^*$ for $n \geq 0$, is a solution of Eq.(2.1), or equivalently, x^* is a fixed point.

Definition 3. Let x^* be an equilibrium point of Eq.(2.1).

1. The equilibrium point x^* of Eq.(2.1) is called **locally stable** if for $\epsilon > 0$, there exist $\delta > 0$ such that for all $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2.1) and

$$|x_{-k} - x^*| + |x_{-k+1} - x^*| + ... + |x_0 - x^*| < \delta,$$

then

$$|x_n - x^*| < \epsilon \quad forall \quad n \geq 0.$$

2. The equilibrium point x^* of Eq.(2.1) is called **locally asymptotically stable** if it is locally stable and if there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2.1) and
$$|x_{-k} - x^*| + |x_{-k+1} - x^*| + ... + |x_0 - x^*| < \gamma,$$
then
$$\lim_{n\to\infty} = x^*.$$

3. The equilibrium point x^* of Eq.(2.1) is called **global attractor** if for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2.1) we have $\lim_{n\to\infty} = x^*$.

4. The equilibrium point x^* of Eq.(2.1) is called **globally asymptotically stable** if it is locally stable and global attractor of Eq.(2.1).

5. The equilibrium point x^* of Eq.(2.1) is called **unstable** if x^* is not locally stable.

Definition 4. The linearized equation of Eq.(2.1) about the equilibrium point x^* is the linear difference equation
$$y_{n+1} = \sum_{j=0}^{k} \frac{\partial F(x^*, x^*, ..., x^*)}{\partial x_{n-j}} y_{n-j}. \tag{2.2}$$
The characteristic equation associated with Eq.(2.2) is
$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0,$$
where
$$p_j = \frac{\partial F(x^*, x^*, ..., x^*)}{\partial x_{n-j}}.$$
Theorem 1. [34] Assume that $p_i \in R$, $i = 1, 2, ...$ and $k \in \{0, 1, 2, ...\}$.
Then
$$\sum_{i=1}^{k} |p_i| < 1,$$
is a sufficient condition for the asymptotic stability of the difference equation
$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0, \qquad n = 0, 1, 2, ...$$
Next, to prove the global attractor of the fixed points we will introduce a fundamental theorem.

Theorem 2. [34] Let $g : [a, b]^{K+1} \to [a, b]$ be a continuous function, where k is a positive integer and [a, b] is an interval of real numbers. Consider the difference equation
$$x_{n+1} = g(x_n, x_{n-1,...,}x_{n-k}), \qquad n = 0, 1, ... \tag{2.3}$$
Suppose that g satisfies the following conditions:

1. For each integer i with $1 \leq i \leq k+1$, the function $g(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

2. If m, M is a solution of the system
$$m = g(m_1, m_2, ..., m_{k+1}), \quad M = g(M_1, M_2, ..., M_{k+1}),$$
then $m = M$, where for each $i = 1, 2, ..., k+1$, we set
$$m_i = \left\{ \begin{array}{c} m, if \text{ g is non-decreasing in } z_i, \\ M, if \text{ g is non-increasing in } z_i \end{array} \right\},$$
$$M_i = \left\{ \begin{array}{c} M, if \text{ g is non-decreasing in } z_i, \\ m, if \text{ g is non-increasing in } z_i \end{array} \right\}.$$

Then, there exists exactly one equilibrium point x* of Eq.(2.3), and every solutionof Eq.(2.3) converges to x*.

3. Periodic solutions

In this section we Introduce the following theorem states the necessary and sufficient conditions to study the existence of periodic solutions of Eq.(1.1).

Theorem 3. There are a positive prime period two solutions of Eq.(1.1) if and only if

$$(c+d)(a+1) > 4d, \quad d > ac. \tag{3.1}$$

Proof. Firstly, Let there exists a prime period two solution

$$..., \tau, t, \tau, t, ...,$$

of Eq.(1.1). We will prove that Condition (3.1) holds.

