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Common Fixed Point and Common Coupled Fixed Point Theorems for Weakly Monotone Mappings

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ABSTRACT: In this paper, we have established two common fixed point theorems that are variants of the fixed point theorem of Boyd and Wong. By applying the newly established fixed point theorems, two common coupled fixed point theorems have been proved for a pair of weakly increasing mappings. Examples are given to substantiate our theorems.

Key Words: Coupled fixed point, coupled common fixed point, weakly increasing mappings, partially ordered set.

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1. Introduction

The most celebrated fixed point theorem in metric fixed point theory, known as the Banach contraction principle, has undergone several extensions and generalizations due to the simple nature of the statement and its wide range of applicability. Some of its significant generalizations are done by introducing classes of generalized contraction mappings due to Rakotch, Boyd and Wong, Browder, Geraghty, Caristi, and Jaggi [17,10,6,7,31,22]. In 2003, Ran and Reuruings [32] started a new direction for the study of fixed points by establishing an analogous result of the Banach fixed point theorem in partially ordered metric spaces (a metric space with a partial order on it). Followed by this, Nieto and Lopez [27,28] established several fixed point theorems in partially ordered metric spaces using the continuity of function as well as the order completion property of the domain. Nowadays, a large number of research projects have been carried out in this direction [2,3,8,11,12].
In 1988, the concept of coupled fixed point was introduced by Guo and Lakshmikantham [19] as an extension of fixed point. They have established some coupled fixed point theorems for both continuous and discontinuous operators.The coupled fixed point theorems proposed by Gnana Bhaskar and Lakshmikantham [18] in 2006 gained more attention from researchers. In 2009, Lakshmikantham and Ciric generalized the results in [18] by introducing the concepts of coupled coincidence and coupled common fixed points in [26]. Following these research works, several authors carried out studies on coupled fixed points, coupled coincidence points, and coupled common fixed points [4,5,13,14,23,24,21,34,20,1,9,30,29].
In this paper, we establish common fixed point theorems for a pair of weakly increasing mappings that are variants of Boyd and Wong's fixed point theorem in partially ordered metric spaces. Using these results, we have proved common coupled fixed point theorems for a pair of weakly increasing mappings in partially ordered metric spaces.
Some useful definitions and results follow:

Definition 1.1. *An element $x \in X$ is said to be a common fixed point of the mappings $f, g : X \to X$ if $f(x) = x = g(x)$.*

Definition 1.2. *An element $(x, y) \in X \times X$ is said to be a common coupled fixed point of the mappings $F, G : X \times X \to X$ if $F(x, y) = x = G(x, y)$ and $F(y, x) = y = G(y, x)$.*

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Definition 1.3. *[15,16] Let $(X, \leq)$ be a partially ordered set. Let $f, g : X \to X$ and $F, G : X \times X \to X$ be mappings on X.*

1. *f, g are said to be weakly increasing if $f(x) \leq g(f(x))$ and $g(x) \leq f(g(x))$ for all $x \in X$.*
2. *f, g are said to be weakly decreasing if $f(x) \geq g(f(x))$ and $g(x) \geq f(g(x))$ for all $x \in X$.*
3. *F, G are said to be weakly increasing if*
$$F(x, y) \leq G(F(x, y), F(y, x))$$
and
$$G(x, y) \leq F(G(x, y), G(y, x))$$
for all $x, y \in X$.
4. *F, G are said to be weakly decreasing if*
$$F(x, y) \geq G(F(x, y), F(y, x))$$
and
$$G(x, y) \geq F(G(x, y), G(y, x))$$
for all $x, y \in X$.

In the above definitions 1 and 2, if $f = g$, then we say that f is weakly increasing and weakly decreasing respectively. Similarly in definitions 3 and 4 if $F = G$ then we say that F is weakly increasing and weakly decreasing respectively.

