Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–13.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.62936

Three weak solutions for a class of fourth order $p(x)$-Kirchhoff type problem with Leray-Lions operators

Khalid Soualhine, Mohammed Filali, Mohamed Talbi and Najib Tsouli

ABSTRACT: In this work, we study the multiplicity of a weak solution for a fourth order $p(x)$-Kirchhoff type problem involving the Leray-Lions type operators with no flux boundary condition. By using variational approach and critical point theory, we determine an open interval of parameters for which our problem admits at least three distinct weak solutions.

Key Words: Kirchhoff type problem, Leray-Lions type operators, variable exponent, no-flux boundary condition.

Contents

1. Introduction

The purpose of the present paper is to study the following Kirchhoff type problem with no flux boundary condition

$$(P_\lambda)\begin{cases} M(L(u))\Big(\Delta(a(x,\Delta u))+b(x)|u|^{p(x)-2}u\Big)=\lambda\Big[f(x,u)-g(x,u)\Big] & \text{in} \quad \Omega\\ u=constant,\ \ \Delta u=0 & \text{on} \quad \partial\Omega,\\ \displaystyle\int_{\partial\Omega}\frac{\partial}{\partial n}a(x,\Delta u)ds=0, \end{cases}$$

where $\Omega\subset\mathbb{R}^N(N\geq 2)$ is a bounded domain with a smooth boundary $\partial\Omega$, λ is a positive parameter, $p(x)\in C_+(\overline{\Omega})=\left\{h\in C(\overline{\Omega}) \text{ and } h^-=\min\limits_{x\in\overline{\Omega}}h(x)>1\right\}$ is a log-Hölder continuous function, that is, there exists $C>0$ such that

$$|p(x)-p(y)|\leq\frac{C}{-log|x-y|}\ \text{for all } x,y\in\Omega,\ \text{with } 0<|x-y|\leq\frac{1}{2}.$$

The functions $f,\ g:\Omega\times\mathbb{R}\to\mathbb{R}$ will be specified later, $b(x)$ is a weight function and $a:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying certain conditions which will be stated later on. The functional L is defined by

$$L(u)=\int_\Omega\left[A(x,\Delta u)+\frac{b(x)}{p(x)}|u|^{p(x)}\right]dx. \tag{1.1}$$

The research on Kirchhoff-type problems has aroused great interest over recent years because of their applications to many fields. Indeed, this type of equation arise in various models of physical and biological systems. Specifically, in [16], Kirchhoff has introduced the model given by the equation

$$\rho\frac{\partial^2u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\right)\frac{\partial^2u}{\partial x^2}=0, \tag{1.2}$$

2010 *Mathematics Subject Classification:* 35D30, 35J35, 35J60, 35G30.
Submitted March 17, 2022. Published May 25, 2022

which extends the classical D'Alembert's wave equation, by considering the effect of the changing in the length of the string during the vibration. A distinct feature is that the (1.2) problem contains a nonlocal coefficient $\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx$ which depends on the average $\frac{1}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx$. The parameters in this equation have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. After the work by Lions [19], various equation of the Kirchhoff type have been studied in the case p-Laplacian and $p(x)$-Laplacian operator (see [1,7,24]). Other authors have been interested in the Kirchhoff problems governed by the $p(x)$-biharmonic operator (see for example [12,22,23]).

In this work, we are going to consider more general operators, namely, the Leray-Lions operators that are bearing the names of the mathematicians who introduced them to the mathematical literature [18]. More precisely, we are interested in the Kirchhoff problems involving the operators

$$\Delta(a(u, \Delta u)),$$

where a is a function which satisfies some suitable conditions. Boureanu in [4] is the first who treated one class of fourth-order elliptic problem with variable exponent and Leray-Lions type operators. Kefi et al. in [13,14] were interested in the problems that involve the latter kind of operators, where they treated the following problems

$$\Delta(a(u, \Delta u)) = \lambda f(x, u) \quad \text{and} \quad \Delta(a(u, \Delta u)) = \lambda V(x)|u|^{q(x)-2}u,$$

with Navier boundary condition

$$u = \Delta u = 0 \text{ on } \partial\Omega,$$

they showed the existence of at least three weak solutions by using the variational methods and by applying a multiplicity Theorem of Bonanno and Marano [3].
Recently, Al-Shomrani et al. [15] have studied the following problem

$$(P)\begin{cases} \Delta(a(x, \Delta u)) + b(x)|u|^{q(x)-2}u = \lambda\Big(V_1(x)|u|^{l(x)-2}u - V_2(x)|u|^{\beta(x)-2}u\Big) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where V_1 and V_2 are two functions satisfying
(H) There exist s_1, $s_2 \in C(\Omega)$, such that $V_i \in L^{s_i(x)}(\Omega)$, $(i = 1, 2)$ with $V_2 \geq 0 \in \Omega$ and $V_1 > 0 \in \Omega_0 \subset\subset \Omega$, with $|\Omega_0| > 0$,
with this latter hypothesis and by using variational methods the authors proved the existence of a weak solution to problem (P).

Inspired by the above mentioned papers, we discuss in this article the multiplicity of weak solutions of the fourth order problem (P_λ) which combines the Kirchhoff function, the Leray-Lions type operators and the no flux boundary condition.
Noting that our problem is different from the work [15] and, more importantly, our different strategy gave us three weak solutions; moreover, we have another strong point which is the two weights functions m and n change signs in Ω.

