Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–15.
ISSN-0037-8712 in press
doi:10.5269/bspm.63218

Stability Analysis of a Delayed SEIRQ Epidemic Model with Diffusion

E. Anaama, C. Allalou and K. Hilal

ABSTRACT: In this paper, we investigate the effect of spatial diffusion and delay on the dynamical behavior of the SEIRQ epidemic model. The introduction of the delay in this model makes it more realistic and modelizes the latency period. In addition, the consideration of an SEIRQ model with diffusion aims to better understand the impact of the spatial heterogeneity of the environment and the movement of individuals on the persistence and extinction of disease. First, we determined a threshold value R_0 of the delayed SEIRQ model with diffusion. Next, By using the theory of partial functional differential equations, we have shown that the unique disease-free equilibrium is asymptotically stable , what is proven by the numericals scchema. Moreover,we search under their condition the endemic equilibrium is asymptotically stable.

Key Words: SEIRQ epidemic model, incidence rate, ordinary differential equations, delayed differential equations, partial differential equations.

Contents

1 Introduction **1**

2 The well-posedness **3**

3 Basic reproduction number and existence of equilibirum **4**

4 Local stability of the equilibria for the SEIRQ models **5**
4.1 Stability of disease-free equilibirum P 7
4.2 Stability of endemic equilibirum P* 8

5 Numerical **10**

6 Conclusion **14**

1. Introduction

The Kermack-McKendrick model is the first one to provide a mathematical description of the kinetic transmission of an epidemic in an unstructured population [9]. In this model the total population is assumed to be constant and divided into three classes: susceptible, infected (and infective), and removed (recovered with permanent immunity) and assuming that the transfers between these classes are instantaneous. The spread of an infection governed by this simple model that integrates neither diseases that have a latency period nor the influence of space on the dynamics of this model, has allowed many scientists to participate in the improvement of this model and to present more realistic models to describe the evolution of various types of epidemics. Recently, several extensions of the Kermack-McKendrick model have been proposed and analyzed, trying to take into consideration diseases that have a latency period. In reality, the transfers between the different classes (susceptible, infected and removed) are not instantaneous, because many diseases such as influenza and tuberculosis have an incubation period, that is to say the time elapsing between the moment when a susceptible individual is infected and the moment when he becomes infectious and can transmit this disease. Motivated by these reasons that characterize most diseases, Cooke [3] proposed a mathematical model formulated by delay differential equations (DDEs) to describe the spread of communicable diseases. This delayed model is an extension of [9] that incorporates a bilinear incidence function. The bilinear incidence is based on the law of mass action, which is more appropriate for communicable diseases, such as influenza, but not suitable for sexually transmitted diseases. This prompted researchers to improve the incidence function by considering

2010 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted April 10, 2022. Published October 28, 2022

a more general function. Several authors have contributed to this improvement by proposing a delayed SIR model with a more general incidence function (see, e.g., [2,23] and references cited therein). The models mentioned above have concentrated only on the temporal dimension with out diffusion. As we know, in many cases the spatial variation of population plays an important role in the disease spreading model and the time variation governs the dynamical behavior of the disease spreading, see [12]. Just as pointed in [12], an infectious case is first found at one location and then the disease spreads to other areas. However, due to the large mobility of people within a country or even worldwide, spatially uniform models are not sufficient to give a realistic picture of disease diffusion. For this reason, the spatial effects cannot be neglected in studying the spread of epidemics. Focusing on the influence of space on the qualitative behavior of the SIR epidemic model, several improvements are made (see,e.g., [20,21] and references cited therein).
In this paper, we generalize all the DDE and DDEs models PDE presented in [1,24] by proposing the following delayed SEIRQ epidemic model with spatial diffusion and bilinear incidence function:

$$\begin{aligned}
&\frac{\partial S}{\partial t}(x,t) = d\Delta S(x,t) + \Lambda - \beta \tfrac{S(x,t)(I(x,t)+qE(x,t))}{N(x,t)} - \mu S(x,t),\\
&\frac{\partial E}{\partial t}(x,t) = d\Delta E(x,t) + \beta \tfrac{S(x,t-\tau)(I(x,t-\tau)+qE(x,t-\tau))}{N(x,t-\tau)} - (\sigma+\mu)E(x,t),\\
&\frac{\partial I}{\partial t}(x,t) = d\Delta I(x,t) + \sigma E(x,t) - \left(\gamma+\delta_q+\mu\right) I(x,t),\\
&\frac{\partial R}{\partial t}(x,t) = d\Delta R(x,t) + \gamma I(x,t) + \gamma_q Q(x,t) - \mu R(x,t),\\
&\frac{\partial Q}{\partial t}(x,t) = d\Delta Q(x,t) + \delta_q I(x,t) - \left(\gamma_q+\mu\right) Q(x,t),
\end{aligned} \tag{1.1}$$

where Δ denotes the Laplacian operator, $S(x,t), E(x,t), I(x,t), Q(x,t), R(x,t)$ are the numbers of susceptible, infectious but not yet symptomatic , quarantined (or isolated) infected, and recovered individuals at location x and time t, respectively. μ is positive constants representing the natural mortality rate of the population. Λ is the recruitment rate of new individuals into the susceptible class. The positive constant d indicates the diffusion rate, β is the transmission rate, and q is the fraction of transmission rate for exposed. The exposed individuals develop symptoms at a rate σ, so $1/\sigma$ is the latent period, as many σE exposed will be infected. The number of exposed increases as many $\beta\frac{S(x,t)(I(x,t)+qE(x,t))}{N(x,t)}$ individuals after direct contact between susceptible and exposed or infected.
Likewise the infected symptomatically can be quarantined at rate δ_q, also they recover at rates γ and after quarantine (isolation) recover at rate γ_q.
Throughout this paper, we consider the system (1.1) with initial conditions

$$\begin{cases}
S(x,t) = \psi_1(x,t) \geq 0,\\
E(x,t) = \psi_2(x,t) \geq 0,\\
I(x,t) = \psi_3(x,t) \geq 0, \qquad \text{for } (x,t) \in \bar{\Omega} \times [-\tau, 0],\\
Q(x,t) = \psi_4(x,t) \geq 0,\\
R(x,t) = \psi_5(x,t) \geq 0,
\end{cases} \tag{1.2}$$

and zero-flux boundary conditions

$$\frac{\partial S}{\partial v} = \frac{\partial E}{\partial v} = \frac{\partial I}{\partial v} = \frac{\partial Q}{\partial v} = \frac{\partial R}{\partial v} = 0, t \geq 0, x \in \partial\Omega, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial v}$ represents the outside normal derivative on $\partial\Omega$. The boundary condition in (1.3) implies that susceptible,exposed, infectious, quarantined and recovered individuals do not across the boundary $\partial\Omega$.
The paper is organized as follows. In next section, we study the well-posedness for model (1.1). Section 3 is devoted to investigate to the local stability of the disease-free equilibrium and the endemic through the study of associated characteristic equations. equilibrium. In Sect. 5, to support our theoretical predictions, some numerical simulations are given. Finally, a brief conclusion is given to conclude this work.

2. The well-posedness

In this section, we focus on the well-posedness of solutions for (1.1) by establishing the global existence, uniqueness, nonnegativity and boundedness of solutions. In the following, we need some notations. Let $\mathbb{X} = \mathcal{C}\left(\overline{\Omega}, \mathbb{R}^5\right)$ be the Banach space of continuous functions from $\overline{\Omega}$ into $\mathbb{R}^5$, and $\mathcal{C}_{\mathbb{X}} = \mathcal{C}([-\tau, 0], \mathbb{X})$ denotes the Banach space of continuous $\mathbb{X}$ -valued functions on $[-\tau, 0]$ equipped with the supremum norm. For any real numbers $a \leq b, t \in [a, b]$ and any continuous function $u : [a-\tau, b] \to \mathbb{X}$, u_t is the element of $\mathcal{C}_{\mathbb{X}}$ given by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-\tau, 0]$. Moreover, we identify any element $\psi \in \mathcal{C}_{\mathbb{X}}$ as a function from $\overline{\Omega} \times [-\tau, 0]$ in $\mathbb{R}^5$ defined by $\psi(x,t) = \psi(t)(x)$ The next theorem gives us the existence and uniqueness of the global positive solution.

