Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–4.
ISSN-0037-8712 in press
doi:10.5269/bspm.64969

Derivation Alternator Rings with $S(a,b,c)=0$

Dr. P. Sarada Devi, Dr. K. Hari Babu, Dr. Y. Suresh Kumar

ABSTRACT: In this paper, we discuss about the derivation alternator rings which are nonassociative but not (-1,1) rings. By assuming some additional conditions, we prove that derivation alternator rings are (-1,1) rings. Here we validate a semiprime derivation alternator ring with commutators in the left nucleus satisfies the identity $S(a,b,c)=0$. By using this we show that a semiprime derivation alternator ring with commutators in the left nucleus is a (-1,1) ring.

Key Words: (-1,1) rings, derivation alternator rings, nonassociative rings, semiprime rings.

Contents

1 Introduction **1**

2 Preliminaries **1**

3 Main Results **3**

1. Introduction

Hentzel [3] studied about derivation alternator rings. Thedy [6] proved that the rings R satisfying the identifies $((R,R),R,R)=0,((a,b,a),b)=0$ for all a,b in R, and also $(R,(R,R),R)=0$ or $(R,R,(R,R))=0$. The main solution is that prime rings following these identities are also commutative or associative. Kleinfeld [4] proved the same result by considering the rings with (a,b,a) and commutators in the left nucleus. In [1] the rings with $((R,R),R,R)=0$ and $((a,b,a),R)=0$ have been studied. Here we validate that semiprime derivation alternator ring with commutators in the left nucleus is a (-1,1) ring.

2. Preliminaries

A 2-divisible nonassociative ring is said to be a derivation alternator ring if it satisfies the following properties [3]

$$(a,a,a)=0 \tag{2.1}$$

$$(bc,a,a)=b(c,a,a)+(b,a,a)c \tag{2.2}$$

and

$$(a,a,bc)=b(a,a,c)+(a,a,b)c \tag{2.3}$$

A (-1,1) ring is a nonassociative ring in which the right alternative law [2] $(a,b,b)=0, i.e., (a,b,c)+(a,c,b)=0$ and $(a,b,c)+(b,c,a)+(c,a,b)=0$ hold. The left nucleus N_l of a ring R is defined as $N_l = n \in R/(n,R,R)=0$. Throughout this paper, R denotes a derivation alternator ring with commutators in the left nucleus. That is,

$$((R,R),R,R)=0 \tag{2.4}$$

Using (2),(3) and linearized (1) that such rings also satisfy

$$(a,bc,a)=b(a,c,a)+(a,b,a)c \tag{2.5}$$

By using Teichmuller identity [2]

$$(da,b,c)\check{}(d,ab,c)+(d,a,bc)=d(a,b,c)+(d,a,b)c \tag{2.6}$$

2010 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted September 10, 2022. Published October 13, 2022

As in [3] the identity (6) along with (5),(3) and (1) gives
$(a^2, b, a) = (a, ab, a)\check{}(a, a, ba) + a(a, b, a) + (a, a, b)a$
$a(a, b, a)\check{}(a, a, b)a + a(a, b, a) + (a, a, b)a$
$= 2a(a, b, a)$.

Thus we have proved

$$(a^2, b, a) = 2a(a, b, a) \tag{2.7}$$

Also we know that (7) becomes

$$(a, b, a^2) = 2(a, b, a)a \tag{2.8}$$

In [5] the properties of flexile derivation alternator rings have been studied. It is shown that a derivation alternator ring follows the flexible law,

$$(a, b, a) = 0 \tag{2.9}$$

The following properties hold in an arbitrary ring: [2]

$$(ab, c)\check{}a(b, c)\check{}(a, c)b = (a, b, c) + (b, a, c)\check{}(a, c, b) \tag{2.10}$$

$$(ab, c) + (bc, a) + (ca, b) = S(a, b, c) \tag{2.11}$$

and

$$((a, b), c) + ((b, c), a) + ((c, a), b) = S(a, b, c)\check{}S(a, c, b) \tag{2.12}$$

where $S(a, b, c) = (a, b, c) + (b, c, a) + (c, a, b)$.

$$\text{Using } (R, R) \subseteq N_l \text{ in the identity (11), we get } S(a, b, c) \subseteq N_l \tag{2.13}$$

Moreover, in all ring we have the identity

$$(a, b, c) = a(b, c) + (a, c)b + S(a, b, c)\check{}(a, c, b)\check{}(b, c, a). \tag{2.14}$$

A linearization of (9) implies $(a, b, c) + (c, b, a) = 0$. Then connecting this with (14), (13) and (4) we obtain

$$a(b, c) + (a, c)b \subseteq N_l \tag{2.15}$$

Assume that $n \in N_l$. Then with $d = n$ in (6), we obtain $(na, b, c) = n(a, b, c)$. Combining this with (4) leads

$$(na, b, c) = n(a, b, c) = (an, b, c) \tag{2.16}$$

A combination of (15) and (16) yields

$$(b, c)(a, r, s) = -(a, c)(b, r, s) \tag{2.17}$$

If we substitute a commutator v for b and using (8), we get

$$(v, c)(a, r, s) = 0 \tag{2.18}$$

If we linearize $(a, a, a) = 0$, we get

$$S(a, b, a) + S(a, c, b) = 0. \tag{2.19}$$

By multiplying (12) and (19) with (p, q, r) and using (18) we obtain,
$S(a, b, c)(p, q, r)\check{}S(a, c, b)(p, q, r) = 0$
$S(a, b, c)(p, q, r) + S(a, c, b)(p, q, r) = 0$
By adding the above two equations, we have $2S(a, b, c)(p, q, r) = 0$. Since R is 2-divisible,

$$S(a, b, c)(p, q, r) = 0 \tag{2.20}$$

3. Main Results

Lemma 3.1. *Let $T = t \in N_l / t(R,R,R) = 0$. Then T is an ideal of R and $T(R,R,R) = 0$.*