In the sequel, if $h \in C_+(\overline{\Omega})$, we denote by $h'(x)$ the conjugate exponent of $h(x)$.
We assume that the function a satisfies:
(A_1) $a : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $a(x, 0) = 0$ for a.e. $x \in \overline{\Omega}$;
(A_2) a satisfies the growth condition $|a(x,t)| \leq c_0(d(x) + |t|^{p(x)-1})$ for a.e. $x \in \overline{\Omega}$ and for all $t \in \mathbb{R}$, for some constant $c_0 > 0$ and nonnegative function $d(x) \in L^{p'(x)}(\Omega)$ (where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$);
(A_3) The monotonicity condition $[a(x, s) - a(x, t)](s - t) \geq 0$ for a.e. $x \in \overline{\Omega}$ and for all $s, t \in \mathbb{R}$ with equality if and only if $s = t$;
(A_4) There exists $0 < c_1 < 3min\{1, c_0\}$ such that

$$c_1|t|^{p(x)} \leq a(x,t)t \leq p(x)A(x,t) \text{ for a.e. } x \in \overline{\Omega} \text{ and all } t \in \mathbb{R},$$

where c_0 is the one from (A_2) and $A:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is the antiderivative of a, that is,

$$A(x,t)=\int_0^t a(x,s)ds.$$

This set of hypotheses is only natural when we think at the operators that can be obtained from it by making the suitable choices. Indeed, for $A(x,s)=\dfrac{1}{p(x)}|s|^{p(x)}$ we deduce that $a(x,s)=|s|^{p(x)-2}s$ and for $s=\Delta u$ we arrive to the $p(\cdot)$-biharmonic operator

$$\Delta(a(x,\Delta u))=\Delta(|\Delta u|^{p(x)-2}\Delta u)=\Delta^2_{p(x)}u.$$

A second well-known example of operator arises when we choose

$$A(x,s)=\frac{1}{p(x)}\left[\left(1+|s|^2\right)^{\frac{p(x)}{2}}-1\right],$$

thus $a(x,s)=\left(1+|s|^2\right)^{\frac{p(x)-2}{2}}s$ and for $s=\Delta u$ we obtain the operator

$$\Delta(a(x,\Delta u))=\Delta\Big(\left(1+|\Delta u|^2\right)^{\frac{p(x)-2}{2}}\Delta u\Big).$$

Now we give the assumptions concerning the function M and the weight b:
(H_1) $M:\mathbb{R}^+\to\mathbb{R}^+$ is continuous and verifies

$$a_1t^{\alpha-1}\le M(t) \text{ for all } t\ge 0 \text{ where } \alpha>1 \text{ and } a_1>0.$$

(H_2) $b\in L^\infty(\Omega)$ and there exists $b_0>0$ such that $b(x)\ge b_0$ for all $x\in\Omega$.

The functions $f,g:\Omega\times\mathbb{R}\to\mathbb{R}$ are defined by :

$$f(x,t)=\begin{cases} m(x)|t|^{q_1(x)-2}t & |t|\ge 1\\ m(x)|t|^{r_1(x)-2}t & |t|<1,\end{cases}$$

and

$$g(x,t)=\begin{cases} n(x)|t|^{q_2(x)-2}t & |t|\ge 1\\ n(x)|t|^{r_2(x)-2}t & |t|<1,\end{cases}$$

such that the weights m, n and the variable exponents $q_i(x),\ r_i(x)(i=1,2)$ check
(H_3) $m\in L^{\beta_1(x)}(\Omega)$, where $\beta_1,\ q_1,\ r_1\in C_+(\overline{\Omega})$ and $\dfrac{1}{p_2^*(x)}+\dfrac{1}{\beta_1(x)}<\dfrac{1}{r_1(x)}<\dfrac{1}{q_1(x)}$ for all $x\in\overline{\Omega}$, where

$$p_2^*(x)=\begin{cases} \dfrac{Np(x)}{N-2p(x)} & if \quad p(x)<\dfrac{N}{2},\\ +\infty & if \quad p(x)\ge\dfrac{N}{2}.\end{cases}$$

(H_4) $n\in L^{\beta_2(x)}(\Omega)$, where $\beta_2,\ q_2,\ r_2\in C_+(\overline{\Omega})$ and $\dfrac{1}{p_2^*(x)}+\dfrac{1}{\beta_2(x)}<\dfrac{1}{r_2(x)}<\dfrac{1}{q_2(x)}$ for all $x\in\overline{\Omega}$.
(H_5) There exists a ball $B_\eta(x_0)$ of centre $x_0\in\Omega$ and Radius $\eta>0$ such that $B_\eta(x_0)\subset\subset\Omega$ and $m(x)>0$, $n(x)<0$ for a.e. x in $B_\eta(x_0)$.

Now, we give our main results.

Theorem 1.1. *Assume hypotheses (A_1)-(A_4) and (H_1)-(H_5) are fulfilled, such that $q_1^+<q_2^+<\alpha p^-<\alpha p^+<r_1^-<r_2^-$.*
Then there exist an open interval $\Lambda\subset[0,\delta]$ and a constant $\sigma>0$ such that for all $\lambda\in\Lambda$ problem (P_λ) has at least three weak solutions whose norms are less than σ, where δ will given later one.

The remainder of this paper is arranged as follows: in Section 2, we review some basic knowledge of the variable exponent Lebesgue and Sobolev spaces. In Section 3, we give the proof of our main result. We denote by $X^{'}$ the dual of Banach space X and by $\langle .,.\rangle$ the duality pairing between $X^{'}$ and X .

2. Preliminaries

We recall the definitions of the variable exponent Lebesgue and Sobolev spaces and some of their basic properties, but much more details can be found in the monograph by Rădulescu and Repovš [21] and the references therein.
Let $p(x)$ be a log-Hölder continuous with $1 < p^- \leq p^+ < \infty$.
We define the Lebesgue variable exponent space as

$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable function such that } : \int_\Omega |u|^{p(x)}\, \mathrm{d}x < \infty \right\},$
which is endowed with the following norm

$$|u|_{p(x)} = \inf\left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

$\left(L^{p(x)}(\Omega),\ |\cdot|_{p(x)}\right)$ is a separable and reflexive Banach space, (see [17]).
The modular on the space $L^{p(x)}(\Omega)$ is the map $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega |u(x)|^{p(x)} dx \quad \text{for all} \quad u \in L^{p(x)}(\Omega).$$

The next proposition states the relation between norms and the modular.