Definition 1.4. *[25] The function $\phi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties hold:*

1. *ϕ is continuous and non decreasing*
2. *$\phi(t) = 0$ if and only if $t = 0$.*

We use the following notations:
$\Phi = \{\phi : [0, \infty) \to [0, \infty) \mid \phi \text{ is an altering distance function}\}$,
and $\Psi = \{\psi : [0, \infty) \to [0, \infty) \mid \psi$ is right upper semi-continuous with the condition $\psi(0) = 0$ *and* $\psi(t) < \phi(t)$ for all $t > 0$ *where* $\phi \in \Phi\}$.

Definition 1.5. *[33] An ordered metric space $(X, \preccurlyeq, d)$ is said to have the sequential monotone property if it verifies:*

(i) If $\{x_m\}$ is a non decreasing sequence and $\{x_m\} \xrightarrow{d} x$, then $x_m \preccurlyeq x$ for all m.

(ii) If $\{y_m\}$ is a non increasing sequence and $\{y_m\} \xrightarrow{d} y$, then $y \preccurlyeq y_m$ for all m.

Definition 1.6. *A function $\psi : [0, \infty) \to [0, \infty)$ is said to be right upper semi continuous, if $r_n \downarrow r \geq 0$ then $\limsup_{n\to\infty} \psi(r_n) \leq \psi(r)$.*

The following is a generalization of Banach fixed point theorem by Boyd and Wong.

Theorem 1.7. *[6] Let (X, d) be a complete metric space and suppose $f : X \to X$ satisfies*
$$d(f(x), f(y)) \leq \psi(d(x, y)), \ \ for \ each \ x, y \in X,$$
where $\psi : \bar{P} \to [0, \infty)$ is upper semi continuous from the right on $\bar{P}$ and satisfies $\psi(t) < t$ for all $t \in \bar{P} \setminus \{0\}$. Then f has a unique fixed point x_0 and $f^n(x) \to x_0$ for each $x \in X$.

In the above theorem P denote the range of the metric d.
Note: In this paper we use $\mathcal{F}(f, g)$, $\mathcal{CF}(f, g)$ to denote the set of all fixed points and set of coupled fixed points of the mappings f and g respectively.

2. Main Results

In this section four theorems are established in which two are common fixed point theorems and two are common coupled fixed point theorems for weakly monotone mappings.

Theorem 2.1. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, f and g be self maps on X and the pair (f, g) be weakly increasing with respect to $\preceq$ such that*

$$\phi(d(fx, gy)) \leq \psi(d(x, y)), \tag{2.1}$$

for all comparable $x, g(y) \in X$ where $\phi \in \Phi$ and $\psi \in \Psi$. Suppose either

(a) f (or g) is continuous or

(b) $(X, \preceq, d)$ satisfies the property that if $\{x_n\}$ is an increasing sequence in X and $x_n \to x$ as $n \to \infty$ then $x_n \preceq x,\ \forall\ n$,

then f and g have a common fixed point. Moreover, if x^* and y^* are comparable whenever $x^*, y^* \in \mathcal{F}(f, g)$, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary element.
Define $x_1 = f(x_0),\ x_2 = g(x_1)$.
Continuiung like this we get a sequence $\{x_n\}$ in X such that $\forall\ n \in \mathbb{N} \cup \{0\}$

$$x_{2n+1} = f(x_{2n}) \text{ and } x_{2n+2} = g(x_{2n+1}).$$

Since f and g are weakly increasing, we have

$$x_{2n+1} = f(x_{2n}) \preceq g(f(x_{2n})) = g(x_{2n+1}) = x_{2n+2}$$

and

$$x_{2n+2} = g(x_{2n+1}) \preceq f(g(x_{2n+1})) = f(x_{2n+2}) = x_{2n+3}.$$

Therefore sequence $\{x_n\}$ is monotone increasing in X.
Case 1: $x_m = x_{m+1}$ for some $m \in \mathbb{N} \cup \{0\}$.
Then either $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N} \cup \{0\}$ or $x_{2n} = x_{2n-1}$ for some $n \in \mathbb{N}$.
Case 1(i): Suppose $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N} \cup \{0\}$.
That is $x_{2n} = f(x_{2n})$.
Since $\psi(0) = 0$, we have

$$\begin{aligned} \phi[d(x_{2n+1}, x_{2n+2})] &= \phi[d(f(x_{2n}), g(x_{2n+1}))] \\ &\leq \psi(d(x_{2n}, x_{2n+1})) \\ &= 0 \end{aligned}$$