Theorem 2.1. *For any given initial condition $\psi \in \mathcal{C}_{\mathbb{X}}$ satisfying (1.2), the system (1.1)-(1.3) admits a unique nonnegative solution. Moreover, this solution is global and remains nonnegative.*

Proof: Let $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5) \in \mathcal{C}_{\mathbb{X}}$ and $x \in \overline{\Omega}$. We define $f = (f_1, f_2, f_3, f_4, f_5) : \mathcal{C}_{\mathbb{X}} \to \mathbb{X}$ by

$$\begin{aligned}
f_1(\psi)(x) &= A - \mu\psi_1(x,0) - \tfrac{\beta\psi_1(x,0)(\psi_3(x,0)+q\psi_2(x,0))}{N(x,0)},\\
f_2(\psi)(x) &= \tfrac{\beta\exp(-\mu\tau)\psi_1(x,-\tau)(\psi_3(x,-\tau)+q\psi_2(x,-\tau))}{N(x,-\tau)} - (\mu+\sigma)\psi_2(x,0),\\
f_3(\psi)(x) &= \sigma\psi_2(x,0) - (\mu+\delta_q+\gamma)\psi_3(x,0),\\
f_4(\psi)(x) &= \gamma\psi_3(x,0) + \gamma_q\psi_5(x,0) - \mu\psi_4(x,0),\\
f_5(\psi)(x) &= \delta_q\psi_3(x,0) - (\mu+\gamma_q)\psi_5(x,0).
\end{aligned}$$

Then the system (1.1)-(1.3) can be rewritten as an abstract differential equation in the phase space $\mathcal{C}_{\mathbb{X}}$ in the form

$$\left\{\begin{array}{ll} \dot{u} = Bu + f\left(u_t\right), & t \geq 0 \\ u(0) = \psi \in \mathcal{C}_{\mathbb{X}}, & \end{array}\right. \tag{2.1}$$

where $u(t) = (S(.,t), E(.,t), I(.,t), R(.,t), Q(.,t))^{\top}, \psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)$ and $Bu = (d\Delta S, d\Delta E, d\Delta I, d\Delta R, d\Delta Q)$. We can easily show that f is locally Lipschitz in $\mathcal{C}_{\mathbb{X}}$. According to [5,10,11,18,22] , we deduce that the system (2.1) admits a unique local solution on its maximal interval of existence $[0, t_{\max})$.
Since $0 = (0,0,0,0,0)$ is a lower-solution of the problem (1.1)-(1.3), we have $S(x,t) \geq 0$, $E(x,t) \geq 0$, $I(x,t) \geq 0$, $R(x,t) \geq 0$ and $Q(x,t) \geq 0$.
In the following, our goal is to show that the maximum solution of the problem (1.1)-(1.3), is global. Let's first consider the first equation of the system (1.1), then we have

$$\left\{\begin{array}{l} \frac{\partial S(x,t)}{\partial t} - d\Delta S(x,t) \leq A - \mu S(x,t), \\ \frac{\partial S}{\partial v} = 0, \\ S(x,0) = \psi_1(x,0) \geq 0. \end{array}\right. \tag{2.2}$$

By the comparison principle [17], we have $S(x,t) \leq \tilde{S}(t)$, where $\tilde{S}(t) = \tilde{S}(0)e^{-\mu t} + \frac{A}{\mu}\left(1 - e^{-\mu t}\right)$ is the solution of the following ordinary equation:

$$\left\{\begin{array}{l} \frac{d\tilde{S}}{dt} = A - \mu\tilde{S}, \\ \tilde{S}(0) = \max_{x\in\overline{\Omega}}(\psi_1(x,0)). \end{array}\right. \tag{2.3}$$

Hence,

$$S(x,t) \leq \max\left\{\frac{A}{\mu}, \max_{x\in\overline{\Omega}}(\psi_1(x,0))\right\}, \forall (x,t) \in \overline{\Omega} \times [0.t_{\max}),$$

this implies that S is bounded.
Let $\quad L(x,t) = e^{-\mu\tau}S(x,t-\tau) + E(x,t) + I(x,t) + R(x,t) + Q(x,t),$
thus,

$$\frac{\partial L(x,t)}{\partial t} = e^{-\mu\tau}d\Delta S(x,t-\tau) + d\Delta E(x,t) + d\Delta I(x,t) + d\Delta R(x,t) + d\Delta Q(x,t) + e^{-\mu\tau}A - \mu L(x,t).$$

Then, we have

$$\left\{\begin{array}{l}\frac{\partial L(x,t)}{\partial t}-d\Delta L(x,t)\leq e^{-\mu\tau}A-\mu L(x,t),\\ \frac{\partial L}{\partial v}=0,\\ L(x,0)=e^{-\mu\tau}\psi_1(x,-\tau)+\psi_2(x,0)+\psi_3(x,0)+\psi_4(x,0)+\psi_5(x,0).\end{array}\right. \tag{2.4}$$

Applying the comparison principle to the system (2.4) we obtain

$$L(x,t)\leq\max\left\{\frac{e^{-\mu\tau}A}{\mu},\max_{x\in\overline{\Omega}}L(x,0)\right\},\forall(x,t)\in\overline{\Omega}\times[0.t_{\max}).$$

Therefore, E, I, R and Q are bounded. So, we proved that S, E, I, R and Q are bounded on $\overline{\Omega}\times[0,t_{\max})$. By the standard theory for semilinear parabolic systems [7], we deduce that $t_{\max}=+\infty$. This completes the proof. □

3. Basic reproduction number and existence of equilibirum

In this section we determine the equilibrium of the SEIRQ models, for that we solve the following system

$$\left\{\begin{array}{l}\Lambda-\beta\frac{S}{N}(I+qE)-\mu S=0,\\ \beta e^{-\mu\tau}\frac{S}{N}(I+qE)-(\sigma+\mu)E=0,\\ \sigma E-(\gamma+\delta_q+\mu)\,I=0,\\ \gamma I+\gamma_q Q-\mu R=0,\\ \delta_q I-\left(\gamma_q+\mu\right)Q=0.\end{array}\right.$$

Then the disease-free equilibrium is given by

$$\mathcal{P}_0=\left(S^0,E^0,I^0,R^0,Q^0\right)$$

where $E^0=I^0=R^0=Q^0=0$ and $S^0=\frac{\Lambda}{\mu}$.
Furthermore, the system (1.1) has a unique endemic equilibirum

$$\mathcal{P}^*=\left(S^*,E^*,I^*,R^*,Q^*\right),$$

where

$$\left\{\begin{array}{l}S^*=\frac{(\sigma+\mu)(\gamma+\delta_q+\mu)N}{e^{-\mu\tau}(q\beta(\gamma+\delta_q+\mu)+\sigma\beta)},\\ E^*=\frac{\Lambda e^{-\mu\tau}}{\sigma+\mu}-\frac{\mu N(\gamma+\delta_q+\mu)}{q\beta(\gamma+\delta_q+\mu)+\beta\sigma},\\ I^*=\frac{\sigma}{\gamma+\delta_q+\mu}E^*,\\ R^*=\frac{\sigma\gamma\left(\gamma_q+\mu\right)+\sigma\delta_q\gamma_q\mu}{\mu\left(\gamma_q+\mu\right)(\gamma+\delta_q+\mu)}E^*,\\ Q^*=\frac{\delta_q\sigma}{\left(\gamma_q+\mu\right)(\gamma+\delta_q+\mu)}E^*.\end{array}\right. \tag{3.1}$$

Now let's determine the expression of basic reproduction number denoted by $\mathcal{R}_0$, by using the method presented in [19], using the same notations, the matrix F and V are given by

$$F=\begin{pmatrix}\frac{q\beta e^{-\mu\tau}S^0}{N^0} & \frac{\beta e^{-\mu\tau}S^0}{N^0} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad\text{and}\quad V=\begin{pmatrix}\sigma+\mu & 0 & 0\\ -\sigma & \gamma+\delta_q+\mu & 0\\ 0 & -\delta_q & \gamma_q+\mu\end{pmatrix}.$$

Thus

$$FV^{-1}=\begin{pmatrix}A & B & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

where

$$A=\frac{q\beta e^{-\mu\tau}\left(\gamma+\delta_q+\mu\right)+\beta e^{-\mu\tau}\sigma}{\left(\sigma+\mu\right)\left(\gamma+\delta_q+\mu\right)},\quad \text{and } B=\frac{\beta e^{-\mu\tau}}{\gamma+\delta_q+\mu}.$$

If ρ is the spectral radius of FV^{-1}, then the expression of the basic reproduction number is as follows

$$\mathcal{R}_0=\rho\left(FV^{-1}\right)=\frac{e^{-\mu\tau}\left(q\beta\left(\gamma+\delta_q+\mu\right)+\beta\sigma\right)}{\left(\sigma+\mu\right)\left(\gamma+\delta_q+\mu\right)}.$$

So, we can rewrite (3.1) as follows

$$\begin{cases} S^*=\frac{(\sigma+\mu)(\gamma+\delta_q+\mu)N}{e^{-\mu\tau}(q\beta(\gamma+\delta_q+\mu)+\sigma\beta)},\\ E^*=\frac{e^{-\mu\tau}(\mathcal{R}_0\Lambda-\mu N)}{\mathcal{R}_0(\sigma+\mu)},\\ I^*=\frac{\sigma}{\gamma+\delta_q+\mu}E^*,\\ R^*=\frac{\sigma\gamma\left(\gamma_q+\mu\right)+\sigma\delta_q\gamma_q\mu}{\mu\left(\gamma_q+\mu\right)(\gamma+\delta_q+\mu)}E^*,\\ Q^*=\frac{\delta_q\sigma}{\left(\gamma_q+\mu\right)(\gamma+\delta_q+\mu)}E^*. \end{cases}\tag{3.2}$$

Then, $\mathcal{P}^*=(S^*,E^*,I^*,R^*,Q^*)$ exist if $\mathcal{R}_0>\frac{\mu N}{\Lambda}$.