Proposition 2.1. *(see [11], Theorem 1.3 and Theorem 1.4). For $u \in L^{p(x)}(\Omega)$ and $(u_n) \subset L^{p(x)}(\Omega)$, we have*

(i) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;

(ii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;

(iii) $|u_n|_{p(x)} \to 0 (\to \infty) \Leftrightarrow \rho(u_n) \to 0 (\to \infty)$;

(iv) $|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho(u_n - u) \to 0$.

From the previous proposition we get the Hölder type inequality.

Proposition 2.2. *(see [17], Theorem 2.1). For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the Hölder type inequality*

$$\left| \int_\Omega u(x)v(x)\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)},$$

where $L^{p'(x)}(\Omega)$ is a conjugate space of $L^{p(x)}(\Omega)$.

Similarly, if $r_1,\ r_2,\ r_3 \in C_+(\overline{\Omega})$ such that $\dfrac{1}{r_1(x)} + \dfrac{1}{r_2(x)} + \dfrac{1}{r_3(x)} = 1$, then

$$\left| \int_\Omega u(x)v(x)w(x)\, dx \right| \leq 3|u|_{r_1(x)}|v|_{r_2(x)}|w|_{r_3(x)}, \tag{2.1}$$

for all $u \in L^{r_1(x)}(\Omega)$, $v \in L^{r_2(x)}(\Omega)$ and $w \in L^{r_3(x)}(\Omega)$, (see [10], Proposition 2.5).

Theorem 2.3. *([20], Theorem 2). Let $h : \Omega \times \mathbb{R}^M \to \mathbb{R}$ be a Carathéodory function which satisfies the growth condition*

$$|h(x,u)| \leq c_1(x) + c(x) \sum_{i=1}^{M} |u_i|^{\frac{p_1(x)}{p_2(x)}}, \quad x \in \Omega \quad, u \in \mathbb{R}^M,$$

where $c_1 \in L^{p_2(\cdot)}(\Omega)$ and c is a non negative function with $1 \leq c^- \leq c^+ < \infty$.
Then the Nemytsky operator N_h that maps an M-tuple of functions $(u_1, u_2, \cdots u_M)$ into

$$N_h(u_1, u_2, \cdots, u_M)(x) = h(x, u_1(x), \cdots, u_M(x)) \quad x \in \Omega,$$

is well-defined, bounded and continuous from $[L^{p_1(\cdot)}(\Omega)]^M$ into $L^{p_2(\cdot)}(\Omega)$.

We also recall the following proposition, which will be needed later.

Proposition 2.4. (*see [8], Lemma 2.1*). *Let p and q be measurable functions such that $p \in L^{\infty}(\Omega)$ and $1 < p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$ such that $u \neq 0$. Then*

(i) $|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)}$;

(ii) $|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \leq \left||u|^{p(x)}\right|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}$.

In the sequel, we give the definition of the Sobolev spaces $W^{k,p(x)}(\Omega)$:

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : D^{\alpha}u \in L^{p(x)}(\Omega), |\alpha| \leq k \right\},$$

where $D^{\alpha}u = \dfrac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_N^{\alpha_N}}$, with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index such that $|\alpha| = \sum_{i=1}^{N}\alpha_i$. The space $W^{k,p(x)}(\Omega)$ endowed with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha|\leq k} |D^{\alpha}u|_{p(x)},$$

becomes a separable and reflexive Banach space. For more details we can see ([17]). The log-Hölder continuity of $p(x)$ plays a decisive role in the following density results.

Theorem 2.5. (*see ([6], Section 6.5.3). Assume that $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with Lipschitz boundary and $p(x)$ is log-Hölder continuous with $1 < p^- \leq p^+ < \infty$. Then $C^{\infty}(\overline{\Omega})$ is dense in $W^{k,p(x)}(\Omega)$.*

Moreover, the following embedding theorem takes place.

Theorem 2.6. (*see ([6], Section 6) and ([11], Theorem 2.3)). Let us consider $q \in C(\overline{\Omega})$ such that $1 < q^- \leq q^+ < \infty$ and $q(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, where*

$$p_k^*(x) = \begin{cases} \dfrac{Np(x)}{N - kp(x)} & if \quad p(x) < \dfrac{N}{k}, \\ +\infty & if \quad p(x) \geq \dfrac{N}{k}, \end{cases}$$

for any $x \in \overline{\Omega}$ and $k \geq 1$. Then there is a continuous embedding

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

If we have $q(x) < p_k^*(x)$ for all $x \in \overline{\Omega}$ the embedding becomes compact.

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$.
Due to the log-Hölder continuity of the exponent $p(x)$, the space $W_0^{1,p(x)}(\Omega)$ coincides with

$$W_0^{1,p(x)}(\Omega) = \left\{ u \in W^{1,p(x)}(\Omega) : u = 0 \text{ on } \partial\Omega \right\}.$$

As a consequence of the Poincaré inequality, $\|u\|_{W^{1,p(x)}(\Omega)}$ and $|\nabla u|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. Therefore, for any $u \in W_0^{1,p(x)}(\Omega)$ we can define an equivalent norm $\|u\|_{W_0^{1,p(x)}(\Omega)}$ such that

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = |\nabla u|_{p(x)},$$

and which makes $W_0^{1,p(x)}(\Omega)$ a separable and reflexive Banach space (see [11]).
The space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ equipped with the norm

$$\begin{aligned}\|u\|_{W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)} &= \|u\|_{W^{2,p(x)}(\Omega)} + \|u\|_{W_0^{1,p(x)}(\Omega)} \\ &= |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|\alpha|=2} |D^{\alpha}u|_{p(x)}\end{aligned}$$

is a separable and reflexive Banach space.
Moreover, we know that $\|u\|_{W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)}$ and $|\Delta u|_{p(x)}$ are equivalent norms on $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$, (see [25], Theorem 4.4).
Let

$$\|u\|_b = \inf\left\{\omega > 0 : \int_\Omega \left(\left|\frac{\Delta u(x)}{\omega}\right|^{p(x)} + b(x)\left|\frac{u(x)}{\omega}\right|^{p(x)}\right) dx \leq 1\right\},$$