Therefore by the property of ϕ we get $x_{2n+1} = x_{2n+2}$.
That is, $x_{2n} = x_{2n+1} = x_{2n+2} = g(x_{2n+1})$, which gives that $x_{2n} = g(x_{2n})$.
Therefore x_{2n} is a common fixed point of f and g.
Case 1(ii): Suppose $x_{2n} = x_{2n-1}$ for some $n \in \mathbb{N}$.
In similar steps as in Case 1(i) we get x_{2n-1} is a common fixed point of f and g.
Case 2: $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.
Consider,

$$\begin{aligned} \phi[d(x_{2n+1}, x_{2n+2})] &= \phi[d(f(x_{2n}), g(x_{2n+1}))] \\ &\leq \psi(d(x_{2n}, x_{2n+1})) \qquad (2.2) \\ &< \phi[d(x_{2n}, x_{2n+1})] \end{aligned}$$

also,

$$\begin{aligned}\phi[d(x_{2n+3}, x_{2n+2})] &= \phi[d(f(x_{2n+2}), g(x_{2n+1}))] \\ &\leq \psi(d(x_{2n+2}, x_{2n+1})) \\ &< \phi[d(x_{2n+2}, x_{2n+1})].\end{aligned}$$

Since ϕ is monotone increasing we have

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}) \text{ and } d(x_{2n+3}, x_{2n+2}) < d(x_{2n+2}, x_{2n+1}).$$

That is, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of non negative reals, so there exist $s \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = s.$$

Suppose $s > 0$.
By (2.2) we have

$$\phi[d(x_{2n+1}, x_{2n+2})] \leq \psi(d(x_{2n}, x_{2n+1})).$$

By taking upper limit on both sides we get

$$\phi[s] \leq \psi(s),$$

a contradiction. Therefore $s = 0$.
Hence, to prove that $\{x_n\}$ is a Cauchy sequence in X, it is enough to prove that $\{x_{2n}\}$ is a Cauchy sequence in X.
On the contrary assume that there exist $\epsilon > 0$ and two sub sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n(k) > m(k) \geq k$ such that

$$d(x_{2m(k)}, x_{2n(k)}) \geq \epsilon, \ \forall\ k \in \mathbb{N} \tag{2.3}$$

and by choosing $n(k)$ to be the smallest number exceeding $m(k)$ for which (2.3) holds we get,

$$d(x_{2m(k)}, x_{2n(k)-2}) < \epsilon. \tag{2.4}$$

By (2.3) and (2.4) we get

$$\begin{aligned}\epsilon &\leq d(x_{2m(k)}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2n(k)-2}) + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) \\ &< \epsilon + d(x_{2n(k)-2}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}).\end{aligned}$$

By taking limit as $k \to \infty$ we get

$$\lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)}) = \epsilon \tag{2.5}$$

$$\text{and } \lim_{k\to\infty} d(x_{2m(k)}, x_{2n(k)-2}) = \epsilon. \tag{2.6}$$

Next consider

$$\begin{aligned}d(x_{2m(k)}, x_{2n(k)}) &\leq d(x_{2m(k)}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2n(k)}) \\ &\leq d(x_{2m(k)}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2m(k)}) + d(x_{2m(k)}, x_{2n(k)}).\end{aligned}$$

By taking limit as $k \to \infty$ and by (2.5) we get

$$\lim_{k\to\infty} d(x_{2m(k)+1}, x_{2n(k)}) = \epsilon. \tag{2.7}$$

Now consider

$$\begin{aligned} d(x_{2m(k)+1}, x_{2n(k)}) &\leq d(x_{2m(k)+1}, x_{2m(k)+2}) + d(x_{2m(k)+2}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)}) \\ &\leq d(x_{2m(k)+1}, x_{2m(k)+2}) + d(x_{2m(k)+2}, x_{2m(k)+1}) + d(x_{2m(k)+1}, x_{2n(k)}) \\ &\quad + d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2n(k)}). \end{aligned}$$