4. Local stability of the equilibria for the SEIRQ models

Let $\tilde{S}=S-S^*,\tilde{E}=E-E^*,\tilde{I}=I-I^*,\tilde{R}=R-R^*$ and $\tilde{Q}=Q-Q^*$, where $(S^*,E^*,I^*R^*,Q^*)^\top$ is an arbitrary equilibrium point, and drop bars for simplicity. Then the system (1.1) can be transformed into the following form

$$\begin{cases} \frac{\partial S}{\partial t}(x,t)=d\Delta S(x,t)+\Lambda-\beta\frac{(S(x,t)+S^*)((I(x,t)+I^*)+q(E(x,t)+E^*))}{N(x,t)}-\mu(S(x,t)+S^*),\\ \frac{\partial E}{\partial t}(x,t)=d\Delta E(x,t)+e^{-\mu\tau}\beta\frac{(S(x,t-\tau)+S^*)((I(x,t-\tau)+I^*)+q(E(x,t-\tau)+E^*))}{N(x,t-\tau)}-(\sigma+\mu)(E(x,t)+E^*),\\ \frac{\partial I}{\partial t}(x,t)=d\Delta I(x,t)+\sigma(E(x,t)+E^*)-\left(\gamma+\delta_q+\mu\right)\left(I(x,t)+I^*\right),\\ \frac{\partial R}{\partial t}(x,t)=d\Delta R(x,t)+\gamma(I(x,t)+I^*)+\gamma_q(Q(x,t)+Q^*)-\mu(R(x,t)+R^*),\\ \frac{\partial Q}{\partial t}(x,t)=d\Delta Q(x,t)+\delta_q(I(x,t)+I^*)-\left(\gamma_q+\mu\right)\left(Q(x,t)+Q^*\right). \end{cases}\tag{4.1}$$

Thus, the arbitrary equilibrium point $\mathcal{P}^*=(S^*,E^*,I^*,R^*,Q^*)^\top$ of the system (1.1) is transformed into the zero equilibrium point $(0,0,0,0,0)^\top$ of the system (4.1).

In the following, we will analyze stability of the zero equilibrium point of the system (4.1). Denote $u(t)=(S(.,t),E(.,t),I(.,t),R(.,t),Q(.,t))^\top$ and $\psi=(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5)\in\mathcal{C}_{\mathrm{X}}$, then the system (4.1) can be rewritten as an abstract differential equation in the phase space $\mathcal{C}_{\mathrm{X}}$ of the form

$$\dot{u}(t)=D\Delta u(t)+L\left(u_t\right)+g\left(u_t\right),\tag{4.2}$$

where $D=\operatorname{diag}\{d,d,d,d,d\}, L:\mathcal{C}_{\mathbb{X}}\to\mathbb{X}$ and $g:\mathcal{C}_{\chi}\to\mathbb{X}$ are given, respectively, by

$$L(\psi)(x)=\begin{pmatrix} -\left(\mu+\frac{\beta(I^*+qE^*)}{N}\right)\psi_1(x,0)-\frac{\beta S^*(\psi_3(x,0)+q\psi_2(x,0))}{N}\\ \frac{\beta e^{-\mu\tau}(I^*+qE^*)}{N}\psi_1(x,-\tau)+\frac{\beta e^{-\mu\tau}S^*(\psi_3(x,-\tau)+q\psi_2(x,-\tau))}{N}-(\mu+\sigma)\psi_2(x,0)\\ \sigma\psi_2(x,0)-(\mu+\delta_q+\gamma)\psi_3(x,0)\\ \gamma\psi_3(x,0)+\gamma_q\psi_5(x,0)-\mu\psi_4(x,0)\\ \delta_q\psi_3(x,0)-(\mu+\gamma_q)\psi_5(x,0) \end{pmatrix},\tag{4.3}$$

and

$$g(\psi)(x)=\begin{pmatrix} g_1(\psi)(x)\\ g_2(\psi)(x)\\ g_3(\psi)(x)\\ g_4(\psi)(x)\\ g_5(\psi)(x) \end{pmatrix},$$

where

$$\begin{cases} g_1(\psi)(x) = & \dfrac{\beta\left(I^*+qE^*\right)}{N}\psi_1(x,0)+\dfrac{\beta S^*\left(\psi_3(x,0)+q\psi_2(x,0)\right)}{N} \\ & +\Lambda-\dfrac{\beta\left(\psi_1(x,0)+S^*\right)\left(\left(\psi_3(x,0)+I^*\right)+q\left(\psi_2(x,0)+E^*\right)\right)}{N}-\mu S^*, \\ g_2(\psi)(x) = & -\dfrac{\beta e^{-\mu\tau}\left(I^*+qE^*\right)}{N}\psi_1(x,0)-\dfrac{\beta e^{-\mu\tau}S^*\left(\psi_3(x,0)+q\psi_2(x,0)\right)}{N} \\ & +\dfrac{\beta e^{-\mu\tau}\left(\psi_1(x,0)+S^*\right)\left(\left(\psi_3(x,0)+I^*\right)+q\left(\psi_2(x,0)+E^*\right)\right)}{N}-(\mu+\sigma)E^*, \\ g_3(\psi)(x) = & \sigma E^*-(\mu+\gamma+\delta_q)I^*, \\ g_4(\psi)(x) = & \gamma I^*+\gamma_q Q^*-\mu R^*, \\ g_5(\psi)(x) = & \delta_q I^*-(\mu+\gamma_q)Q^*. \end{cases} \tag{4.4}$$

For $\psi=u_t, \psi=\left(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5\right)^\top\in\mathcal{C}_\times$, the linearized system of (4.2) at the zero equilibrium point is

$$\dot{u}=D\Delta u(t)+L\left(u_t\right),$$

and its characteristic equation is

$$\lambda\omega-D\Delta\omega-L\left(e^{\lambda\cdot}\omega\right)=0, \tag{4.5}$$

where $\omega\in\operatorname{dom}(\Delta)$, and $\omega\neq 0, \operatorname{dom}(\Delta)\subset\mathbb{X}$.
Let $0=\eta_0<\eta_1<\cdots$ be the sequence of eigenvalues for the elliptic operator $-\Delta$ subject to the Neumann boundary condition on Ω, and $E\left(\eta_i\right)$ be the eigenspace corresponding to η_i in $L^2(\Omega)$.
Let $\left\{\phi_{ij}, j=1,\ldots,\dim E\left(\eta_i\right)\right\}$ be an orthonormal basis of $E\left(\eta_i\right)$, and $\mathbb{Y}_{ij}=\left\{a\phi_{ij}, a\in\mathbb{R}\right\}$.
Then

$$L^2(\Omega)=\bigoplus_{i=0}^{+\infty}\mathbb{Y}_i \text{ and } \mathbb{V}_i=\bigoplus_{j=1}^{\dim E(\eta_i)}\mathbb{Y}_{ij}.$$

Moreover, we put

$$\begin{cases} \beta_{ij}^1=\begin{pmatrix}\phi_{ij}\\0\\0\\0\\0\end{pmatrix}, \quad \beta_{ij}^2=\begin{pmatrix}0\\\phi_{ij}\\0\\0\\0\end{pmatrix}, \quad \beta_{ij}^3=\begin{pmatrix}0\\0\\\phi_{ij}\\0\\0\end{pmatrix}, \quad \beta_{ij}^4=\begin{pmatrix}0\\0\\0\\\phi_{ij}\\0\end{pmatrix}, \\ \textit{and} \quad \beta_{ij}^5=\begin{pmatrix}0\\0\\0\\0\\\phi_{ij}\end{pmatrix}, i=0,1,2,\ldots, j=1,2,\ldots,\dim E\left(\eta_i\right). \end{cases} \tag{4.6}$$

