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Existence of Solutions for Elliptic Systems Involving the Fractional $p(x)$-Laplacian

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ABSTRACT:
In this paper, we consider a class of non-homogeneous fractional $(p(.), q(.))-$Laplacian systems of the form

$$(\mathcal{S}^s) \qquad \begin{cases} \left(-\Delta_{p(x)}\right)^s u = \frac{\partial F}{\partial u}(x,u,v) & \text{in} \quad \Omega, \\ \left(-\Delta_{q(x)}\right)^s v = \frac{\partial F}{\partial v}(x,u,v) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, $\left(-\Delta_{p(x)}\right)^s$ is the fractional $p(x)$-Laplacian and the gradient F is a nonlinear C^1-functional. Using variational methods and combined with the theory of Lebesgue and fractional Sobolev spaces with variable exponents, we prove the existence of solutions to system $(\mathcal{S}^s)$.

Key Words: Fractional $p(x,.)$-Laplacian, Generalized fractional Sobolev spaces, Critical points, Variational system.

Contents

1. Introduction

Since the fractional Sobolev spaces $W^{s,q(.),p(.,.)}(\Omega)$ where thoroughly studied by Kaufmann [24], Lebesgue and fractional Sobolev spaces with variable exponents have been used very recently to models various phenomena which describe clearly the complex terms in our concrete real word. Such that, the thin obstacle problems [27], anomalous diffusion [25], finance [13], optimization [19], phase transition [1,28], game theory and Lévy processes [10] etc., also we refer to [3,5,14,21,22] for some properties researches of such spaces. These later become a best framework to deal with a special kinds of problems which are usually called nonlocal problems, taken this name due to the presence of the integral over the entire domain. These problems are driven by the so called fractional $p(x)$-Laplacian *i.e* $\left(-\Delta_{p(x)}\right)^s$ defined by

$$\left(-\Delta_{p(x)}\right)^s u(x) := \lim_{\varepsilon \to 0} \int_{\Omega \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y))}{|x-y|^{N+sp(x,y)}} dy, \quad \text{for } x \in \Omega,$$

where $B_\varepsilon(x) := \{y \in \Omega, |x-y| \leq \varepsilon\}$, for $s \in (0,1)$.

The fractional $p(x)$-Laplacian namely in general, pseudo-differential or integro-differential operator. On the other hand, we remark that it is a fractional version of the well known $p(x)$-Laplacian. This operator possesses more complicated nonlinearity then the non local p-Laplacian due to the fact that $\left(-\Delta_{p(x)}\right)^s$ is not homogeneous. This fact implies difficulties in transposing the results obtained in both local and nonlocal p-Laplacian to the problems arising the nonlocal $p(x)$-Laplacian.

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Using different approaches, many efforts have been devoted to study such problems, among which we mention ([2], [4], [5], [7], [8], [11], [12], [26]). We also refer to the recent monographs by Bisci and Dipierro ([6], [15]), for related problems concerning the variational analysis of solutions of some classes of nonlocal problems.

In the present paper, we are concerned with the following system

$$(\mathcal{S}^s) \qquad \begin{cases} \left(-\Delta_{p(x)}\right)^s u = \frac{\partial F}{\partial u}(x,u,v) & \text{in} \quad \Omega, \\ \left(-\Delta_{q(x)}\right)^s v = \frac{\partial F}{\partial v}(x,u,v) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary $\partial\Omega$. Furthermore, the nonlinearities on the right side are the gradient of a C^1-functional F supposed to achieve some growth conditions, and $p, q : \Omega \times \Omega \to]1, +\infty[$ are two continuous bounded functions verifying

$$1 < \min\left\{p^-, q^-\right\} < \max\left\{p^+, q^+\right\} < \min\left\{p_s^*(x), q_s^*(x)\right\},$$

and

$$p^- < q^+ \text{ and } q^- < p^+,$$

where

$$p^- := \inf_{(x,y)\in\Omega\times\Omega} p(x,y), p^+ := \sup_{(x,y)\in\Omega\times\Omega} p(x,y),$$

$$q^- := \inf_{(x,y)\in\Omega\times\Omega} q(x,y), q^+ := \sup_{(x,y)\in\Omega\times\Omega} q(x,y).$$