By taking limit as $k \to \infty$ and by (2.7) we get

$$\lim_{k\to\infty} d(x_{2m(k)+2}, x_{2n(k)+1}) = \epsilon. \tag{2.8}$$

Next consider

$$\begin{aligned} \phi[d(x_{2n(k)+1}, x_{2m(k)+2})] &= \phi[d(f(x_{2n(k)}), g(x_{2m(k)+1}))] \\ &\leq \psi(d(x_{2n(k)}, x_{2m(k)+1})) \end{aligned}$$

By (2.7) and (2.8) we have

$$\phi[\epsilon] < \psi(\epsilon),$$

a contradiction.
Thus $\{x_{2n}\}$ is a Cauchy sequence in X, hence $\{x_n\}$ is a Cauchy sequence in X.
Since X is a complete metric space, there exist some $x \in X$ such that

$$\lim_{n\to\infty} x_n = x.$$

(a) Suppose that f is continuous.
We have $x_{2n+1} = f(x_{2n})$.
By taking limit as $n \to \infty$, we get $x = f(x)$
Since f and g are weakly increasing we get x and $g(x)$ are comparable.
Now consider

$$\begin{aligned} \phi[d(x, g(x))] &= \phi[d(f(x), g(x))] \\ &\leq \psi(d(x, x)) \\ &= 0. \end{aligned}$$

By the property of ϕ we get $x = g(x)$.
Thus x is a common fixed point of f and g.
Similarly, when g is continuous we get x is a common fixed point of f and g.
(b) Suppose X satisfies the property that if $\{x_n\}$ is an increasing sequence in X and $x_n \to x$ as $n \to \infty$ then $x_n \preceq x, \ \forall\ n$.
By the property of X, we have $x_n \preceq x$ for all n, which gives that $x_{2n+2} \preceq x$ for all n.
Now consider

$$\begin{aligned} \phi[d(f(x), x_{2n+2})] &= \phi[d(f(x), g(x_{2n+1}))] \\ &\leq \psi(d(x, x_{2n+1})). \end{aligned}$$

By taking the upper limit as $n \to \infty$ and by the property of ϕ and ψ we get

$$\phi[d(f(x), x)] = 0,$$

which gives that $x = f(x)$.
Since f and g are weakly increasing we get x and $g(x)$ are comparable.
Now consider

$$\begin{aligned} \phi[d(x, g(x))] &= \phi[d(f(x), g(x))] \\ &\leq \psi(d(x, x)) \\ &= 0. \end{aligned}$$

Thus $x = g(x)$.
Therefore x is a common fixed point of f and g.

Suppose $x, y \in X$ are two different common fixed points of f and g.
Assume that x and y are comparable.
Now consider

$$\begin{aligned}\phi[d(x,y)] &= \phi[d(f(x),g(y))]\\ &\leq \psi(d(x,y))\\ &< \phi[d(x,y)],\end{aligned}$$

a contradiction.
Therefore $x = y$.
Thus there exist a unique common fixed point of f and g.
Hence the proof. □

We illustrate the theorem with the following example.

Example 2.2. *Let $X = [0,1]$ with usual metric.*
Define $f : X \to X$ and $g : X \to X$ as

$$f(x) = \frac{x}{2} \text{ and } g(x) = \begin{cases} \frac{x}{2}, & \text{if } x \neq 1 \\ \frac{1}{4}, & \text{if } x = 1. \end{cases}$$

Define a partial order $\preceq$ on X as

$$\preceq\, := \left\{(x,x), \left(\frac{x}{2}, \frac{x}{2^n}\right) \middle|\ x \in X,\ n \in \mathbb{N}\right\}.$$

The pair of mappings (f,g) is weakly increasing and satisfy the contraction type condition (2.1) for $\phi(t) = t,\ \psi(t) = \frac{t}{2},\ \forall t$.
Here 0 is a common fixed point of f and g.