From Eq.(1.1) we see that

$$\tau = at + \frac{bt}{ct - d\tau},$$

and

$$t = a\tau + \frac{b\tau}{c\tau - dt}.$$

Then

$$c\tau t - d\tau^2 = act^2 - ad\tau t + bt, \tag{3.2}$$

and

$$c\tau t - dt^2 = ac\tau^2 - ad\tau t + b\tau. \tag{3.3}$$

Subtract (3.3) from (3.2) we get

$$\begin{aligned} d(t^2 - \tau^2) &= ac(t^2 - \tau^2) + b(t - \tau), \\ d(t-\tau)(t+\tau) &= ac(t-\tau)(t+\tau) + b(t-\tau) \end{aligned}$$

Since $\tau \neq t$, it follows that

$$t + \tau = \frac{b}{d - ac}. \tag{3.4}$$

Now, adding (3.2) and (3.3) gives

$$2c\tau t - d(\tau^2 + t^2) = ac(\tau^2 + t^2) - 2ad\tau t + b(\tau + t). \tag{3.5}$$

We will use the following relation with (3.4), (3.5)

$$\tau^2 + t^2 = (\tau + t)^2 - 2\tau t \quad for\ all\ \tau, t \in R,$$

yields that

$$\tau t = \frac{b^2 d}{(d-ac)^2 (c+d)(a+1)}. \tag{3.6}$$

From Eq.(3.4) and Eq.(3.6) it is clear that τ and t are the two positive distinct roots of the quadratic equation

$$(d - ac)\lambda^2 - b\lambda + \frac{b^2 d}{(d-ac)(c+d)(a+1)} = 0. \tag{3.7}$$

$$\lambda_{1,2} = \frac{b + \sqrt{b^2 - \frac{4b^2 d}{(c+d)(a+1)}}}{2(d-ac)},$$

where $\beta = \sqrt{b^2 - \frac{4b^2 d}{(c+d)(a+1)}}$.

If $\beta > 0$,we get

$$b^2 > \frac{4b^2 d}{(c+d)(a+1)}.$$

Therefore the inequality (3.1) holds.

Secondly, suppose that the inequality (3.1) is true. We will prove that Eq.(1.1) has a prime period two solution.

$$\tau = \frac{b+\beta}{2(d-ac)},$$

and

$$t = \frac{b-\beta}{2(d-ac)}.$$

β is a real positive numbers, therefore τ and t are distinct positive real numbers.

Put

$$z_{-1} = \tau \quad \text{and} \quad z_0 = t.$$

We will show that $z_1 = z_{-1} = \tau \; and \; z_{-2} = z_0 = t.$

From Eq.(1.1)

$$\begin{aligned} z_1 &= at + \frac{bt}{ct - d\tau} = \frac{act^2 - ad\tau t + bt}{ct - d\tau} \\ &= \frac{ac[\frac{b-\beta}{2(d-ac)}]^2 - ad[\frac{b^2 d}{(d-ac)^2(c+d)(a+1)}] + b[\frac{b-\beta}{2(d-ac)}]}{c[\frac{b-\beta}{2(d-ac)}] - d[\frac{b+\beta}{2(d-ac)}]}. \end{aligned}$$

We Multiply the numerator and denominator by $4(d-ac)^2$

$$z_1 = \frac{2b^2 d - \frac{4ab^2cd + 4ab^2d^2}{(c+d)(a+1)} - 2bd\beta}{2(d-ac)\{cb - bd - (c+d)\beta\}}.$$

Multiplying the numerator and denominator by $\{(c+d)(a+1)\}\{cb - bd + (c+d)\beta\}$ gives

$$z_1 = \frac{[(4b^3d^3 + 4b^3cd^2 - 4ab^3cd^2 - 4ab^3c^2d) + (4b^2cd^2 + 4b^2d^3 - 4ab^2cd^2 - 4ab^2c^2d)\beta}{2(d-ac)\{4b^2cd^2 + 4b^2d^3 - 4ab^2cd^2 - 4ab^2c^2d\}}.$$

Dividing the numerator and denominator by $\{4b^2cd^2 + 4b^2d^3 - 4ab^2cd^2 - 4ab^2c^2d\}$ yields

$$z_1 = \frac{b+\beta}{2(d-ac)} = \tau.$$

Similarly we can show that

$$z_2 = t.$$

Then by induction we get that

$$z_{2n} = t \;\; and \;\; z_{2n+1} = \tau \; for \; all \; n \geq -5.$$

Finally, Therfore Eq.(1.1) has the positive prime period two solution

$$..., \tau, t, \tau, t, ...,$$

Considering that the distinct roots of the quadratic equation (3.7) are τ and t, and the proof is complete.