The critical fractional Sobolev exponent is given by

$$m_s^*(x) := \begin{cases} \frac{Nm(x,x)}{N-sm(x,x)} & \text{if} \quad N > sm(x,x), \\ +\infty & \text{if} \quad N \leq sm(x,x). \end{cases}$$

To the best of our knowledge, systems structured as $(\mathcal{S}^s)$ are few. But in our context, we note the following recent work due by Boumazourh et Al. [8]

$$\begin{cases} \left(-\Delta_{p(x)}\right)^s u = \frac{\partial F}{\partial u}(x,u,v) & \text{in} \quad \Omega, \\ \left(-\Delta_{q(x)}\right)^s v = \frac{\partial F}{\partial v}(x,u,v) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The authors have obtained an existence result using mountain pass theorem, extended by that the work of El Hamidi [20] to the nonlocal case.

Moreover, in the local case $(s = 1)$, Djellit, Youbi and Tas [18] have been investigated the existence of solutions to the following sublinear system

$$\begin{cases} -\Delta_{p(x)} u = \frac{\partial F}{\partial u}(x,u,v) & \text{in} \quad \mathbb{R}^N, \\ -\Delta_{q(x)} v = \frac{\partial F}{\partial v}(x,u,v) & \text{in} \quad \mathbb{R}^N, \end{cases}$$

and generalized by that, the results obtained in the local form with (p, q)-Laplacian operators founded in [17].

Motivated by the several results above mentioned, the research work in this paper is the continuation of these papers, where it involves a more general class of operator than given in [17] and [18]. Specifically, our aim in this paper is to establish the existence of solutions for $(\mathcal{S}^s)$ by invoking variational methods.

The article has the following structure, Section 2 devoted to introduce some notations needed for the framework on the paper, and we recall some tools defined by the theory of variable exponents Lebesgue and fractional Sobolev spaces, Section 3 states the main results and its proofs.

2. Notations and Preliminaries

In this section, we introduce some useful properties on Lebesgue and fractional Sobolev spaces with variable exponents found in ([5], [14], [21], [22], [24]) and references therein for more details.

Set

$$C_+(\overline{\Omega}) = \{h \in C(\Omega); h(x) > 1, \quad \forall\, x \in \Omega\}.$$

For $p \in C_+(\Omega)$, let $L^{p(x)}(\Omega)$ the variable exponent Lebesgue space defined as

$$L^{p(x)}(\Omega) := \left\{ u : u \text{ is a measurable real-valued function, } \varrho(u) = \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{p(.),\Omega} := \|u\|_{L^{p(.)}(\Omega)} = \inf \left\{ \lambda > 0 : \varrho(\frac{u(x)}{\lambda}) < 1 \right\}.$$

The space $\left(L^{p(x)}(\Omega), \|u\|_{p(.),\Omega}\right)$ is a reflexive, uniformly convex and separable Banach space (see [21]), and its conjugate space is $L^{q'(x)}(\Omega)$ where $q'(x) = \frac{q(x)}{q(x)-1}$.

Proposition 2.1. ([21], [22]) *For $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the Hölder's inequality*

$$\int_\Omega |uv| dx \leq (\frac{1}{p^-} + \frac{1}{p'^-}) \|u\|_{p(.),\Omega} \|v\|_{p'(.),\Omega} \leq 2 \|u\|_{p(.),\Omega} \|v\|_{p'(.),\Omega}.$$

Proposition 2.2. ([21], [22]) *If $u \in L^{p(x)}(\Omega)$, then*

$$\min\{\|u\|^{p^-}_{p(.),\Omega}, \|u\|^{p^+}_{p(.),\Omega}\} \leq \varrho(u) \leq \max\{\|u\|^{p^-}_{p(.),\Omega}, \|u\|^{p^+}_{p(.),\Omega}\}.$$

In addition, we have

1. $\|u\|_{p(.),\Omega} < 1 (resp. = 1, > 1) \Leftrightarrow \varrho(u) < 1 (resp. = 1, > 1)$,
2. $\|u\|_{p(.),\Omega} < 1 \Rightarrow \|u\|^{p^+}_{p(.),\Omega} \leq \varrho(u) \leq \|u\|^{p^-}_{p(.),\Omega}$,
3. $\|u\|_{p(.),\Omega} > 1 \Rightarrow \|u\|^{p^-}_{p(.),\Omega} \leq \varrho(u) \leq \|u\|^{p^+}_{p(.),\Omega}$,
4. $\varrho(\frac{u}{\|u\|_{p(.),\Omega}}) = 1$.