where $b(x)$ satisfies (H_2), this norm represents a norm on both $W^{2,p(x)}(\Omega)$ and $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$ and it is equivalent to the usual norm defined here, (see [9], Remark 2.1). In this work, we need to choose a variable exponent space that is appropriate for our study. Therefore, we introduce the following space

$$V = \left\{u \in W^{2,p(x)}(\Omega) : \quad u|_{\partial\Omega} \equiv \mathit{constant}\right\},$$

this space is a closed subspace of the separable and reflexive Banach space $W^{2,p(x)}(\Omega)$ equipped with the usual norm, so $\left(V, \|\cdot\|_{W^{2,p(x)}(\Omega)}\right)$ is a separable and reflexive Banach space (see [5], Theorem 4).
Thereafter, we consider the modular $\Phi : V \to \mathbb{R}$ defined by

$$\Phi(u) = \int_\Omega \left[|\Delta u|^{p(x)} + b(x)|u|^{p(x)}\right] dx,$$

which verifies the following properties

Proposition 2.7. *(See [5], Proposition 1). For $u \in W^{2,p(x)}(\Omega)$and $(u_n) \in W^{2,p(x)}(\Omega)$, we have*

(i) $\|u\|_b < 1(= 1; > 1) \Leftrightarrow \Phi(u) < 1(= 1; > 1)$;

(ii) $\|u\|_b \geq 1 \Rightarrow \|u\|_b^{p^-} \leq \Phi(u) \leq \|u\|_b^{p^+}$;

(iii) $\|u\|_b \leq 1 \Rightarrow \|u\|_b^{p^+} \leq \Phi(u) \leq \|u\|_b^{p^-}$;

(iv) $\|u_n\|_b \to 0(\to \infty) \Leftrightarrow \Phi(u_n) \to 0(\to \infty)$.

Proposition 2.8. (*See [4], Proposition* 3, 5 *and Theorem* 3.2). *Let the functional $L : V \to \mathbb{R}$ defined in (1.1) by*

$$L(u) = \int_\Omega \left[A(x, \Delta u) + \frac{b(x)}{p(x)}|u|^{p(x)}\right] dx.$$

Then we have

(i) The functional L is of class C^1, with derivative defined by :

$$\langle L'(u), h\rangle = \int_\Omega a(x, \Delta u)\Delta h dx + \int_\Omega b(x)|u|^{p(x)-2}uh dx \quad \text{for all } h \in V.$$

(ii) L is sequentially weakly lower semicontinuous, that is, for any $u \in V$ and any subsequence $(u_n) \subset V$ such that $u_n \rightharpoonup u$ in V, there holds

$$L(u) \leq \liminf_{n\to\infty} L(u_n).$$

(iii) The mapping $L' : V \to V'$ is of type (S_+), that is, $u_n \rightharpoonup u$ and

$$\limsup_{n\to\infty}\langle L'(u_n), u_n - u\rangle \leq 0 \text{ imply that } u_n \longrightarrow u.$$

The remark below of the embedding will be very useful for us in the following.

Remark 2.9. *Assume that assumption (H_3)-(H_4) are fulfilled, then $r_i(x)\beta_i'(x) < p_2^*(x)$, $q_i(x)\beta_i'(x) < p_2^*(x)$ and $\theta_i(x) = \dfrac{\beta_i(x)q_i(x)}{\beta_i(x) - q_i(x)} < p_2^*(x)$ $(i = 1, 2)$ for all $x \in \overline{\Omega}$. Therefore, the embeddings $V \hookrightarrow L^{r_i(x)\beta_i'(x)}(\Omega)$, $V \hookrightarrow L^{q_i(x)\beta_i'(x)}(\Omega)$ and $V \hookrightarrow L^{\theta_i(x)}(\Omega)$ $(i = 1, 2)$ are compact.*

3. Main result

In this part, we present the proof of Theorem 1.1. First, we introduce the definition of a weak solution to our problem. Applying Green's formula and taking into account the fact that V is a closed subspace of $(W^{2,p(x)}(\Omega), \|\cdot\|_{W^{2,p(x)}(\Omega)})$ together with the density result in Theorem 2.5 and the boundary conditions, we arrive at the following formulation.

Definition 3.1. *We say that the function $u \in V$ is a weak solution of problem (P_λ) if and only if*

$$\begin{aligned} M(L(u)) &\left(\int_\Omega a(x, \Delta u)\Delta h dx + \int_\Omega b(x)|u|^{p(x)-2} u h dx \right) \\ &= \lambda \left[\int_\Omega f(x,u) h dx - \int_\Omega g(x,u) h dx \right], \quad \textit{for all} \quad h \in V. \end{aligned}$$

Consider the functional $J_\lambda : V \to \mathbb{R}$ defined by

$$J_\lambda(u) = J_1(u) - \lambda J_2(u) + \lambda J_3(u),$$

where

$$J_1(u) = \widehat{M}(L(u)),\ J_2(u) = \int_\Omega F(x,u)dx \quad \text{and} \quad J_3(u) = \int_\Omega G(x,u)dx \tag{3.1}$$

such that

$$\widehat{M}(t) = \int_0^t M(s)ds,\ F(x,t) = \int_0^t f(x,s)ds \quad \text{and} \quad G(x,t) = \int_0^t g(x,s)ds.$$

The proof of Theorem 1.1 requires the following results.

Lemma 3.2. *The operators : $u \in L^{q_1(x)}(\Omega) \mapsto k(.,u(.)) \in L^{q_1'(x)}(\Omega)$ and $u \in L^{\beta_1'(x) q_1(x)}(\Omega) \mapsto K(.,u(.)) \in L^{\beta_1'(x)}(\Omega)$ such that*

$$k(x,u(x)) = \begin{cases} |u(x)|^{q_1(x)-2}u(x), & |u(x)| \geq 1 \\ |u(x)|^{r_1(x)-2}u(x), & |u(x)| < 1 \end{cases}$$

and

$$K(x,u(x)) = \begin{cases} \dfrac{1}{q_1(x)}|u(x)|^{q_1(x)}, & |u(x)| \geq 1 \\ \dfrac{1}{r_1(x)}|u(x)|^{r_1(x)}, & |u(x)| < 1 \end{cases}$$

are continuous.