4. Analysis of Local Stability of the Equilibrium Point

This part studied the local stability character of the equilibrium point of Eq.(1.1). The equilibrium point of Eq.(1.1) are given by

$$z^* = az^* + \frac{bz^*}{cz^* - dz^*}.$$

If $(c-d)(1-a) > 0$, then the only positive fixed points of Eq.(1.1) is given by

$$z^* = \frac{b}{(c-d)(1-a)},\ \ c \neq d,\ \ a \neq 1.$$

Suppose that $f : (0,\infty)^2 \to (0,\infty)$ be a continuous function defined by

$$f(r,s) = ar + \frac{br}{cr - ds}. \tag{4.1}$$

Therefore

$$\frac{\partial f(r,s)}{\partial r} = a - \frac{bds}{(cr-ds)^2},$$

$$\frac{\partial f(r,s)}{\partial s} = \frac{bdr}{(cr-ds)^2}.$$

Then

$$\frac{\partial f(z^*,z^*)}{\partial r} = a - \frac{bdz^*}{(cz^*-dz^*)^2} = a - \frac{d(1-a)}{(c-d)} = p_0,$$

$$\frac{\partial f(z^*,z^*)}{\partial s} = \frac{bdz^*}{(cz^*-dz^*)^2} = \frac{d(1-a)}{(c-d)} = p_1.$$

Then, the linearized equation of Eq.(1.1) about the equilibrium is

$$y_{n+1} - p_0 y_{n-4} - p_1 y_{n-9} = 0.$$

Theorem 4. Let

$$|ac - d| + d\,|1-a| < |c-d|\,.$$

Then the fixed point of Eq.(1.1) is locally asymptotically stable.

Proof. By using Theorem (1), The Eq.(1.1) is asymptotically stable if

$$|p_0| + |p_1| < 1.$$

This gives

$$\left| a - \frac{d(1-a)}{(c-d)} \right| + \left| \frac{d(1-a)}{(c-d)} \right| < 1,$$

it can be written as

$$|a(c-d) - d(1-a)| + |d(1-a)| < |c-d|\,.$$

Hence,

$$|ac - d| + d\,|1-a| < |c-d|\,.$$

The proof is complete.

5. Global Attractivity of the Equilibrium Point of Eq.(1.1)

In this part, we examine the global attractivity character of solutions of Eq.(1.1).

Theorem 5. If $ac > d, a \neq 1$, then equilibrium point z^* of Eq.(1.1) is a global attractor.

Proof. Let a, b are a real numbers and assume that $f : [a, b]^2 \to [a, b]$ be a function defined by Eq.(4.1).Therefore

$$\begin{aligned}\frac{\partial f(r,s)}{\partial r} &= a - \frac{bds}{(cr-ds)^2},\\ \frac{\partial f(r,s)}{\partial s} &= \frac{bds}{(cr-ds)^2}.\end{aligned}$$

Case (1) If $a - \frac{bds}{(cr-ds)^2} > 0$, then it is clear that the function f(r, s) is increasing in r, s. Let (ω, ζ) is a solution of the system where M=ζ and m=ω as following Theorem (2).

$$\omega = g(\omega,\omega) \ \ and \ \ \zeta = g(\zeta,\zeta).$$

Then from Eq.(1.1), we get that

$$\begin{aligned}\omega &= a\omega + \frac{b\omega}{c\omega - d\omega},\\ \zeta &= a\zeta + \frac{b\zeta}{c\zeta - d\zeta}.\end{aligned}$$

Then

$$(\zeta - \omega) = a(\zeta - \omega), a \neq 1.$$

Hence

$$\zeta = \omega.$$

Then by Theorem (2) z* is a global attractor of Eq.(1.1).