Clearly, the family $\left(\beta_{ij}^1,\beta_{ij}^2,\beta_{ij}^3,\beta_{ij}^4,\beta_{ij}^5\right)$ is a basis of $\left(L^2(\Omega)\right)^5$. Therefore, any element ω of $\mathbb{X}$ can be written in the in the following form

$$\begin{aligned} \omega &= \left(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5\right) \\ &= \sum_{i=0}^{+\infty}\sum_{j=1}^{\dim E(\eta_i)}\left\langle\omega_1,\phi_{ij}\right\rangle\beta_{ij}^1+\left\langle\omega_2,\phi_{ij}\right\rangle\beta_{ij}^2+\left\langle\omega_3,\phi_{ij}\right\rangle\beta_{ij}^3+\left\langle\omega_4,\phi_{ij}\right\rangle\beta_{ij}^4+\left\langle\omega_5,\phi_{ij}\right\rangle\beta_{ij}^5. \end{aligned} \tag{4.7}$$

Next, from a straightforward analysis and using (4.6) and (4.7) we show that (4.5) is equivalent to

$$\left(\lambda I_5+\eta_i D-M\right)\begin{pmatrix}\left\langle\omega_1,\phi_{ij}\right\rangle\\\left\langle\omega_2,\phi_{ij}\right\rangle\\\left\langle\omega_3,\phi_{ij}\right\rangle\\\left\langle\omega_4,\phi_{ij}\right\rangle\\\left\langle\omega_5,\phi_{ij}\right\rangle\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}, i=0,1,2,\ldots, \quad j=1,2,\ldots,\dim E\left(\eta_i\right), \tag{4.8}$$

where M is given by

$$M=\begin{pmatrix} -\mu-\frac{\beta(I^*+qE^*)}{N} & -\frac{\beta qS^*}{N} & -\frac{\beta S^*}{N} & 0 & 0\\ \frac{\beta(I^*+qE^*)}{N}e^{-\mu\tau}e^{-\lambda\tau} & -(\mu+\sigma)+\frac{\beta qS^*}{N}e^{-\mu\tau}e^{-\lambda\tau} & \frac{\beta S^*}{N}e^{-\mu\tau}e^{-\lambda\tau} & 0 & 0\\ 0 & \sigma & -(\mu+\gamma+\delta_q) & 0 & 0\\ 0 & 0 & \gamma & -\mu & \gamma_q\\ 0 & 0 & \delta_q & 0 & -(\mu+\gamma_q) \end{pmatrix}.$$

Thus the characteristic equation is

$$(\lambda+d\eta_i+\mu+\delta_q)\,(\lambda+d\eta_i+\mu)\left(\lambda^3+a\lambda^2+b\lambda+c+(x\lambda^2+y\lambda+z)e^{-\lambda\tau}\right)=0,\quad i=0,1,\ldots,\qquad(4.9)$$

where

$$\begin{aligned} a&=3(\eta_i d+\mu)+\sigma+\gamma+\delta_q+\tfrac{\beta(I^*+qE^*)}{N},\\ b&=(\eta_i d+\mu+\sigma)(\eta_i d+\mu+\tfrac{\beta(I^*+qE^*)}{N})+(\eta_i d+\mu+\gamma+\delta_q)(2(\eta_i d+\mu)+\sigma+\tfrac{\beta(I^*+qE^*)}{N}),\\ c&=(\eta_i d+\mu+\gamma+\delta_q)(\eta_i d+\mu+\sigma)(\eta_i d+\mu+\tfrac{\beta(I^*+qE^*)}{N}),\\ x&=\tfrac{(-q\beta S^*)}{N}e^{-\mu\tau},\\ y&=\tfrac{(-q\beta S^*)}{N}e^{-\mu\tau}(2(\eta_i d+\mu)+\gamma+\delta_q+\tfrac{\sigma}{q}),\\ z&=\tfrac{-q\beta(\eta_i d+\mu)S^*}{N}e^{-\mu\tau}(\eta_i d+\mu+\gamma+\delta_q+\tfrac{\sigma}{q}). \end{aligned}$$

4.1. Stability of disease-free equilibirum P

Using the above analysis, in this part, we take $(S^*,E^*,I^*,R^*,Q^*)=P=\left(\frac{A}{\mu},0,0,0,0\right)$. Thus, the characteristic equation (4.9) becomes for $i=0,1,\ldots$

$$(\lambda+d\eta_i+\mu)^2\,(\lambda+d\eta_i+\mu+\delta_q)\left[\lambda^2+\lambda(C+B)+BC-q\beta e^{-\mu\tau}e^{-\lambda\tau}(\lambda+C+\sigma/q)\right]=0,\qquad(4.10)$$

where

$$\begin{cases} B=d\eta_i+\mu+\sigma,\\ C=d\eta_i+\mu+\delta_q+\gamma.\end{cases}$$

Theorem 4.1. *If $\mathcal{R}_0\le 1$, then the disease-free equilibrium P is locally asymptotically stable for all $\tau\ge 0$.*

Proof: For $\tau=0$, the Eq. (4.10) is equivalent to the following cubic equation

$$(\lambda+d\eta_i+\mu)^2\,(\lambda+d\eta_i+\mu+\delta_q)\left[\lambda^2+\lambda(C+B-q\beta)+BC-(q\beta C+\beta\sigma)\right]=0,\quad i=0,1,\ldots\qquad(4.11)$$

where

$$\begin{cases} B=d\eta_i+\mu+\sigma,\\ C=d\eta_i+\mu+\delta_q+\gamma.\end{cases}$$

As $\mathcal{R}_0\le 1$, we have

$$C+B-q\beta=C+d\eta_i+\frac{q(\mu+\sigma)(\mu+\delta_q+\gamma)(1-\mathcal{R}_0)+(\mu+\sigma)\sigma}{q(\mu+\delta_q+\gamma)+\sigma}>0,$$

and

$$\begin{aligned} CB-(q\beta C+\beta\sigma)&=(d\eta_i)^2+d\eta_i\left[\mu+\gamma+\delta_q+\frac{q(\mu+\sigma)(\mu+\delta_q+\gamma)(1-\mathcal{R}_0)+(\mu+\sigma)\sigma}{q(\mu+\delta_q+\gamma)+\sigma}\right]\\ &\quad+(\mu+\sigma)(\mu+\gamma+\delta_q)(1-\mathcal{R}_0)>0. \end{aligned}$$

According to the Routh-Hurwitz criteria, all the roots of equation (4.11) have negative real parts. Therefore, when $\tau=0$, the disease-free equilibrium point P is locally asymptotically stable.
Next, Since all the roots of equation (4.11) have negative real parts for $\tau=0$. it follows that if instability

occurs for a particular value of the delay τ, a characteristic root of (4.10) must intersect the imaginary axis. If (4.10) has a purely imaginary root $i\omega$, with $\omega > 0$, then, by separating real and imaginary parts in (4.10), we have

$$\begin{cases} q\omega\beta e^{-\mu\tau}\sin(\omega\tau) - q\beta(C+\dfrac{\sigma}{q})e^{-\mu\tau}\cos(\omega\tau) = \omega^2 - BC, \\ q\omega\beta e^{-\mu\tau}\cos(\omega\tau) + q\beta(C+\dfrac{\sigma}{q})e^{-\mu\tau}\sin(\omega\tau) = \omega(C+B). \end{cases} \tag{4.12}$$

Taking square on both sides of the equations of (4.12) and summing them up, we obtain

$$\omega^4 + \left(C^2+B^2-(\beta e^{-\mu\tau})^2\right)\omega^2 + (BC)^2 - (q\beta e^{-\mu\tau}(qC+\sigma))^2 = 0. \tag{4.13}$$

It is easy to see that $BC - q\beta(C+\frac{\sigma}{q}) > 0$ and as $\mathcal{R}_0 \leq 1$, we deduce that

$$(BC)^2 - (\beta e^{-\mu\tau}(qC+\sigma))^2 > 0.$$

Moreover, as $\mathcal{R}_0 \leq 1$, we have

$$\begin{aligned} C^2+B^2-(q\beta e^{-\mu\tau})^2 &= C^2 + \left(B+(q\beta e^{-\mu\tau})\right)\left(B-(q\beta e^{-\mu\tau})\right) \\ &= C^2 + \left(B+(q\beta e^{-\mu\tau})\right)\left(d\eta_i + \frac{q(\mu+\sigma)(\mu+\delta_q+\gamma)(1-\mathcal{R}_0)+(\mu+\sigma)\sigma}{q(\mu+\delta_q+\gamma)+\sigma}\right) \geq 0. \end{aligned}$$