Proof. we have $k,\ K : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and we can see that

$$|k(x,u(x))| \leq |u(x)|^{q_1(x)-1} = |u(x)|^{\frac{q_1(x)}{q_1'(x)}}$$

and

$$|K(x,u(x))| \leq \frac{1}{q_1^-}|u(x)|^{q_1(x)} = \frac{1}{q_1^-}|u(x)|^{\frac{\beta_1'(x)q_1(x)}{\beta_1'(x)}}.$$

Thus from Theorem 2.3, we conclude our proof. □

By procceding similarly to ([23], Proposition 7), one can establish the following

Proposition 3.3. *The function J_λ defined above is of class C^1 in V and for every $u,\ h \in V$*

$$\begin{aligned} \langle J_\lambda'(u), h \rangle = & M(L(u)) \left(\int_\Omega a(x, \Delta u)\Delta h dx + \int_\Omega b(x)|u|^{p(x)-2} u h dx \right) \\ & - \lambda \Big[\int_\Omega f(x,u) h dx - \int_\Omega g(x,u) h dx \Big]. \end{aligned}$$

Now, we will state the theorem, which will be essential to establish the existence of the weak solution for our main problem.

Theorem 3.4. *([2], Theorem 2.1). Let X be a separable and reflexive real Banach space, and let I, $J : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $I(x_0) = J(x_0) = 0$ and $I(x) \geq 0$ for every $x \in X$ and that there exist $x_1 \in X$, $r_0 > 0$ such that*

(i) $r_0 < I(x_1)$;

(ii) $\sup_{I(x)<r_0} J(x) < r_0 J(x_1)/I(x_1)$.

Further, put

$$\delta = \frac{\gamma r_0}{r_0 \frac{J(x_1)}{I(x_1)} - \sup_{I(x)<r_0} J(x)},$$

with $\gamma > 1$, and assume that the functional $I - \lambda J$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and

(iii) $\lim_{\|x\| \longrightarrow +\infty} (I(x) - \lambda J(x)) = +\infty$ *for every* $\lambda \in [0, \delta]$.

Then there exist an open interval $\Lambda \subseteq [0, \delta]$ and a positive real number σ such that, for every $\lambda \in \Lambda$, the equation

$$I'(x) - \lambda J'(x) = 0,$$

admits at least three distinct solutions in X whose norms are less than σ.

Thereafter, to prove our main result we need the following lemmas

Lemma 3.5. *The functional $J_\lambda$is coercive, that is, $J_\lambda(u) \longrightarrow \infty$ as $\|\cdot\|_b \longrightarrow \infty$.*

Proof. Using the hypothesis (H_1), the definitions of f, g and the fact that $q_i(x) < r_i(x) (i = 1, 2)$, we have for any $u \in V$ with $\|u\|_b \geq 1$

$$\begin{aligned}
J_\lambda(u) \geq & \frac{a_1}{\alpha}(L(u))^\alpha - \lambda \left[\int_{|u(x)|\geq 1} \frac{m(x)|u|^{q_1(x)}}{q_1(x)} dx + \int_{|u(x)|<1} \frac{m(x)|u|^{r_1(x)}}{r_1(x)} dx \right] \\
& + \lambda \left[\int_{|u(x)|\geq 1} \frac{n(x)|u|^{q_2(x)}}{q_2(x)} dx + \int_{|u(x)|<1} \frac{n(x)|u|^{r_2(x)}}{r_2(x)} dx \right] \\
\geq & \frac{a_1}{\alpha}(L(u))^\alpha - \lambda \left[\int_{|u(x)|\geq 1} \frac{|m(x)||u|^{q_1(x)}}{q_1(x)} dx + \int_{|u(x)|<1} \frac{|m(x)||u|^{r_1(x)}}{r_1(x)} dx \right] \\
& - \lambda \left[\int_{|u(x)|\geq 1} \frac{|n(x)||u|^{q_2(x)}}{q_2(x)} dx + \int_{|u(x)|<1} \frac{|n(x)||u|^{r_2(x)}}{r_2(x)} dx \right] \\
\geq & \frac{a_1}{\alpha}(L(u))^\alpha - \frac{\lambda}{q_1^-} \left[\int_\Omega |m(x)||u|^{q_1(x)} dx \right] - \frac{\lambda}{q_2^-} \left[\int_\Omega |n(x)||u|^{q_2(x)} dx \right].
\end{aligned}$$

Considering hypothesis (A_4), (ii) in Proposition 2.7, the Hölder inequality, Proposition 2.4 and Remark 2.9, there exists $c_4, c_5 > 0$ such that

$$\begin{aligned}
J_\lambda(u) \geq & \frac{a_1 \left(min\{1, c_1\}\right)^\alpha}{\alpha(p^+)^\alpha} \|u\|_b^{\alpha p^-} - \frac{2\lambda}{q_1^-} |m|_{\beta_1(x)} \left| |u|^{q_1(x)} \right|_{\beta_1'(x)} - \frac{2\lambda}{q_2^-} |n|_{\beta_2(x)} \left| |u|^{q_2(x)} \right|_{\beta_2'(x)} \\
\geq & \frac{a_1 \left(min\{1, c_1\}\right)^\alpha}{\alpha(p^+)^\alpha} \|u\|_b^{\alpha p^-} - \frac{2\lambda}{q_1^-} |m|_{\beta_1(x)} \left(|u|_{q_1(x)\beta_1'(x)}^{q_1^+} + |u|_{q_1(x)\beta_1'(x)}^{q_1^-} \right) \\
& - \frac{2\lambda}{q_2^-} |n|_{\beta_2(x)} \left(|u|_{q_2(x)\beta_2'(x)}^{q_2^+} + |u|_{q_2(x)\beta_2'(x)}^{q_2^-} \right) \\
\geq & \frac{a_1 \left(min\{1, c_1\}\right)^\alpha}{\alpha(p^+)^\alpha} \|u\|_b^{\alpha p^-} - \frac{2\lambda c_4}{q_1^-} |m|_{\beta_1(x)} \|u\|_b^{q_1^+} - \frac{2\lambda c_5}{q_2^-} |n|_{\beta_2(x)} \|u\|_b^{q_2^+}.
\end{aligned}$$