Case (2) If $a - \frac{bds}{(cr-ds)^2} < 0$,then it is clear that the function f(r, s) decreasing in r and increasing in s. Let (ω, ζ) is a solution of the system where M=ζ and m=ω as following Theorem (2).

$$\zeta = g(\omega,\zeta) \ \ and \ \ \omega = g(\zeta,\omega).$$

Then from Eq.(1.1), we get that

$$\begin{aligned}\omega &= a\zeta + \frac{b\zeta}{c\zeta - d\omega},\\ \zeta &= a\omega + \frac{b\omega}{c\omega - d\zeta},\end{aligned}$$

$$\begin{aligned}c\zeta\omega - ac\zeta^2 - d\omega^2 + ad\zeta\omega &= b\zeta,\\ c\zeta\omega - ac\omega^2 - d\zeta^2 + ad\zeta\omega &= b\omega,\end{aligned}$$

Therefore

$$(\zeta^2 - \omega^2)(d - ac) = b(\zeta - \omega), ac > d.$$

Hence

$$\zeta = \omega.$$

Then by Theorem (2), z* is a global attractor of Eq.(1.1). The proof is complete.

6. Special Cases of Eq.(1.1)

This section studies the following special case of Eq.(1.1)

$$z_{n+1} = z_{n-4} + \frac{z_{n-4}}{z_{n-4} - z_{n-9}}, \tag{6.1}$$

where the initial conditions $z_{-9}, z_{-8}, z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}, z_{-1}, z_0$ are arbitrary non-zero real numbers .

Theorem 6. Assume that $\{z_n\}_{n=-9}^{\infty}$ be the solution of Eq.(6.1) satisfying $z_{-9} = r$, $z_{-8} = k$, $z_{-7} = h$, $z_{-6} = g$, $z_{-5} = f$, $z_{-4} = e$, $z_{-3} = d$,$z_{-2} = c$, $z_{-1} = b$ and $z_0 = a$. Then, for $n = 0, 1, 2....$

$$z_{10n-9} = \frac{[ne - (n-1)r][e - r + n]}{(e - r)},$$

$$z_{10n-8} = \frac{[nd - (n-1)k][d - k + n]}{(d - k)},$$

$$z_{10n-7} = \frac{[nc - (n-1)h][c - h + n]}{(c - h)},$$

$$z_{10n-6} = \frac{[nb - (n-1)g][b - g + n]}{(b - g)},$$

$$z_{10n-5} = \frac{[na - (n-1)f][a - f + n]}{(a - f)},$$

$$z_{10n-4} = \frac{[(n+1)e - nr][e - r + n]}{(e - r)},$$

$$z_{10n-3} = \frac{[(n+1)d - nk][d - k + n]}{(d - k)},$$

$$z_{10n-2} = \frac{[(n+1)c - nh][c - h + n]}{(c - h)},$$

$$z_{10n-1} = \frac{[(n+1)b - ng][b - g + n]}{(b - g)},$$

$$z_{10n} = \frac{[(n+1)a - nf][a - f + n]}{(a - f)}.$$

Proof. It is clear that for $n = 0$, the result holds. Now let $n > 0$ and for $n - 1$ The assumption holds. That is;

$$z_{10n-19} = \frac{[(n-1)e - (n-2)r][e - r + n - 1]}{(e - r)},$$

$$z_{10n-18} = \frac{[(n-1)d - (n-2)k][d - k + n - 1]}{(d - k)},$$

$$z_{10n-17} = \frac{[(n-1)c - (n-2)h][c - h + n - 1]}{(c - h)},$$

$$z_{10n-16} = \frac{[(n-1)b - (n-2)g][b - g + n - 1]}{(b - g)},$$

$$z_{10n-15} = \frac{[(n-1)a-(n-2)f][a-f+n-1]}{(a-f)},$$

$$z_{10n-14} = \frac{[ne-(n-1)r][e-r+n-1]}{(e-r)},$$

$$z_{10n-13} = \frac{[nd-(n-1)k][d-k+n-1]}{(d-k)},$$

$$z_{10n-12} = \frac{[nc-(n-1)h][c-h+n-1]}{(c-h)},$$

$$z_{10n-11} = \frac{[nb-(n-1)g][b-g+n-1]}{(b-g)},$$

$$z_{10n-10} = \frac{[na-(n-1)f][a-f+n-1]}{(a-f)}.$$

Now, it follows that from Eq.(6.1)