Therefore, Eq. (4.12) has no positive roots and characteristic equation (4.10) does not admit any purely imaginary root for all τ. Since P is asymptotically stable for $\tau = 0$, it remains asymptotically stable for all $\tau \geq 0$. □

4.2. Stability of endemic equilibirum P*

In this part, we will discuss the local stability of the endemic equilibrium P^*. First, we take $(S^*, E^*, I^*, R^*, Q^*) = P^*$. Thus, the characteristic equation (4.9) becomes

$$(\lambda + d\eta_i + \mu + \delta_q)(\lambda + d\eta_i + \mu)\left(\lambda^3 + a\lambda^2 + b\lambda + c + (x\lambda^2 + y\lambda + z)e^{-\lambda\tau}\right) = 0, \quad i = 0, 1, \ldots \tag{4.14}$$

where

$$\begin{aligned} a &= 3(\eta_i d+\mu)+\sigma+\gamma+\delta_q+\tfrac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}, \\ b &= (\eta_i d+\mu+\sigma)(\eta_i d+\mu+\tfrac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}) + (\eta_i d+\mu+\gamma+\delta_q)(2(\eta_i d+\mu)+\sigma+\tfrac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}), \\ c &= (\eta_i d+\mu+\gamma+\delta_q)(\eta_i d+\mu+\sigma)(\eta_i d+\mu+\tfrac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}), \\ x &= \tfrac{(-q\beta)}{\mathcal{R}_0}e^{-\mu\tau}, \\ y &= \tfrac{(-q\beta)}{\mathcal{R}_0}e^{-\mu\tau}(2(\eta_i d+\mu)+\gamma+\delta_q+\tfrac{\sigma}{q}), \\ z &= \tfrac{-q\beta(\eta_i d+\mu)}{\mathcal{R}_0}e^{-\mu\tau}(\eta_i d+\mu+\gamma+\delta_q+\tfrac{\sigma}{q}). \end{aligned}$$

Theorem 4.2. *If $\mathcal{R}_0 \geq \max\left(\frac{2q\beta}{\eta_i d}, \frac{q\beta}{\sigma}\right)$ then the endemic equilibrium P^* is locally asymptotically stable for all $\tau \geq 0$.*

Proof: For $\tau = 0$, the characteristic equation (3.15) is transformed into the following form

$$(\lambda + d\eta_i + \mu + \delta_q)(\lambda + d\eta_i + \mu)\left(\lambda^3 + (a+x)\lambda^2 + (b+y)\lambda + c + z\right) = 0, \quad i = 0, 1, \ldots \tag{4.15}$$

where

$$\begin{aligned} a &= 3(\eta_i d+\mu)+\sigma+\gamma+\delta_q+\tfrac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}, \\ b &= (\eta_i d+\mu+\sigma)(\eta_i d+\mu+\tfrac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}) + (\eta_i d+\mu+\gamma+\delta_q)(2(\eta_i d+\mu)+\sigma+\tfrac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}), \\ c &= (\eta_i d+\mu+\gamma+\delta_q)(\eta_i d+\mu+\sigma)(\eta_i d+\mu+\tfrac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}), \\ x &= \tfrac{(-q\beta)}{\mathcal{R}_0}, \\ y &= \tfrac{(-q\beta)}{\mathcal{R}_0}(2(\eta_i d+\mu)+\gamma+\delta_q+\tfrac{\sigma}{q}), \\ z &= \tfrac{-q\beta(\eta_i d+\mu)}{\mathcal{R}_0}(\eta_i d+\mu+\gamma+\delta_q+\tfrac{\sigma}{q}). \end{aligned}$$

As $R_0 > \max\left(\frac{2q\beta}{\eta_i d}, \frac{q\beta}{\sigma}\right)$, we deduce that

$$a + x = \quad 2\eta_i d + 3\mu + \sigma + \gamma + \delta_q + \frac{\beta E^* \left(\sigma + q(\mu + \gamma + \delta_q)\right)}{N(\mu + \gamma + \delta_q)} + \left(\eta_i d - \frac{q\beta}{\mathcal{R}_0}\right) > 0,$$

$$\begin{aligned} b + y =& 3(\eta_i d)^2 + 2\eta_i d(\mu + \gamma + \delta_q + u + \frac{\beta E^* \left(\sigma + q(\mu + \gamma + \delta_q)\right)}{N(\mu + \gamma + \delta_q)}) \\ &+ (\mu + \frac{\beta E^* \left(\sigma + q(\mu + \gamma + \delta_q)\right)}{N(\mu + \gamma + \delta_q)})(\mu + \gamma + \delta_q) \\ &+ 2\eta_i d(\sigma - \frac{q\beta}{\mathcal{R}_0}) + \mu(2\eta_i d - \frac{q\beta}{\mathcal{R}_0}) > 0, \end{aligned}$$

$$\begin{aligned} c + z =& \left(\eta_i d + \mu + \frac{E^*}{N}\mathcal{R}_0\left(\mu + \sigma\right)\right)\left((\eta_i d)^2 + \eta_i d(\mu + \sigma + \mu + \gamma + \delta_q)\right) \\ &+ \left(\mu + \sigma\right)^2\left(\mu + \gamma + \delta_q\right)\frac{E^*}{N}\mathcal{R}_0 > 0, \end{aligned}$$

and

$$\begin{aligned} (a+x)(b+y) - (c+z) =& (\eta_i d + \mu + \sigma)\left(\eta_i d + \mu + \frac{\beta E^* \left(\sigma + q(\mu + \gamma + \delta_q)\right)}{N(\mu + \gamma + \delta_q)}\right)[(\eta_i d + \mu + \sigma) + (\eta_i d + \mu \\ &+ \frac{\beta E^* \left(\sigma + q(\mu + \gamma + \delta_q)\right)}{N(\mu + \gamma + \delta_q)})\Big] + (\eta_i d + \mu)\left(\frac{q\beta\eta_i}{\mathcal{R}_0} + (\eta_i d + \mu + \sigma) \times (\eta_i d + \mu + \gamma + \delta_q)\right) + \\ &\left(\eta_i d + \mu + \sigma + \eta_i d + \mu + \gamma + \delta_q + \eta_i d + \mu + \frac{\beta E^* \left(\sigma + q(\mu + \gamma + \delta_q)\right)}{N(\mu + \gamma + \delta_q)}\right) \times [\\ &\eta_i d\left(\eta_i d - \frac{2q\beta}{\mathcal{R}_0}\right) + \mu\left(\eta_i d - \frac{q\beta}{\mathcal{R}_0}\right) + (\mu + \gamma + \delta_q)\left(\eta_i d + \mu + \frac{\mathcal{R}_0 E^* \left(\sigma + \mu\right)}{N}\right) \\ &+\eta_i d \times \frac{\mathcal{R}_0 E^* \left(\sigma + \mu\right)}{N}\Big] > 0. \end{aligned}$$

According to the Routh-Hurwitz criteria, all the roots of equation (4.15) have negative real parts. Therefore, when $\tau = 0$, the endemic equilibrium point P^* is locally asymptotically stable.
Next, Since all the roots of equation (4.15) have negative real parts for $\tau = 0$. it follows that if instability occurs for a particular value of the delay τ, a characteristic root of (4.14) must intersect the imaginary axis. If (4.14) has a purely imaginary root $i\omega$, with $\omega > 0$, then, by separating real and imaginary parts in (4.14), we have

$$\begin{cases} -y\omega\sin(\omega\tau) + (z - x\omega^2)\cos(\omega\tau) = b\omega - \omega^3, \\ \ \ y\omega\cos(\omega\tau) + (z - x\omega^2)\sin(\omega\tau) = a\omega^2 - c. \end{cases} \tag{4.16}$$

Taking square on both sides of the equations of (4.16) and summing them up, we obtain

$$\omega^6 + \left(a^2 - 2b - x^2\right)\omega^4 + (b^2 - 2ac - y^2 + 2xz)\omega^2 + c^2 - z^2 = 0. \tag{4.17}$$