Since $q_1^+ < q_2^+ < \alpha p^-$, we deduce that J_λ is coercive. $\square$

Lemma 3.6. *The functional J_λ verifies the Palais-Smale condition for every $\lambda > 0$, that is, any sequence $(u_n) \subset V$ which satisfies the properties*

$$|J_\lambda(u_n)| \leq c \quad \text{and} \quad J'_\lambda(u_n) \longrightarrow 0 \quad \text{in} \quad V^{'} \quad \text{as} \quad n \longrightarrow \infty, \tag{3.2}$$

possesses a convergent subsequence in V.

Proof. Let $(u_n) \subset V$ be a sequence verifying (3.2). Then by the fact that J_λ is coercive and by the first relation in (3.2) we can see that (u_n) is bounded in V. Since V is a reflexive Banach space and a closed subspace of $W^{2,p(x)}(\Omega)$.
Therefore, there exists a subsequence still denoted by (u_n) and $u \in V$ such that

$$u_n \rightharpoonup u \text{ in } V \text{ as } n \longrightarrow \infty, \tag{3.3}$$

then

$$\lim_{n \longrightarrow \infty} \langle J'_\lambda(u_n), u_n - u\rangle = 0.$$

So

$$\lim_{n \longrightarrow \infty} M(L(u_n)) \left(\int_\Omega \left[a(x, \Delta u_n)\Delta(u_n - u) + b(x)|u_n|^{p(x)-2}u_n(u_n - u) \right] dx \right)$$
$$-\lambda \int_\Omega f(x, u_n)(u_n - u)dx + \lambda \int_\Omega g(x, u_n)(u_n - u)dx = 0.$$

By (H_3)-(H_4), Hölder inequality (2.1) and Proposition 2.4 we have

$$\begin{aligned} \left| \int_\Omega f(x, u_n)(u_n - u)dx \right| &\leq \left| \int_\Omega m(x)|u_n|^{q_1(x)-2}u_n(u_n - u)dx \right| \\ &\leq 3|m|_{\beta_1(x)} \left| |u_n|^{q_1(x)-1} \right|_{q'_1(x)} |u_n - u|_{\theta_1(x)} \\ &\leq 3|m|_{\beta_1(x)} \left[|u_n|^{q_1^+-1}_{q_1(x)} + |u_n|^{q_1^- -1}_{q_1(x)} \right] |u_n - u|_{\theta_1(x)}, \end{aligned}$$

and

$$\begin{aligned} \left| \int_\Omega g(x, u_n)(u_n - u)dx \right| &\leq \left| \int_\Omega n(x)|u_n|^{q_2(x)-2}u_n(u_n - u)dx \right| \\ &\leq 3|n|_{\beta_2(x)} \left| |u_n|^{q_2(x)-1} \right|_{q'_2(x)} |u_n - u|_{\theta_2(x)} \\ &\leq 3|n|_{\beta_2(x)} \left[|u_n|^{q_2^+-1}_{q_2(x)} + |u_n|^{q_2^- -1}_{q_2(x)} \right] |u_n - u|_{\theta_2(x)}, \end{aligned}$$

due to embeddings in Remark 2.9, we get

$$\lim_{n \longrightarrow \infty} \int_\Omega f(x, u_n)(u_n - u)dx = 0 \quad \text{and} \quad \lim_{n \longrightarrow \infty} \int_\Omega g(x, u_n)(u_n - u)dx = 0.$$

Then, we arrive at

$$\lim_{n \longrightarrow \infty} M(L(u_n)) \left(\int_\Omega \left[a(x, \Delta u_n)\Delta(u_n - u) + b(x)|u_n|^{p(x)-2}u_n(u_n - u) \right] dx \right) = 0.$$

On the other hand, by (H_1), (A_4) and Proposition 2.7, we have

$$\left| M(L(u_n)) \left(\int_\Omega \left[a(x, \Delta u_n)\Delta(u_n - u) + b(x)|u_n|^{p(x)-2}u_n(u_n - u) \right] dx \right) \right|$$
$$\geq \left| \frac{a_1(\min\{1, c_1\})^{\alpha-1}}{(p^+)^{\alpha-1}} \min\{\|u_n\|_b^{(\alpha-1)p^+}, \|u_n\|_b^{(\alpha-1)p^-}\} \right| \left| \int_\Omega a(x, \Delta u_n)\Delta(u_n - u)dx \right.$$
$$\left. + \int_\Omega b(x)|u_n|^{p(x)-2}u_n(u_n - u)dx \right| \geq 0.$$

If $\|u_n\|_b \longrightarrow 0$ then (u_n) converge strongly to $u=0$ in V and the proof is finished.
Otherwise $\|u_n\|_b$ is bounded in V. Then

$$\lim_{n\longrightarrow\infty}\int_\Omega a(x,\Delta u_n)\Delta(u_n-u)dx+\int_\Omega b(x)|u_n|^{p(x)-2}u_n(u_n-u)dx=0.$$

As a conclusion, the weak convergence (3.3) and (iii) in Proposition 2.8 imply that $u_n \longrightarrow u$ in V, then we conclude that J_λ verifies the Palais-Smale condition. □

Lemma 3.7. *The functional J_λ is (sequentially) weakly lower semicontinuous for every $\lambda>0$.*