Proposition 2.3. ([21], [22]) *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$ a.e. in Ω. If $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \leq || u |^{p(x)}|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)},$$

$$|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \leq || u |^{p(x)}|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)}.$$

In particular, if $p(x) = p$ is a constant, then $|| u |^p|_{q(x)} = | u |^p_{pq(x)}$.

Next, by fixing $0 < s < 1$ and let $p \in C(\Omega \times \Omega, (1, \infty))$ be such that p is symmetric, *i.e.* $p(x, y) = p(y, x)\ \forall x, y \in \Omega$.

For $p(x) = p(x, x) \in C^+(\Omega)$, we introduce the fractional variable exponent Sobolev space as follows:

$$W^{s,p(.,.)}(\Omega) := W^{s,p(.),p(.,.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega), \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy < \infty \right\},$$

endowed with the norm

$$\|u\|_{s,p(.),\Omega} := \|u\|_{W^{s,p(.),p(.,.)}(\Omega)} = \|u\|_{p(.),\Omega} + [u]_{s,p(.,.)}\,,$$

where the symbol $[u]_{s,p(.,.)}$ defined by

$$[u]_{s,p} := [u]_{s,p(.,.)} := \inf\left\{\lambda > 0 : \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}\,|x-y|^{N+sp(x,y)}}dxdy < 1\right\},$$

is the corresponding variable exponents Gagliardo semi-norm.

The space $\left(W^{s,p(.,.)}(\Omega), \|u\|_{s,p(.),\Omega}\right)$ is a separable, reflexive Banach space (See [5], [24]) .

Proposition 2.4. ([5], [24]) *Let $s \in]0,1[$ and $p(x) : \Omega \to \mathbb{R}$ is Lipschitz-continuous function, then there exists a positive constant c such that*

$$|u|_{p_s^*(x)} \le c\,\|u\|_{s,p(.),\Omega}\,,\ \forall u \in W^{s,p(.,.)}(\Omega).$$

Proposition 2.5. ([5], [24]) *Let $s \in]0,1[$ and $p, q \in C(\Omega\ )$ be such that $1 \le q(x) \le p_s^*(x)$ for all $x \in \Omega$ with $sp(x) < N$. Assume moreover that the functions p and q are uniformly continuous. Then the embedding $W^{s,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous. Moreover, if $\inf_\Omega\ (p_s^*(x) - q(x)\,) > 0$ then, the embedding $W^{s,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.*

Remark 2.6. *1. The above theorem remains true when we replace $W^{s,p(x)}(\Omega)$ by $W_0^{s,p(x)}(\Omega)$ where $W_0^{s,p(.,.)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{s,p(.,.)}(\Omega)$ with respect to the norm $\|u\|_{s,p(.),\Omega}$.*

2. The embedding of $W^{s,p(x)}(\Omega)$ in $L^{p(x)}(\Omega)$ implies that $[u]_{s,p}$ is a norm on $W_0^{s,p(x)}(\Omega)$ which is equivalent to $\|u\|_{s,p(.),\Omega}$.

For $u \in W^{s,p(.,.)}(\Omega)$, we define the modular function $\rho_{p(.,.)} : W^{s,p(.,.)}(\Omega) \to \mathbb{R}$, by

$$\rho_{p(.,.)}(u) = \int_\Omega\int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy.$$

Proposition 2.7. ([24]) *Let $p : \Omega\times\Omega \to]1,+\infty[$ be a continuous bounded functions and $s \in]0,1[$. For any $u \in W^{s,p(.,.)}(\Omega)$, we have*

1. $\|u\|_{s,p} < 1(resp. = 1, > 1) \Leftrightarrow \rho_{p(.,.)}(u) < 1(resp. = 1, > 1)$,
2. $\|u\|_{s,p} < 1 \Rightarrow [u]_{s,p}^{p+} \le \rho_{p(.,.)}(u) \le [u]_{s,p}^{p-}$,
3. $\|u\|_{s,p} \ge 1 \Rightarrow [u]_{s,p}^{p-} \le \rho_{p(.,.)}(u) \le [u]_{s,p}^{p+}$,
4. $\rho_{p(.,.)}(\frac{u}{[u]_{s,p}}) = 1$.