It is easy to see that $c - z > 0$, we deduce that $c^2 - z^2 > 0$. Moreover, we have

$$\begin{aligned} a^2 - 2b - x^2 =& \left(\eta_i d + \mu + \sigma - \frac{q\beta}{\mathcal{R}_0}\right)\left(\eta_i d + \mu + \sigma + \frac{q\beta}{\mathcal{R}_0}\right) + \left(\eta_i d + \mu + \frac{\beta E^* \left(\sigma + q(\mu + \gamma + \delta_q)\right)}{N(\mu + \gamma + \delta_q)}\right)^2 \\ &+ (\eta_i d + \mu + \gamma + \delta_q)^2 + 2 > 0, \end{aligned}$$

$$
\begin{aligned}
b^2-2ac-y^2+2xz = \Big(& (\eta_i d+\mu+\sigma)^2(\eta_i d+\mu+\gamma+\delta_q)^2 - 2(\eta_i d+\mu+\sigma)(\eta_i d+\mu+\gamma+\delta_q) \\
& + \Big(\eta_i d+\mu+\sigma\Big)^2\Big(\eta_i d+\mu+\frac{\beta E^*\Big(\sigma+q(\mu+\gamma+\delta_q)\Big)}{N(\mu+\gamma+\delta_q)}\Big)^2 \\
& - 2\Big(\eta_i d+\mu+\sigma\Big)\Big(\eta_i d+\mu+\frac{\beta E^*\Big(\sigma+q(\mu+\gamma+\delta_q)\Big)}{N(\mu+\gamma+\delta_q)}\Big) \\
& + (\eta_i d+\mu+\gamma+\delta_q)^2\left(\eta_i d+\mu+\frac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}\right)^2 \\
& - 2(\eta_i d+\mu+\gamma+\delta_q)\left(\eta_i d+\mu+\frac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}\right) \\
& +2\left(\frac{(-q\beta)}{\mathcal{R}_0}e^{-\mu\tau}\right)^2(\eta_i d+\mu+\gamma+\delta_q+\frac{\sigma}{q}) - \left(\frac{(-q\beta)}{\mathcal{R}_0}e^{-\mu\tau}(2(\eta_i d+\mu)+\gamma+\delta_q+\frac{\sigma}{q})\right)^2\Big) > 0,
\end{aligned}
$$

$$
\begin{aligned}
&(a^2-2b-x^2)\left(b^2-2ac-y^2+2xz\right)-c^2+z^2 \\
&= \left(\left(\eta_i d+\mu+\sigma+\frac{q\beta}{\mathcal{R}_0}\right)\left(\eta_i d+\mu+\sigma-\frac{q\beta}{\mathcal{R}_0}\right)+(\eta_i d+\mu+\gamma+\delta_q)^2\right. \\
&\left. + \left(\frac{-q\beta\,(\eta_i d+\mu)}{\mathcal{R}_0}(\eta_i d+\mu+\gamma+\delta_q+\frac{\sigma}{q})\right)^2+2\right) + \left(\left(\eta_i d+\mu+\sigma-\frac{(q\beta)}{\mathcal{R}_0}e^{-\mu\tau}\right)\left(\eta_i d+\mu+\sigma+\frac{(q\beta)}{\mathcal{R}_0}e^{-\mu\tau}\right)\right. \\
&\left. + (\eta_i d+\mu+\gamma+\delta_q)^2+\left(\eta_i d+\mu+\frac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}\right)^2+2\right) + \Big((\eta_i d+\mu+\sigma)^2(\eta_i d+\mu+\gamma+\delta_q)^2 \\
& -2(\eta_i d+\mu+\sigma)(\eta_i d+\mu+\gamma+\delta_q)+(\eta_i d+\mu+\sigma)^2\left(\eta_i d+\mu+\frac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}\right)^2 \\
& -2(\eta_i d+\mu+\sigma)\left(\eta_i d+\mu+\frac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}\right) \\
& + (\eta_i d+\mu+\gamma+\delta_q)^2\left(\eta_i d+\mu+\frac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}\right)^2 \\
& -2(\eta_i d+\mu+\gamma+\delta_q)\left(\eta_i d+\mu+\frac{\beta E^*(\sigma+q(\mu+\gamma+\delta_q))}{N(\mu+\gamma+\delta_q)}\right) \\
& +2\left(\frac{(-q\beta)}{\mathcal{R}_0}e^{-\mu\tau}\right)^2(\eta_i d+\mu+\gamma+\delta_q+\frac{\sigma}{q}) - \left(\frac{(-q\beta)}{\mathcal{R}_0}e^{-\mu\tau}(2(\eta_i d+\mu)+\gamma+\delta_q+\frac{\sigma}{q})\right)^2\Big) > 0.
\end{aligned}
$$

Therefore, Eq. (4.17) has no positive roots and characteristic equation (4.14) does not admit any purely imaginary root for all η_i. Since P^* is asymptotically stable for $\tau = 0$, it remains asymptotically stable for all $\tau \geq 0$. □

5. Numerical

In this section, we perform some numerical simulations to illustrate the theoretical results. For the sake of simplicity, we consider a one-dimensional bounded spatial domain $\Omega = [0, 1]$. Thus, we propose system (1.1) with Neumann boundary conditions

$$\frac{\partial S}{\partial v} = \frac{\partial E}{\partial v} = \frac{\partial I}{\partial v} = \frac{\partial R}{\partial v} = \frac{\partial Q}{\partial v} = 0, \quad t \geq 0,\ x = 0, 1$$

and initial conditions

$$
\begin{aligned}
&S(x,t) = |\cos(3\pi x)| \geq 0,\ E(x,t) = |\cos(3\pi x)| \geq 0,\ I(x,t) = |\sin(2\pi x)| \geq 0, \\
&R(x,t) = |\sin(2\pi x)| \geq 0,\ Q(x,t) = |\cos(3\pi x)| \geq 0, \quad (x,t) \in [0,1]\times[-\tau,0].
\end{aligned}
$$

Moreover, to solve system (1.1) using a numerical algorithm, we must discretize each equation of system (1.1) as a finite difference equation. The Crank-Nicolson method [4] is a finite difference method used for numerically solving a partial differential equation. It is a second-order method in time and space, and is numerically stable. Thereafter, a brief description of the Crank-Nicolson method applied to our problem will be provided below. We first start by partitioning the spatial interval $[0, 1]$ and temporal interval $[0, t_f]$ into respective finite grids as follows.

$$\begin{cases} x_i = (i-1)\Delta x, & i = 1, 2, \dots, N_x + 1 \text{ where } \Delta x := \frac{1}{N_x}, \\ t_j = (j-1)\Delta t, & j = 1, 2, \dots, N_t + 1 \text{ where } \Delta t := \frac{t_f}{N_t}. \end{cases}$$

Therefore, using discretization, we can describe $S(x,t)$ as $S_{i,j}$ $(i = 1, \dots, N_x + 1, j = 1, \dots, N_t + 1)$, $E(x,t)$ as $E_{i,j}$ $(i = 1, \dots, N_x + 1, j = 1, \dots, N_t + 1)$, $I(x,t)$ as $I_{i,j}$ $(i = 1, \dots, N_x + 1, j = 1, \dots, N_t + 1)$, $Q(x,t)$ as $Q_{i,j}$ $(i = 1, \dots, N_x + 1, j = 1, \dots, N_t + 1)$ and $R(x,t)$ as $R_{i,j}$ $(i = 1, \dots, N_x + 1, j = 1, \dots, N_t + 1)$.
In addition, we can discretize the system (1.1) as follows:

$$\begin{cases} \dfrac{S_{i,j+1} - S_{i,j}}{\Delta t} = & \dfrac{d}{2}\left(\dfrac{S_{i+1,j+1} - 2S_{i,j+1} + S_{i-1,j+1}}{\Delta x^2} + \dfrac{S_{i+1,j} - 2S_{i,j} + S_{i-1,j}}{\Delta x^2}\right) \\ & + A - \mu S_{i,j} - \dfrac{\beta S_{i,j}(I_{i,j} + qE_{i,j})}{N_{i,j}}, \\ \dfrac{E_{i,j+1} - E_{i,j}}{\Delta t} = & \dfrac{d}{2}\left(\dfrac{E_{i+1,j+1} - 2E_{i,j+1} + S_{i-1,j+1}}{\Delta x^2} + \dfrac{E_{i+1,j} - 2E_{i,j} + E_{i-1,j}}{\Delta x^2}\right) \\ & + \dfrac{e^{-\mu\tau}\beta S_{i,j-\tau/\Delta t}(I_{i,j-\tau/\Delta t} + qE_{i,j-\tau/\Delta t})}{N_{i,j-\tau/\Delta t}} - (\mu + \sigma)E_{i,j}, \\ \dfrac{I_{i,j+1} - I_{i,j}}{\Delta t} = & \dfrac{d}{2}\left(\dfrac{I_{i+1,j+1} - 2I_{i,j+1} + I_{i-1,j+1}}{\Delta x^2} + \dfrac{I_{i+1,j} - 2I_{i,j} + I_{i-1,j}}{\Delta x^2}\right) \\ & + \sigma E_{i,j} - (\gamma + \mu + \delta_q)I_{i,j}, \\ \dfrac{R_{i,j+1} - R_{i,j}}{\Delta t} = & \dfrac{d}{2}\left(\dfrac{R_{i+1,j+1} - 2R_{i,j+1} + R_{i-1,j+1}}{\Delta x^2} + \dfrac{R_{i+1,j} - 2R_{i,j} + R_{i-1,j}}{\Delta x^2}\right) \\ & + \gamma I_{i,j} + \gamma_q Q_{i,j} - \mu R_{i,j}, \\ \dfrac{Q_{i,j+1} - Q_{i,j}}{\Delta t} = & \dfrac{d}{2}\left(\dfrac{Q_{i+1,j+1} - 2Q_{i,j+1} + Q_{i-1,j+1}}{\Delta x^2} + \dfrac{Q_{i+1,j} - 2Q_{i,j} + Q_{i-1,j}}{\Delta x^2}\right) \\ & + \delta_q I_{i,j} - (\gamma_q + \mu)Q_{i,j}. \end{cases} \tag{5.1}$$