Proof. We already know that L is weakly lower semicontinuous by (ii) in Proposition 2.8. Moreover, the continuity and growth of $\widehat{M}$ allow us to deduce that J_1 is weakly lower semicontinuous. It remains to verify that this proprety is also true for J_2 and J_3.
Let $(u_n)\subset V$ such that $u_n \rightharpoonup u$ in V. Due to Remark 2.9, $u_n \longrightarrow u$ in $L^{\beta_1'(x)q_1(x)}(\Omega)$ as $n\longrightarrow\infty$. By using Lemma 3.2 and the Hölder inequality, we get

$$\begin{aligned}|J_2(u_n)-J_2(u)| &\leq \int_\Omega |F(x,u_n)-F(x,u)|dx\\ &\leq \int_\Omega |m(x)||K(x,u_n(x))-K(x,u(x))|dx\\ &\leq 2|m|_{\beta_1(x)}|K(x,u_n(x))-K(x,u(x))|_{\beta_1'(x)} \longrightarrow 0, \text{ as } n\longrightarrow\infty,\end{aligned}$$

hence the functional J_2 is weakly continuous in V. Similarly we can show that J_3 is also weakly continuous. So J_λ is sequentially weakly lower semicontinuous. □

***Proof of Theorem* 1.1.** Given the ball $B_\eta(x_0)$ in hypothesis (H_5), we can construct the following $C_c^\infty(\Omega)$ function

$$\varphi(x)=\begin{cases}1 & in \quad B_{\frac{\eta}{2}}(x_0)\\ 0 & in \quad \Omega\setminus B_\eta(x_0).\end{cases}$$

Then, we have

$$\begin{aligned}&\int_\Omega F(x,\varphi)dx-\int_\Omega G(x,\varphi)dx\\ &=\int_\Omega \frac{m(x)}{r_1(x)}|\varphi(x)|^{r_1(x)}dx-\int_\Omega \frac{n(x)}{r_2(x)}|\varphi(x)|^{r_2(x)}dx\\ &=\int_{\Omega\setminus B_\eta(x_0)} \frac{m(x)}{r_1(x)}|\varphi(x)|^{r_1(x)}dx+\int_{B_\eta(x_0)\setminus B_{\frac{\eta}{2}}(x_0)} \frac{m(x)}{r_1(x)}|\varphi(x)|^{r_1(x)}dx\\ &\quad+\int_{B_{\frac{\eta}{2}}(x_0)} \frac{m(x)}{r_1(x)}|\varphi(x)|^{r_1(x)}dx-\int_{\Omega\setminus B_\eta(x_0)} \frac{n(x)}{r_2(x)}|\varphi(x)|^{r_2(x)}dx\\ &\quad-\int_{B_\eta(x_0)} \frac{n(x)}{r_2(x)}|\varphi(x)|^{r_2(x)}dx\\ &\geq \int_{B_{\frac{\eta}{2}}(x_0)} \frac{m(x)}{r_1(x)}|\varphi(x)|^{r_1(x)}dx \geq \frac{1}{r_1^+}\int_{B_{\frac{\eta}{2}}(x_0)} m(x)dx.\end{aligned}$$

So, it is easy to see that

$$\varphi\in V \qquad \text{and} \qquad \frac{\int_\Omega [F(x,\varphi)-G(x,\varphi)]\,dx}{J_1(\varphi)}>0. \tag{3.4}$$

We take $0 < r_0 < \left\{ \frac{a_1\, (min\{1, c_1\})^{\alpha}}{\alpha(p^+)^{\alpha}}, J_1(\varphi) \right\}$ and we notice that, whenever $J_1(u) < r_0$, we obtain by (H_1) and (A_4) that

$$\frac{a_1\, (min\{1, c_1\})^{\alpha}}{\alpha(p^+)^{\alpha}} \left(\int_{\Omega} \left[|\Delta u|^{p(x)} + b(x)|u|^{p(x)} \right] dx \right)^{\alpha} < r_0, \tag{3.5}$$

then

$$\int_{\Omega} \left[|\Delta u|^{p(x)} + b(x)|u|^{p(x)} \right] dx < 1,$$

and by (i) in Proposition 2.7, $\|u\|_b < 1$ which implies by (3.5) that

$$\|u\|_b < (c_6 r_0)^{\frac{1}{\alpha p^+}}, \tag{3.6}$$

where $c_6 > 0$, on the other hand by definitions of f, g, and the fact that $q_i(x) < r_i(x)$ for $(i = 1, 2)$, we get

$$\int_{\Omega} \left[F(x, u) - G(x, u) \right] dx \leq \frac{1}{q_1^-} \int_{\Omega} |m(x)||u|^{r_1(x)} dx + \frac{1}{q_2^-} \int_{\Omega} |n(x)||u|^{r_2(x)} dx.$$

Then by using Hölder inequality and (3.6), there exist $c_7, c_8 > 0$ such that

$$\int_{\Omega} \left[F(x, u) - G(x, u) \right] dx \leq c_7 r_0^{\frac{r_1^-}{\alpha p^+}} + c_8 r_0^{\frac{r_2^-}{\alpha p^+}}.$$

We know that $\alpha p^+ < r_1^- < r_2^- < p_2^*(x)$. Therefore, relation (3.4) allows us to choose r_0 small enough to have

$$\frac{sup_{J_1(u) < r_0} \int_{\Omega} [F(x, u) - G(x, u)]\, dx}{r_0} < \frac{\int_{\Omega} [F(x, \varphi) - G(x, \varphi)]\, dx}{J_1(\varphi)}. \tag{3.7}$$

Note that $J_1(u) \geq 0$ for all $u \in V$ and

$$J_1(0) = \int_{\Omega} \left[F(x, 0) - G(x, 0) \right] dx = 0.$$

Moreover, since

$$J_{\lambda}(u) = J_1(u) - \lambda \int_{\Omega} \left[F(x, u) - G(x, u) \right] dx,$$

Lemma 3.5, Lemma 3.6 and Lemma 3.7 and relation (3.7) allow us to apply Theorem 3.4. Thus we obtain the existence of a constant $\sigma > 0$ and an open interval $\Lambda \subseteq [0, \delta]$,
where

$$\delta = \frac{\gamma r_0}{r_0 \frac{\int_{\Omega} [F(x,\varphi) - G(x,\varphi)] dx}{J_1(\varphi)} - \sup_{J_1(u) < r_0} \int_{\Omega} [F(x, u) - G(x, u)]\, dx}$$

with $\gamma > 1$, such that for all $\lambda \in \Lambda$ problem (P_{λ}) admits at least three distinct weak solutions in V of norms less than σ. This ends the proof of theorem 1.1. $\square$

Acknowledgments

The authors would like to thank the anonymous referees and the editor for their careful reading and valuable comments and suggestions on this article.