3. Main results

The goal of this section is to establish the existence of solutions for the system $(\mathcal{S}^s)$ in X, where X denote the product space $W_0^{s,p(.,.)}(\Omega) \times W_0^{s,q(.,.)}(\Omega)$ which is equipped with norm

$$\|(u,v)\|_X = \max\left([u]_{s,p}\,, [v]_{s,q}\right).$$

The space X^* is the topological dual of X endowed with the usual dual norm.

Throughout the rest of the paper the letters $c_i, i = 1, 2, ...,$ denote positive constants which may vary from line to line.

Definition 3.1. *The pair $(u,v) \in X$ is called a weak solution for $(\mathcal{S}^s)$ if*

$$
\begin{aligned}
&\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}dxdy \\
&+\int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{q(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sq(x,y)}}dxdy \\
&-\int_{\Omega} \frac{\partial F}{\partial u}(x,u,v)\varphi(x)\,dx - \int_{\Omega} \frac{\partial F}{\partial v}(x,u,v)\psi(x)\,dx = 0,
\end{aligned}
$$

for all (u,v) and (φ,ψ) in X.

3.1. Hypotheses

In this part, we give that the nonlinearity F satisfies the following hypotheses

$(\mathcal{F}_1)$ $F \in C^1(\Omega,\mathbb{R},\mathbb{R})$ and $F(x,0,0)=0$.

$(\mathcal{F}_2)$ For all $U=(u,v)\in\mathbb{R}^2$ and for almost every $x\in\Omega$

$$
\begin{aligned}
\left|\frac{\partial F}{\partial u}(x,U)\right| &\leq a_1(x)|U|^{p_1^+-1}+a_2(x)|U|^{p_2^--1},\\
\left|\frac{\partial F}{\partial v}(x,U)\right| &\leq b_1(x)|U|^{q_1^+-1}+b_2(x)|U|^{q_2^--1},
\end{aligned}
$$

where $1<p_1(x),p_2(x)<\min(p(x),q(x))<\frac{N}{S}$ and $1<q_1(x),q_2(x)<q(x)$. The weight functions a_i and b_i, $(i=1,2)$ belong respectively to the generalized Lebesgue spaces $L^{\alpha_i(x)}(\Omega)\cap L^{\beta_i(x)}(\Omega)$ and $L^{\gamma_i(x)}(\Omega)\cap L^{\delta_i(x)}(\Omega)$, where

$$
\begin{aligned}
\alpha_i(x) &= \frac{p_s^*(x)}{p_s^*(x)-p_i^j}, \beta_i(x)=\frac{p_s^*(x)q_s^*(x)}{p_s^*(x)q_s^*(x)-p_s^*(x)\left(p_i^j-1\right)-q_s^*(x)},\\
\gamma_i(x) &= \frac{q_s^*(x)}{q_s^*(x)-q_i^j}, \delta_i(x)=\frac{p_s^*(x)q_s^*(x)}{p_s^*(x)q_s^*(x)-q_s^*(x)\left(q_i^j-1\right)-p_s^*(x)}.
\end{aligned}
$$

with $j=-$ or $+$.

$(\mathcal{F}_3)$ There exist $M>0$ and $1<l_1,l_2<\min(p^-,q^-)$, such that, for $|s|+|t|\geq 2M$, we have

$$
\liminf_{(s,t)\to 0}\frac{F(x,s,t)}{|s|^{l_1}+|t|^{l_2}}>0, \text{ uniformly for } x\in\Omega.
$$

Assumption $(\mathcal{F}_3)$ implies that the potential function F is sufficiently positive in a neighborhood of zero.

Now, we are ready to state our result.

Theorem 3.2. *Under the assumptions $(\mathcal{F}_1)-(\mathcal{F}_3)$, the problem $(\mathcal{S}^s)$ has at least one nontrivial solution.*

3.2. Proof of main result

In this subsection, we aim to prove Theorem 3.2. For that, we need the following lemmas.