Applying the central difference formula to approximate the Neumann boundary condition (1.3), we see that (5.1) yields the following system:

$$\begin{cases} MS_{j+1} = NS_j + U_j \\ ME_{j+1} = NE_j + V_j \\ MI_{j+1} = NI_j + W_j \\ MR_{j+1} = NR_j + Y_j \\ MQ_{j+1} = NQ_j + Z_j, \end{cases} \tag{5.2}$$

where

$$S_j = \begin{bmatrix} S_{1,j} \\ S_{2,j} \\ \vdots \\ S_{N_x,j} \\ S_{N_x+1,j} \end{bmatrix}, \quad E_j = \begin{bmatrix} E_{1,j} \\ E_{2,j} \\ \vdots \\ E_{N_x,j} \\ E_{N_x+1,j} \end{bmatrix}, \quad I_j = \begin{bmatrix} I_{1,j} \\ I_{2,j} \\ \vdots \\ I_{N_x,j} \\ I_{N_x+1,j} \end{bmatrix}, \quad R_j = \begin{bmatrix} R_{1,j} \\ R_{2,j} \\ \vdots \\ R_{N_x,j} \\ R_{N_x+1,j} \end{bmatrix}, \quad Q_j = \begin{bmatrix} Q_{1,j} \\ Q_{2,j} \\ \vdots \\ Q_{N_x,j} \\ Q_{N_x+1,j} \end{bmatrix},$$

$$U_j = 2\Delta t \cdot \begin{bmatrix} A - \mu S_{1,j} - \frac{\beta S_{1,j}(I_{1,j}+qE_{1,j})}{N_{1,j}} \\ A - \mu S_{2,j} - \frac{\beta S_{2,j}(I_{2,j}+qE_{2,j})}{N_{2,j}} \\ \vdots \\ A - \mu S_{i,j} - \frac{\beta S_{N_x,j}(I_{N_x,j}+qE_{N_x,j})}{N_{N_x,j}} \\ A - \mu S_{N_x+1,j} - \frac{\beta S_{N_x+1,j}(I_{N_x+1,j}+qE_{N_x+1,j})}{N_{N_x+1,j}} \end{bmatrix},$$

$$V_j = 2\Delta t \cdot \begin{bmatrix} \frac{\beta e^{-\mu\tau} S_{1,j-\tau/\Delta t}(I_{1,j-\tau/\Delta t}+qE_{1,j-\tau/\Delta t})}{N_{1,j-\tau/\Delta t}} - (\mu+\sigma) E_{1,j} \\ \frac{\beta e^{-\mu\tau} S_{2,j-\tau/\Delta t}(I_{2,j-\tau/\Delta t}+qE_{2,j-\tau/\Delta t})}{N_{2,j-\tau/\Delta t}} - (\mu+\sigma) E_{2,j} \\ \vdots \\ \frac{\beta e^{-\mu\tau} S_{N_x,j-\tau/\Delta t}(I_{N_x,j-\tau/\Delta t}+qE_{N_x,j-\tau/\Delta t})}{N_{N_x,j-\tau/\Delta t}} - (\mu+\sigma) E_{N_x,j} \\ \frac{\beta e^{-\mu\tau} S_{N_x+1,j-\tau/\Delta t}(I_{N_x+1,j-\tau/\Delta t}+qE_{N_x+1,j-\tau/\Delta t})}{N_{N_x+1,j-\tau/\Delta t}} - (\mu+\sigma) E_{N_x+1,j} \end{bmatrix},$$

$$W_j = 2\Delta t \cdot \begin{bmatrix} \sigma E_{1,j} - (\gamma+\mu+\delta_q) I_{1,j} \\ \sigma E_{2,j} - (\gamma+\mu+\delta_q) I_{2,j} \\ \vdots \\ \sigma E_{N_x,j} - (\gamma+\mu+\delta_q) I_{N_x,j} \\ \sigma E_{N_x+1,j} - (\gamma+\mu+\delta_q) I_{N_x+1,j} \end{bmatrix}, \quad Y_j = 2\Delta t \cdot \begin{bmatrix} \gamma I_{1,j} + \gamma_q Q_{1,j} - \mu R_{1,j} \\ \gamma I_{2,j} + \gamma_q Q_{2,j} - \mu R_{1,j} \\ \vdots \\ \gamma I_{N_x,j} + \gamma_q Q_{N_x,j} - \mu R_{N_x,j} \\ \gamma I_{N_x+1,j} + \gamma_q Q_{N_x+1,j} - \mu R_{N_x+1,j} \end{bmatrix},$$

and

$$Z_j = 2\Delta t \cdot \begin{bmatrix} \delta_q I_{1,j} - (\gamma_q+\mu) Q_{1,j} \\ \delta_q I_{2,j} - (\gamma_q+\mu) Q_{2,j} \\ \vdots \\ \delta_q I_{N_x,j} - (\gamma_q+\mu) Q_{N_x,j} \\ \delta_q I_{N_x+1,j} - (\gamma_q+\mu) Q_{N_x+1,j} \end{bmatrix},$$

and we take $r := d\frac{\Delta t}{\Delta x^2}$, then the tridiagonal matrices M and N are given by:

$$M = \begin{bmatrix} 2+2r & -2r & 0 & 0 & \cdots & 0 \\ -r & 2+2r & -r & 0 & \ddots & \vdots \\ 0 & -r & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & -r & 0 \\ \vdots & \ddots & 0 & -r & 2+2r & -r \\ 0 & \cdots & 0 & 0 & -2r & 2+2r \end{bmatrix}, \quad N = \begin{bmatrix} 2-2r & 2r & 0 & 0 & \cdots & 0 \\ r & 2-2r & r & 0 & \ddots & \vdots \\ 0 & r & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & r & 0 \\ \vdots & \ddots & 0 & r & 2-2r & r \\ 0 & \cdots & 0 & 0 & 2r & 2-2r \end{bmatrix}.$$

Consequently, it follows from (5.2) that

$$\begin{cases} S_{j+1} = M^{-1}\{NS_j + U_j\}, \\ E_{j+1} = M^{-1}\{NE_j + V_j\}, \\ I_{j+1} = M^{-1}\{NI_j + W_j\}, \\ R_{j+1} = M^{-1}\{NR_j + Z_j\}, \\ Q_{j+1} = M^{-1}\{NQ_j + Y_j\}. \end{cases}$$