Theorem 2.3. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, f and g be self maps on X and the pair (f,g) be weakly decreasing with respect to $\preceq$ such that*

$$\phi(d(fx, gy)) \leq \psi(d(x,y)),$$

for all comparable $x, g(y) \in X$ where $\phi \in \Phi$ and $\psi \in \Psi$. Suppose either

(a) f (or g) is continuous or

(b) $(X, \preceq, d)$ satisfies the property that if $\{x_n\}$ is a decreasing sequence in X and $x_n \to x$ as $n \to \infty$ then $x \preceq x_n,\ \forall\ n$,

then f and g have a common fixed point. Moreover, if x^* and y^* are comparable whenever $x^*, y^* \in \mathcal{F}(f,g)$, then f and g have a unique common fixed point.

Proof. The proof is similar to that of Theorem 2.1. Here since the pair of mappings f, g are weakly decreasing we obtain a decreasing sequence $\{x_n\}$ instead of an increasing sequence. □

By taking X to be a totally ordered set, $f = g$ and $\phi \in \Phi$ the identity function on $[0, \infty)$, in Theorem 2.1 and Theorem 2.3 we get two new fixed point theorems. The following corollary is proposed by combining the two theorems.

Corollary 2.4. *Let $(X, \preceq, d)$ be a totally ordered complete metric space and suppose $f : X \to X$ satisfies*

$$d(f(x), f(y)) \leq \psi(d(x, y)), \ \ for\ each\ x, y \in X,$$

where $\psi : [0, \infty) \to [0, \infty)$ is upper semi continuous from the right and satisfies $\psi(t) < t$ for all $t > 0$. Then f has a unique fixed point provided either $f(x) \preceq f(f(x))$ or $f(x) \succeq f(f(x))$ for all $x \in X$.

Remark 2.5. *The above Corollary is the fixed point theorem of Boyd and Wong [Theorem 1.7] for weakly increasing (decreasing) mappings.*

Now we define a partial order and a metric on the product space by using the partial order and metric on the underlying space as follows:
Let $(X, \preceq, d)$ be a partially ordered metric space.
Define a partial order $\leq$ and a metric D on the set $X \times X$ as follows:
For all $(x, y), (u, v) \in X \times X$,

$$(x, y) \leq (u, v) \ \Leftrightarrow \ x \preceq u \text{ and } y \preceq v,$$

and

$$D((x, y), (u, v)) = d(x, u) + d(y, v)$$

.

Note 2.6. *It can be easily shown that*

1. *(X, d) is complete if and only if $(X \times X, D)$ is complete.*
2. *$(X, \preceq, d)$ has sequential monotone property if and only if $(X \times X, \leq, D)$ has sequential monotone property.*
3. *$F, G : X \times X \to X$ are weakly increasing (decreasing) with respect to the partial order $\preceq$ if and only if the mappings $T_F, T_G : X \times X \to X \times X$ defined by $T_F(x, y) = (F(x, y), F(y, x))$ and $T_G(x, y) = (G(x, y), G(y, x))$ are weakly increasing (decreasing) with respect to the partial order $\leq$.*
4. *The mappings F and G are continuous if and only if the mappings T_F and T_G are continuous.*
5. *The mappings F and G have common coupled fixed point if and only if the mappings T_F and T_G have common fixed points.*

Theorem 2.7. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, $F, G : X \times X \to X$ be the given mappings and the pair (F, G) be weakly increasing with two variables with respect to $\preceq$ such that*

$$\phi(d(F(x, y), G(u, v)) + d(F(y, x), G(v, u))) \leq \psi(d(x, u) + d(y, v)), \tag{2.9}$$

for all comparable $(x, y), (G(u, v), G(v, u)) \in X \times X$ where $\phi \in \Phi$ and $\psi \in \Psi$. If the following conditions hold:

(a) F (or G) is continuous or

(b) $(X, \preceq, d)$ satisfies the property that if $\{x_n\}$ is an increasing sequence in X and $x_n \to x$ as $n \to \infty$ then $x_n \preceq x, \ \forall\ n$,

then F and G have a common coupled fixed point. Moreover, if (x_1, y_1) and (x_2, y_2) are comparable whenever $(x_1, y_1), (x_2, y_2) \in \mathcal{CF}(F, G)$, then F and G have a unique common coupled fixed point.