Lemma 3.3. *Under the assumptions $(\mathcal{F}_1)-(\mathcal{F}_2)$, the functional F is well defined. Moreover, F is Fréchet differentiable, its derivative is*

$$
F'(u,v)(\varphi,\psi)=\int_{\Omega}\frac{\partial F}{\partial u}(x,u,v)\varphi(x)\,dx-\int_{\Omega}\frac{\partial F}{\partial v}(x,u,v)\psi(x)\,dx, \forall(u,v),(\varphi,\psi)\in X.
$$

Proof. F is well defined on X. Indeed, for all pair $(u,v)\in X$ and by the assumptions $(\mathcal{F}_1)$, we can write

$$\begin{aligned} F(x,u,v) &= \int_0^u \frac{\partial F}{\partial s}(x,s,v)ds + F(x,0,v), \\ &= \int_0^u \frac{\partial F}{\partial s}(x,s,v)ds + \int_0^v \frac{\partial F}{\partial s}(x,0,s)ds + F(x,0,0), \end{aligned}$$

then by $(\mathcal{F}_2)$, we have the following estimation

$$\begin{aligned} F(x,u,v) \leq\ & c_1[a_1(x)|u|^{p_1^+} + a_1(x)|v|^{p_1^+-1}|u| + a_2(x)|u|^{p_2^-} + a_2(x)|v|^{p_2^- -1}|u| \\ & +b_1(x)|v|^{q_1^+} + b_2(x)|v|^{q_2^-}], \end{aligned} \tag{3.1}$$

Now, by utilizing the propositions 2.1-2.4, Sobolev's imbedding and $(\mathcal{F}_2)$; we obtain

$$\begin{aligned} F(u,v) &= \int_\Omega F(x,u,v)dx, \\ &\leq c_2[|a_1|_{\alpha_1(x)}\|u\|^{p_1^+}_{p_s^*(x),\Omega} + |a_1|_{\beta_1(x)}\|v\|^{p_1^+-1}_{q_s^*(x),\Omega}\|u\|_{p_s^*(x),\Omega} + |a_2|_{\alpha_2(x)}\|u\|^{p_2^-}_{p_s^*(x),\Omega}, \\ &\quad + |a_2|_{\beta_2(x)}\|v\|^{p_2^- -1}_{q_s^*(x),\Omega}\|u\|_{p_s^*(x),\Omega} + |b_1|_{\gamma_1(x)}\|v\|^{q_1^+}_{q_s^*(x),\Omega} \\ &\quad + |b_2|_{\gamma_2(x)}\|v\|^{q_2^-}_{q_s^*(x),\Omega}], \\ &< \infty. \end{aligned}$$

□

Lemma 3.4. *Under the assumptions $(\mathcal{F}_1)-(\mathcal{F}_2)$, the functional F is weakly lower semicontionuous.*

Proof. Let (u_n,v_n) is a bounded sequence in X, up to a subsequence still denoted (u_n,v_n) weakly convergent to (u,v) in X. Let B_R be the ball of Ω centred at the origin with radius R. We denote C_{B_R} the complement of B_R in Ω and the functional F_R defined on $W_0^{s,p(.,.)}(B_R)\times W_0^{s,q(.,.)}(B_R)$ expressed by

$$F_R(u,v) = \int_{B_R} F\left(x,u\left(x\right),v\left(x\right)\right)dx,$$

It is well known that F_R is lower semicontinuous. Set

$$F(u_n,v_n) := F_R(u_n,v_n) + \int_{C_{B_R}} F\left(x,u_n,v_n\right)dx,$$

Using $(\mathcal{F}_2)$ and properties of the weights a_i and b_i, we have for R sufficiently large

$$\int_{C_{B_R}} F\left(x,u_n\left(x\right),v_n\left(x\right)\right) \to 0 \text{ as } n\to+\infty.$$

Hence , F is weakly lower semicontinuous. □

Using standard arguments as those used in [8], we show that $J\in C^1(X,\mathbb{R})$, and it is weakly lower semicontinuous, where J is defined by

$$\begin{aligned} J(u,v) &= \int_{\Omega\times\Omega}\frac{1}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dxdy \\ &\quad + \int_{\Omega\times\Omega}\frac{1}{q(x,y)}\frac{|v(x)-v(y)|^{q(x,y)}(v(x)-v(y))}{|x-y|^{N+sq(x,y)}}dxdy \end{aligned}$$

and its Fréchet derivative is given by

$$\begin{aligned} J'(u,v)(\varphi,\psi) &= \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}dxdy \\ &+ \int_{\Omega\times\Omega} \frac{|v(x)-v(y)|^{q(x,y)-2}(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+sq(x,y)}}dxdy. \end{aligned}$$

We associate to the system $(\mathcal{S}^s)$ the following Euler-Lagrange functional

$$\begin{aligned} I(u,v) &= \int_{\Omega\times\Omega} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}dxdy \\ &+ \int_{\Omega\times\Omega} \frac{1}{q(x,y)} \frac{|v(x)-v(y)|^{q(x,y)}(v(x)-v(y))}{|x-y|^{N+sq(x,y)}}dxdy \\ &- \int_\Omega \frac{\partial F}{\partial u}(x,u,v)\,dx - \int_\Omega \frac{\partial F}{\partial v}(x,u,v)\,dx. \end{aligned}$$

In other words, $I(u,v) = J(u,v) - F(u,v)$. Moreover, weak solutions of system $(\mathcal{S}^s)$ is equivalent to being critical points of the energy functional I.