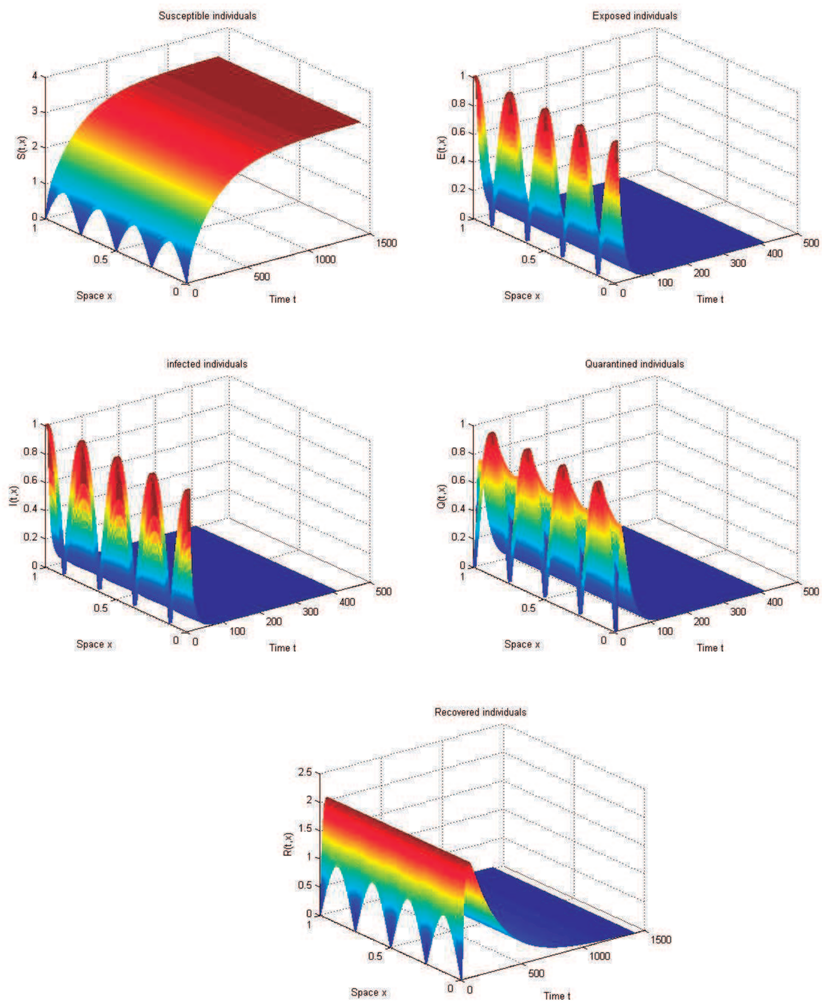
Therefore, we get a recursive schema, with is numerically stable. The parameters employed in the numerical simulations are summarized in Table 1 .

Table 1: List of parameters and their values used in numerical simulations

Parameter	Description	Value
Λ	Recruitment rate of the population	Varied
μ	Natural death of the population	0.004
δ_q	isolation Rate of infected	0.1547
σ	Rate of exposed individuals to the infected	0.0714
β	Transmission rate	0.081
q	the fraction of transmission rate for exposed	0.007
γ	Recovery rate	0.0714
γ_q	Recovery rate of isolated infected	0.0573
d	Rate of diffusion	0.00008
τ	Time incubation	8

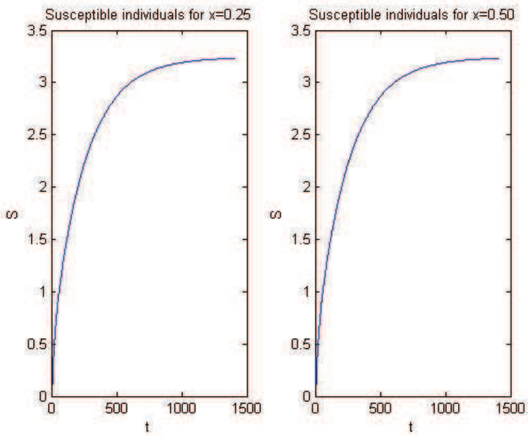
Now, if we choose the values from table 1 , then we have $\mathcal{R}_0 = 0.3301$. By Theorem (4.1), the disease-free equilibrium $P(3.2522, 0, 0, 0, 0)$ is locally asymptotically stable. This means that the disease dies out (see Fig. 1).

Figure 1: Spatiotemporal solution found by numerical integration of system (1.1) under conditions (1.2) and (1.3) when $\mathcal{R}_0 = 0.3301$



To better understand figure 1 we propose figure 2 where we present the curve of S in the case of $x = \frac{1}{2}$ and $x = \frac{1}{4}$.

Figure 2: the curve of S in the case of $x = \frac{1}{2}$ and $x = \frac{1}{4}$.



6. Conclusion

By comparing the results in Theorems , and 4.1 with the propositions 1, 2 of [8] and the proposition 2 of [1], we affirm that we have obtained the same results, but for a more general class of population models. In reality, we have extended these results to contain our model of reaction-diffusion epidemic. Firstly, by analyzing the corresponding characteristic equations, we discussed the local stability of the disease-free equilibrium P and the endemic equilibrium P^* of system (1.1) under homogeneous Neumann boundary conditions. Since $\mathcal{R}_0$ has no relation to the diffusion coefficient d, we have shown in Theorem (4.1) and Theorem (4.2) that spatial diffusion has no effect on the local stability of the steady states of our SEIQR model. Which indicates that, whatever the choice of the diffusion coefficient d, the stability of the equilibrium points remains invariant when the system passes from the dynamics governed by the ordinary differential equations ODE [1] to that governed by the partial differential equations PDE.

References

1. Abta, A., Kaddar, A., Alaoui, H. T. (2012).*Global stability for delay SIR and SEIR epidemic models with saturated incidence rates.* Electronic Journal of Differential Equations, 2012(23), 1-13.

2. Capasso, V., Serio, G. (1978).*A generalization of the Kermack-McKendrick deterministic epidemic model.* Mathematical biosciences, 42(1-2), 43-61.

3. Cooke, K. L. (1979).*Stability analysis for a vector disease model.* The Rocky Mountain Journal of Mathematics, 9(1), 31-42.

4. Crank, J., Nicolson, P. (1947, January).*A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type.* In Mathematical Proceedings of the Cambridge Philosophical Society (Vol. 43, No. 1, pp. 50-67). Cambridge University Press.

5. Fitzgibbon, W. E. (1978). *Semilinear functional differential equations in Banach space.* Journal of Differential Equations, 29(1), 1-14.

6. Hale, J.K.(1993). Verduyn Lunel, S.M.: *Introduction to Functional Differential Equations.*Springer-Verlag, New York, NY

7. Henry, D. (1993) .*Geometric Theory of Semilinear Parabolic Equations.* Lecture Notes in Mathematics, vol. 840. Springer-Verlag, Berlin

8. Kaddar, A., Abta, A., Alaoui, H. T. (2011).*A comparison of delayed SIR and SEIR epidemic models.* Nonlinear Analysis: Modelling and Control, 16(2), 181-190.

9. Kermack, W. O., McKendrick, A. G. (1927).*A contribution to the mathematical theory of epidemics.* Proceedings of the royal society of london. Series A, Containing papers of a mathematical and physical character, 115(772), 700-721.

10. Martin, R. H., Smith, H. L. (1990).*Abstract functional-differential equations and reaction-diffusion systems*. Transactions of the American Mathematical Society, 321(1), 1-44.

11. Smith, H. L., Martin, R. J. (1991).*Reaction-diffusion systems with time delays: monotonicity, invariance, comparison and convergence.*

12. Murray, J.D.(2002). *Mathematical Biology, I and II, 3rd edn*. Springer, New York .

13. Pao, C. V. (1996). *Dynamics of nonlinear parabolic systems with time delays*. Journal of Mathematical Analysis and Applications, 198(3), 751-779.

14. Pao, C.V. (2002).*Convergence of solutions of reaction-diffusion systems with time delays*. Nonlinear Anal. Theory Methods Appl. 48(3), 349-362 .

15. Pao, C.V. *On nonlinear reaction-diffusion systems*,. J. Math. Anal. Appl. 87(1), 165-198 (1982).

16. Pao, C.V. (1992).*Nonlinear Parabolic and Elliptic Equations.*, Springer, New York

17. Weinberger, H. F. (1967). Maximum principles in differential equations.

18. Travis, C. C., Webb, G. (1974). *Existence and stability for partial functional differential equations*. Transactions of the American Mathematical Society, 200, 395-418.

19. Van den Driessche, P., Watmough, J. (2002).*Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission*. Mathematical biosciences, 180(1-2), 29-48.

20. Wang, W., Cai, Y., Wu, M., Wang, K., Li, Z. (2012).*Complex dynamics of a reaction-diffusion epidemic model*. Nonlinear Analysis: Real World Applications, 13(5), 2240-2258.

21. Wang, X. S., Wang, H., Wu, J. (2012).*Traveling waves of diffusive predator-prey systems: disease outbreak propagation*. Discrete Continuous Dynamical Systems-A, 32(9), 3303.

22. Wu, J. (2012).*Theory and applications of partial functional differential equations (Vol. 119)*. Springer Science Business Media.

23. Xu, R., Ma, Z. (2009). *Global stability of a SIR epidemic model with nonlinear incidence rate and time delay*. Nonlinear Analysis: Real World Applications, 10(5), 3175-3189.

24. Yang, J., Liang, S., Zhang, Y. (2011).*Travelling waves of a delayed SIR epidemic model with nonlinear incidence rate and spatial diffusion*. PLoS One, 6(6), e21128.

Essaid Anaama,
Chakir Allalou,
Khalid Hilal,
Laboratory LMACS, FST of Beni-Mellal,
Sultan Moulay Slimane University,
Morocco.
E-mail address: essaidanaama1@gmail.com
E-mail address: chakir.allalou@yahoo.fr
E-mail address: hilalkhalid2005@yahoo.fr