$$\begin{aligned}
z_{10n-9} &= z_{10n-14} + \frac{z_{10n-14}}{z_{10n-14} - z_{10n-19}} \\
&= z_{10n-14}[1 + \frac{1}{\frac{[ne-(n-1)r][e-r+n-1]}{(e-r)} - \frac{[(n-1)e-(n-2)r][e-r+n-1]}{(e-r)}}] \\
&= z_{10n-14}[1 + \frac{(e-r)}{[e-r+n-1][ne-nr+r-ne+e+nr-2r]}] \\
&= z_{10n-14}[1 + \frac{(e-r)}{[e-r+n-1](e-r)}] \\
&= \frac{[ne-(n-1)r][e-r+n-1]}{(e-r)} . \frac{[e-r+n-1+1]}{[e-r+n-1]} \\
&= \frac{[ne-(n-1)r][e-r+n]}{(e-r)}.
\end{aligned}$$

$$\begin{aligned}
z_{10n-8} &= z_{10n-13} + \frac{z_{10n-13}}{z_{10n-13} - z_{10n-18}} \\
&= z_{10n-13}[1 + \frac{1}{\frac{[nd-(n-1)k][d-k+n-1]}{(d-k)} - \frac{[(n-1)d-(n-2)k][d-k+n-1]}{(d-k)}}] \\
&= z_{10n-13}[1 + \frac{(d-k)}{[d-k+n-1][nd-nk+k-nd+d+nk-2k]}] \\
&= z_{10n-13}[1 + \frac{(d-k)}{[d-k+n-1](d-k)}] \\
&= \frac{[nd-(n-1)k][d-k+n-1]}{(d-k)} . \frac{[d-k+n-1+1]}{[d-k+n-1]} \\
&= \frac{[nd-(n-1)k][d-k+n]}{(d-k)}.
\end{aligned}$$

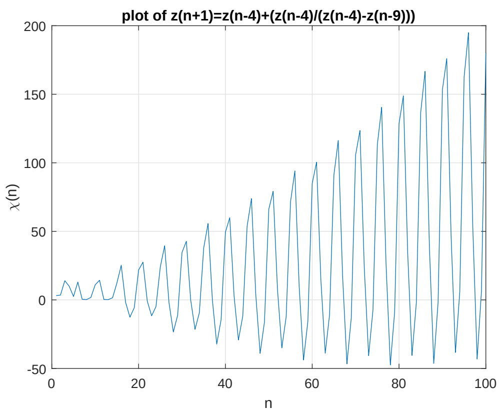
$$\begin{aligned}
z_{10n-1} &= z_{10n-6}+\frac{z_{10n-6}}{z_{10n-6}-z_{10n-11}} \\
&= z_{10n-6}[1+\frac{1}{\frac{[nb-(n-1)g][b-g+n]}{(b-g)}-\frac{[nb-(n-1)g][b-g+n-1]}{(b-g)}}] \\
&= z_{10n-6}[1+\frac{(b-g)}{[nb-(n-1)g][b-g+n-b+g-n+1]}] \\
&= z_{10n-6}[\frac{nb-ng+g+b-g}{nb-(m-1)g}] \\
&= \frac{[nb-(n-1)g][b-g+n]}{(b-g)}.\frac{[(n+1)b-ng]}{[nb-(n-1)g]} \\
&= \frac{[(n+1)b-ng][b-g+n]}{(b-g)}.
\end{aligned}$$

Similarly the other relations can be proofed.