Proof. By substituting t for n in (16), we get

$$(ta,b,c) = t(a,b,c) = (at,b,c) = 0.$$

Thus $tR \in N_l$ and $Rt \in N_l$. First note that $td.(a,b,c) = t.d(a,b,c)$. But (6) multiplied on left by t yields $t.d(a,b,c) = -t.(d,a,b)c = -t(d,a,b)c = 0$. Thus $tw.(a,b,c) = 0$. However (17) yields wt.$(a,b,c) = 0$. Thus, T is an ideal of R and obviously $T(R,R,R) = 0$.
Let A be the associator ideal of R. We know that A is the set of all finite sums of associators and right multiples of associators. We know that R is semiprime, if the only ideal of R which squares to zero is the zero ideal. □

Lemma 3.2. *In a semiprime derivation alternator ring R with commutators in the left nucleus $S(a,b,c) = 0$.*

Proof. By using lemma (1) and equation (6) we establish readily that $T.A = 0$.
But then $T \cap A$ is an ideal of R which squares to zero. Since R is semiprime, then $T \cap A = 0$.
From (13) and (20), we obtain $S(a,b,c) \in T$.
Also

$$S(a,b,c) \text{ is in } A. \tag{3.1}$$

Thus $S(a,b,c) = 0$. Let I be the ideal generated by $\{(b,a,a)/a,b \in R\}$ and we know that the linear span I of the alternators is an ideal. □

Theorem 3.3. *A semiprime derivation alternator ring R with commutators in the left nucleus is a (-1,1) ring.*

Proof. Since $(a,b,a) = 0$, from (7) we have $(aa^2,b,a) = 0$.
This means that by definition a flexible derivation alternator ring is non-commutative Jordan, and non-commutative Jordan rings satisfy

$$(b,a^2,c) = ao(b,a,c).. \tag{3.2}$$

Next from the identity (6)

$$(a^2,b,c)\check{}(a,ab,c) + (a,a,bc) = a(a,b,c) + (a,a,b)c. \tag{3.3}$$

Likewise (6) implies

$$(cb,a,a)\check{}(c,ba,a) + (c,b,a^2) = c(b,a,a) + (c,b,a)a. \tag{3.4}$$

By taking this into our previous equation leads to $2(a^2,b,c)\check{}ao(a,b,c) = 0$. Since R is 2-divisible, we have

$$(a^2,b,c) = ao(a,b,c). \tag{3.5}$$

Flexibility and (25) imply

$$(b,c,a^2) = ao(b,c,a). \tag{3.6}$$

Linearzing (1) and using (9), we get $(b,a,a) + (a,a,b) = 0$.
This implies that $(b,a,a) = -(a,a,b)$.
Then $((b,a,a),a,a) = -((a,a,b),a,a) = ((a,a,a),a,b) = 0$.
Applying (25), we write
$0 =((b^2,a,a),a,a) = (bo(b,a,a),a,a)$
$= bo((b,a,a),a,a) + (b,a,a)o(b,a,a)$
$= 2(b,a,a)^2$.
Since R is 2-divisible, , we get that $(b,a,a)^2 = 0$. Since I is an ideal and R is semiprime, we have $(b,a,a) = 0$.
From lemma (2) and $(b,a,a) = 0$, R is a (-1,1) ring. □

References

1. Hentzel, I. R., Kleinfeld, E. and Smith, H. F. “On rings in the join of associative and commutative”, J. Algebra, Vol. 149, No. 2 (1992), 528-537.

2. Hentzel,I. R. “The characterization of (-1,1) rings”J.Algebra 30(1974),236-258.

3. Hentzel. I. R., Hogben, L. and Smith H. F. “Flexible derivation alternator rings”, Com. in. algebra, 8 (20) (1980), 1997-2014.

4. Kleinfeld, E. “Rings with (x,y,x) and commutators in the left nucleus”, Comm. in. Algebra, 16 (10), (1988), 2023-2029.

5. Kleinfeld, E. “Generalization of alternative rings I”, J. Algebra, 18 (1971), 304-325.

6. Thedy, A. “On rings with commutators in the nucleus”, Math. Z. 119 (1971), 213-218.

Dr. P. Sarada Devi,
Department of Mathematics,
Geethanjali College of Engineering And Technology (Autonomous),
India.
E-mail address: `Sarada.chakireddy@gmail.com`

and

Dr. K Hari Babu,
Department of Mathematics,
Koneru Lakshmaiah Education Foundation,
R.V.S Nagar, Moinabad-Chilkur Rd, Near AP Police Academy,
Aziznagar, Hyderabad, Telangana 500075.
India.
E-mail address: `mathematicshari@gmail.com`

and

Dr. Y. Suresh Kumar,
Department of Mathematics,
Koneru Lakshmaiah Education Foundation,
R.V.S Nagar, Moinabad-Chilkur Rd, Near AP Police Academy,
Aziznagar, Hyderabad, Telangana 500075.
India.
E-mail address: `suresh.mscf@gmail.com`