References

1. Arosio, A., Panizzi, S., *On the well-posedness of the Kirchhoff string*, Trans. Amer. Math. soc. 348, 305-330, (1996).

2. Bonanno, G., *Some remarks on a three critical points theorem*, Nonlinear Anal. TMA. 54, 651-665, (2003).

3. Bonanno, G., Marano, S. A., *On the structure of the critical set of nondifferentiable functions with a weak compactness condition*, Appl. Anal. 89, 1-10, (2010).

4. Boureanu, M.M., *Fourth order problems with Leray-Lions type operators in variable exponent spaces*, Discrete Contin. Dyn. Syst. Ser. S. 12, 231-243, (2019).

5. Boureanu, M.M., Rădulescu, V.D., Repovš, D., *On a $p(.)$-biharmonic problem with no-flux boundary condition*, Comput. Math. App. 72, 2505-2515, (2016).

6. Cruz-Uribe, D.V., Fiorenza, A., *Variable Lebesgue spaces. Foundations and harmonic analysis. Applied and Numerical Harmonic Analysis*, Brikhäuser/Springer, Heidelberg (2013).

7. Dreher, M., *The kirchhoff equation for the p-Laplacian.* Rend. Semin. Mat. Univ. Politec. Torino. 64, 217-238, (2006).

8. Edmunds, D.E., Rákosník, J., *Sobolev embeddings with variable exponent*, studia Math. 143, 267-293,(2000).

9. El Amrouss, A.R., Ourraoui, A.: *Existence of solutions for a boundary problem involving $p(x)$-biharmonic operator*, Bol. Soc. Parana. Mat. (3) 31, 179-192,(2013).

10. Fan, X.L., Han, X., *Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$.* Nonlinear Anal. 59, 173-188, (2004).

11. Fan, X.L., Zhao, D., *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$* , J. Math. Anal. appl. 263, 424-446, (2001).

12. Kefi, K., Rădulescu, V.D., *Small perturbations of nonlocal biharmonic problems with variable exponent and competing nonlinearities*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 29, 439-463, (2018).

13. Kefi, K., Repovš, D.D., Saoudi, K., *On weak solutions for fourth-order problems involving the Leray-Lions type operators*, Math. Meth. Appl. Sci. (2021). https://doi.org/10.1002/mma.7606

14. Kefi, K., Irzi, N., Al-Shomrani, M.M., Repovš, D.D., *On the fourth-order Leray-Lions problem with indefinite weight and nonstandard growth conditions*, Bulletin of Mathematical Sciences. (2021). https://doi.org/10.1142/S1664360721500089

15. Al-Shomrani, M.M., Salah, M.B.M., Ghanmi, A., Kefi, K., *Existence Results for Nonlinear Elliptic Equations with Leray–Lions Operators in Sobolev Spaces with Variable Exponents*, Math Notes 110, 830-841, (2021). https://doi.org/10.1134/S0001434621110201

16. Kirchhoff, G., *Mechanik,* Teubner, Leipzig. (1883).

17. Kováčik, O., Rákosník, J., *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. 41, 592-618, (1991).

18. Leray, J., Lions, J.L., *Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bulletin de la Société Mathématique de France. 93, 97-107, (1965).

19. Lions, J.L., *On some questions in boundary value problems of mathematical physics, in Proceedings of International Symposium on Continuum Mechanics and Partial Differential Equations*, Rio de Janeiro 1977. Math. Stud. (Penha and Medeiros, Eds), North Holland. 30, 284-346, (1978).

20. Matei, P.: *Nemytskij operators in Lebesgue spaces with a variable exponent*, Rom. J. Math. Comput. Sci. 3, 109-118, (2013).

21. Rădulescu, V.D., Repovš, D.D., *Partial Differential Equations with Variable Exponents. Variational Methods and Qualitative Analysis*, Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL. (2015).

22. Soualhine, K., Filali, M., Talbi, M., Tsouli, N., *A critical $p(x)$-biharmonic Kirchhoff type problem with indefinite weight under no flux boundary condition*, Bol. Soc. Mat. Mex. 28, 22,(2022). https://doi.org/10.1007/s40590-022-00419-6

23. Talbi, M., Filali, M., Soualhine, K., Tsouli, N., *On a $p(x)$-biharmonic Kirchhoff type problem with indefinite weight and no flux boundary condition*, Collect. Math. (2021). https://doi.org/10.1007/s13348-021-00316-7

24. Tsouli, N., Haddaoui, M., Hssini EL M., *Multiple Solutions for a Critical $p(x)$-Kirchhoff Type Equations*, Bol. Soc. Paran. Mat. 38 197-211, (2020).

25. Zang, A., Fu, Y., *Interpolation inequalities for derivatives in variable exponent Lebesgue Sobolev spaces*, Nonlinear Anal. 69, 3629-3636, (2008).

Khalid Soualhine,
Department of Mathematics,
University Mohammed I, Faculty of Sciences, Oujda
Morocco.
E-mail address: khalid.slhn2013@gmail.com

and

Mohammed Filali,
Department of Mathematics,
University Mohammed I, Faculty of Sciences, Oujda
Morocco.
E-mail address: filali1959@yahoo.fr

and

Mohamed Talbi,
CRMEF, Oujda
Morocco.
E-mail address: talbijdi@gmail.com

and

Najib Tsouli,
Department of Mathematics,
University Mohammed I, Faculty of Sciences, Oujda
Morocco.
E-mail address: tsouli@hotmail.com