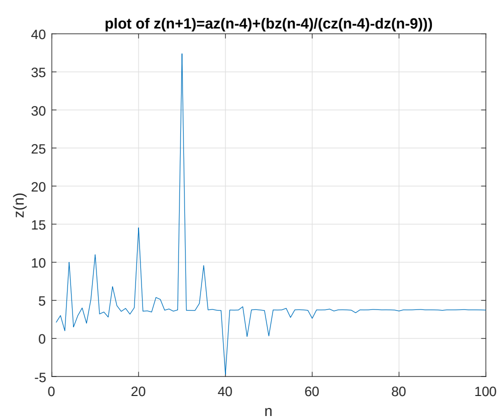
7. Numerical Example

Here we will introduce some numrical examples which represent different types of Eq.(1.1)

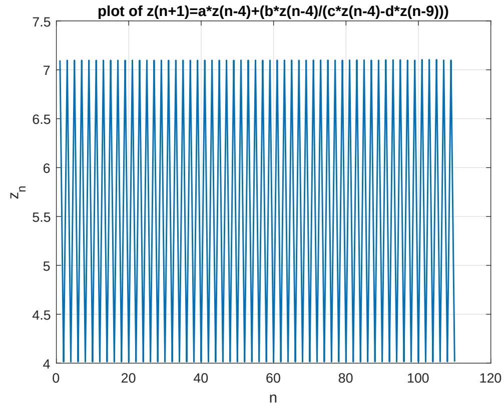
Example 1. In Eq.(1.1), consider $z_{-9}=11$, $z_{-8}=1.75$, $z_{-7}=0.20$, $z_{-6}=.45$, $z_{-5}=13$, $z_{-4}=2.5$, $z_{-3}=10$, $z_{-2}=14$, $z_{-1}=3.5$, $z_0=3.1$, $a=1$, $b=1$, $c=1$, $d=1$. See Figure 1.



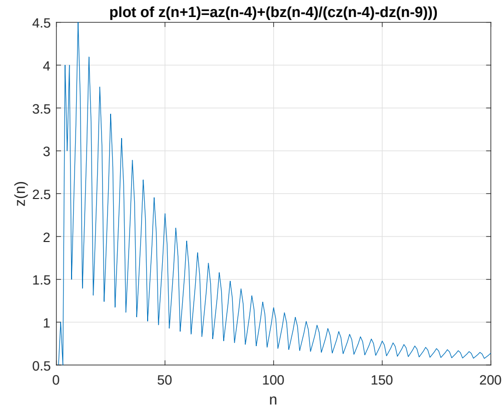
Example 2. Let $z_{-9}=11$, $z_{-8}=5.1$, $z_{-7}=2$, $z_{-6}=4$, $z_{-5}=3$, $z_{-4}=1.5$, $z_{-3}=10$, $z_{-2}=1$, $z_{-1}=3$, $z_0=2.1$, $a=0.2$, $b=6$, $c=3$, $d=1$.We substitute the values into Eq.(1.1). See Figure 2



Example 3. See Figure 3. We put $z_{-9} = z_{-7} =$, $z_{-5} = z_{-3} = z_{-1} = 7.09639 = \tau = (\frac{b+\beta}{2(d-ac)})$, $z_{-8} = z_{-6} = z_{-4} = z_{-2} = z_0 = 4.0147 = t = (\frac{b-\beta}{2(d-ac)})$, $a = 0.3$, $b = 5$, $c = 3.5$, $d = 1.5$, in Eq.(1.1).



Example 4. See the bellow Figure. By using Eq.(1.1), let $z_{-9} = 6$, $z_{-8} = 11.75$, $z_{-7} = 0.20$, $z_{-6} = .5$, $z_{-5} = 12$, $z_{-4} = 3.75$, $z_{-3} = 1.5$, $z_{-2} = 15$, $z_{-1} = 0.9$, $z_0 = 4$, $a = 3$, $b = 1.5$, $c = 3$, $d = 2$.



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E. M. Elsayed,
Department of Mathematics ,
King AbdulAziz University, Faculty of Science,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.

Department of Mathematics,
Faculty of Science, Mansoura University,
Mansoura 35516, Egypt.
E-mail address: emmelsayed@yahoo.com

and

M. T. Alharthi (Corresponding author),
Department of Mathematics,
King AbdulAziz University, Faculty of Science,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.

Department of Mathematics,
Faculty of Science, Jeddah University,
Saudi Arabia.
E-mail address: miami-992@hotmail.com