Proof. By the hypotheses and using the properties in Note 2.6 we see that $(X \times X, \leq, D)$ is a partially ordered complete metric space and the mappings $T_F, T_G : X \times X \to X \times X$ defined by $T_F(x, y) = (F(x, y), F(y, x))$ and $T_G(x, y) = (G(x, y), G(y, x))$ are weakly increasing with respect to the partial order $\leq$.

By the definition of metric D, partial order $\leq$ and the functions T_F, T_G on $X \times X$ we can deduce contractive condition (2.9) as:

$$\phi[D(T_F(x,y), T_G(u,v))] \leq \psi(D((x,y),(u,v))),$$

for all comparable $(x,y), T_G(u,v) \in X \times X$ where $\phi \in \Phi$ and $\psi \in \Psi$.
Again since F (or G) is continuous, T_F (or T_G) is continuous and since $(X, \preceq, d)$ satisfies the property that if $\{x_n\}$ is an increasing sequence in X and $x_n \to x$ as $n \to \infty$ then $x_n \preceq x$, $\forall\ n$, $(X \times X, \leq, D)$ also satisfies the same property.
Now apply Theorem 2.1 for the space $(X \times X, \leq, D)$, and for the mappings T_F and T_G so that we get a unique common fixed point of the mappings T_F and T_G.
Again by the Note 2.6, we get a unique common coupled fixed point of F and G. $\square$

We illustrate the new theorem with the following example.

Example 2.8. *Let $X = [0,1]$ with usual metric.*
Define $F : X \times X \to X$ and $G : X \times X \to X$ as

$$F(x,y) = \frac{x+y}{2} \text{ and } G(x,y) = \begin{cases} \frac{x+y}{2}, & \text{if } (x,y) \neq (1,1), \\ 0, & \text{if } (x,y) = (1,1). \end{cases}$$

Define a partial order $\preceq$ on X as

$$\preceq\ : = \{(x,x)|\ x \in X\}.$$

Corresponding partial order $\sqsubseteq$ on $X \times X$ is:

$$\sqsubseteq\ : = \{((x,y),(x,y))|\ x,y \in X\}.$$

The pair of mappings (F,G) is weakly increasing and satisfy the contraction type condition (2.9) for all $\psi \in \Psi$ and $\phi \in \Phi$
Here $\{(x,x)|\ x \in X \setminus \{1\}\}$ is the set of all common coupled fixed point of F and G.

Theorem 2.9. *Let $(X, \preceq, d)$ be a partially ordered complete metric space, $F, G : X \times X \to X$ be given mappings and the pair (F,G) be weakly decreasing with two variables with respect to $\preceq$ such that*

$$\phi(d(F(x,y), G(u,v)) + d(F(y,x), G(v,u))) \leq \psi(d(x,u) + d(y,v)),$$

for all comparable $(x,y), (G(u,v), G(v,u)) \in X \times X$ where $\phi \in \Phi$ and $\psi \in \Psi$. If the following conditions hold:

(a) F (or G) is continuous or

(b) $(X, \preceq, d)$ satisfies the property that if $\{x_n\}$ is a decreasing sequence in X and $x_n \to x$ as $n \to \infty$ then $x \preceq x_n$, $\forall\ n$,

then F and G have a common coupled fixed point. Moreover, if (x_1, y_1) and (x_2, y_2) are comparable whenever $(x_1, y_1), (x_2, y_2) \in \mathcal{CF}(F,G)$, then F and G have a unique common coupled fixed point.

Proof: Using Theorem 2.3 and continuing as in Theorem 2.7 we get the result.

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