Lemma 3.5. *Assume that $(\mathcal{F}_1)-(\mathcal{F}_2)$ are satisfied, then the functional I is coercive in X.*

Proof. Let $(u,v)\in X, \|(u,v)\|_X > 1$, we have

$$\begin{aligned} I(u,v) &\geq \frac{1}{p^+}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \frac{1}{q^+}\int_\Omega\int_\Omega \frac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy \\ &- \int_\Omega F(x,u(x),v(x))\,dx, \end{aligned}$$

using Young's inequality, propositions 2.1, 2.2 and 2.7, together with the estimation (3.1) we get

$$\begin{aligned} I(u,v) &\geq \frac{1}{p^+}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy + \frac{1}{q^+}\int_\Omega\int_\Omega \frac{|v(x)-v(y)|^{q(x,y)}}{|x-y|^{N+sq(x,y)}}dxdy \\ &-c_3\{|a_1|_{\alpha_1(x)}[u]_{s,p}^{p_1^+} + \frac{p_1^+-1}{p_1^+}|a_1|_{\beta_1(x)}[v]_{s,q}^{p_1^+-1} + \frac{1}{p_1^+}|a_1|_{\beta_1(x)}[u]_{s,p}^{p_1^+} \\ &+|a_2|_{\alpha_2(x)}[u]_{s,p}^{p_2^-} + \frac{p_2^--1}{p_2^-}|a_2|_{\beta_2(x)}[v]_{s,q}^{p_2^--1} + \frac{1}{p_2^-}|a_2|_{\beta_2(x)}[u]_{s,p}^{p_2^-} \\ &+|b_1|_{\gamma_1(x)}[v]_{s,q}^{q_1^+} + |b_2|_{\gamma_2(x)}[v]_{s,q}^{q_2^-}\}, \end{aligned}$$

hence

$$\begin{aligned} I(u,v) &\geq \frac{1}{p^+}[u]_{s,p}^{p^+} + \frac{1}{q^+}[v]_{s,q}^{q^+} - c_3\left(|a_1|_{\alpha_1(x)} + \frac{1}{p_1^+}|a_1|_{\beta_1(x)}\right)[u]_{s,p}^{p_1^+} \\ &-c_3\left(|a_2|_{\alpha_2(x)} + \frac{1}{p_2^-}|a_2|_{\beta_2(x)}\right)[u]_{s,p}^{p_2^-} - c_3\frac{p_1^+-1}{p_1^+}|a_1|_{\beta_1(x)}[v]_{s,q}^{p_1^+-1} \\ &-c_3\frac{p_2^--1}{p_2^-}|a_2|_{\beta_2(x)}[v]_{s,q}^{p_2^--1} - c_3|b_1|_{\gamma_1(x)}[v]_{s,q}^{q_1^+} - c_3|b_2|_{\gamma_2(x)}[v]_{s,q}^{q_2^-}, \\ &= \frac{1}{p^+}[u]_{s,p}^{p^+} + \frac{1}{q^+}[v]_{s,q}^{q^+} - c_4[u]_{s,p}^{p_1^+} - c_5[u]_{s,p}^{p_2^-} \\ &-c_6[v]_{s,q}^{p_1^+} - c_7[v]_{s,q}^{p_2^-} - c_8[v]_{s,p}^{q_1^+} - c_9[v]_{s,q}^{q_2^-}, \\ &\geq \frac{1}{p^+}[u]_{s,p}^{p^+} + \frac{1}{q^+}[v]_{s,q}^{q^+} - c_4[u]_{s,p}^{p_1^+} - c_5[u]_{s,p}^{p_2^-} \\ &-c_6[v]_{s,q}^{p_1^+} - \lambda c_7[v]_{s,q}^{p_2^-} - c_8[v]_{s,p}^{q_1^+} - c_9[v]_{s,q}^{q_2^-}. \end{aligned}$$

Since $1 < p_1(x), p_2(x) < \min(p(x), q(x))$ and $1 < q_1(x), q_2(x) < q(x)$ we infer that

$$I(u,v) \to +\infty \text{ as } \|(u,v)\| \to +\infty.$$

So we conclude that I is coercive on X. □

Lemma 3.6. *If hypotheses $(\mathcal{F}_1) - (\mathcal{F}_3)$ hold. Then there exists $(u_0, v_0) \in X$ such that*

$$\|(u_0, v_0)\| > R \text{ and } I(tu_0, tv_0) < 0.$$

for all t small enough.

Proof. Choose $(u_0, v_0) \in X$, such that $u_0 > 0, v_0 > 0$. For $t \in (0,1)$ small enough we get

$$\begin{aligned} J(tu_0, tv_0) &= [tu_0]_{s,p} + [tv_0]_{s,q}, \\ &\leq t^{\min(p^-,q^-)} \left(\frac{1}{p^-} [u_0]_{s,p}^{p^j} + \frac{1}{q^-} [v_0]_{s,q}^{q^j} \right), \end{aligned}$$

where $j = -$ or $+$.

Now according to $(\mathcal{F}_3)$, we obtain

$$\begin{aligned} I(tu_0, tv_0) &= J(tu_0, tv_0) - \int F(x, tu_0, tv_0)\, dx, \\ &\leq t^{\min(p^-,q^-)} \left(\frac{1}{p^-} [u_0]_{s,p}^{p^j} + \frac{1}{q^-} [v_0]_{s,q}^{q^j} \right) - c_{10} \left(t^{l_1} |u_0|^{l_1} + t^{l_2} |v_0|^{l_2} \right). \end{aligned}$$

Since $l_1 < p^-, l_2 < q^-$, we see that $I_\lambda(tu_0, tv_0) < 0$ for $t < \min(0, A)$, where

$$A = \left(\frac{c_{10} \left(t^{l_1} |u_0|^{l_1} + t^{l_2} |v_0|^{l_2} \right)}{\frac{1}{p^-} [u_0]_{s,p}^{p^j} + \frac{1}{q^-} [v_0]_{s,q}^{q^j}} \right)^{\frac{1}{\min(p^-,q^-)}}.$$

□

Proof. (of Theorem 3.2)

By Lemmas 3.3-3.6 the functional I is weakly lower semicontinuous and coercive on X which is a reflexive Banach space. Consequently, the functional reaches its global minimizer (u_*, v_*) in X. On the other hand, $I \in C^1(X, \mathbb{R})$ (see ([8], Lemma 3.2). Hence, this minimum is necessarily characterized by a critical point of I, which is a weak solution to $(\mathcal{S}^s)$. Moreover, the fact that $I(u_*, v_*) < 0$ implies that (u_*, v_*) is nontrivial. □

Remark 3.7. *In their article, Ho and Kim [23] showed that, $W^{q(x),p(x,y)}(\mathbb{R}^N)$ is continuously embedded in $L^{r(x)}(\mathbb{R}^N)$. Therefore, under the same conditions as in Theorem 3.2, the main result of this paper can be extended to the following problem:*

$$(\mathcal{S}) \qquad \begin{cases} \left(-\Delta_{p(x)}\right)^s u = \frac{\partial F}{\partial u}(x,u,v) & \text{in} \quad \mathbb{R}^N, \\ \left(-\Delta_{q(x)}\right)^s v = \frac{\partial F}{\partial v}(x,u,v) & \text{in} \quad \mathbb{R}^N. \end{cases}$$

Remark 3.8. *Notice that when $p(x) = p$ and $q(x) = q$, the system $(\mathcal{S}^s)$ can be reduced to the following version with nonlocal (p,q)-Laplacian operators*

$$(\mathcal{S}') \qquad \begin{cases} \left(-\Delta_p\right)^s u = \frac{\partial F}{\partial u}(x,u,v) & \text{in} \quad \Omega, \\ \left(-\Delta_q\right)^s v = \frac{\partial F}{\partial v}(x,u,v) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Whereupon, as in Theorem 3.2, and under a similar standard growth conditions, the existence result can be proved using the same variational approach, for $\Omega \subseteq \mathbb{R}^N$.

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