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Further Results on the Hop Domination Number of a Graph

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ABSTRACT: A hop dominating set S in a connected graph G is called a minimal hop dominating set if no proper subset of S is a hop dominating set of G. The upper hop domination number $\gamma_h^+(G)$ of G is the maximum cardinality of a minimal hop dominating set of G. Some general properties satisfied by this concept are studied. It is shown that for every two positive integers a and b where $2 \leq a \leq b$, there exists a connected graph G such that $\gamma_h(G) = a$ and $\gamma_h^+(G) = b$. It is proved that minimal hop dominating set is NP-complete. It is proved that $\gamma_h(G)$ and $\gamma(G)$ are in general incomparable. It is shown that for every pair of positive integers a and b with $a \geq 2$ and $b \geq 1$, there exists a connected graph G such that $\gamma_h(G) = a$ and $\gamma(G) = b$. Finally, we formulate an Integer linear programming problem to compute the hop domination number of G.

Key Words: Distance, hop domination, hop domination number, upper hop domination number, NP-complete.

Contents

1. Introduction

For notation and graph theory terminology we in general,follow [6]. A graph $G = (V, E)$ is a set V of vertices and a set E of edges. Each edge $e \in E$ is associated with two vertices u and v from V, and we write $e = (u, v)$. We say that u is adjacent to v or v is adjacent to u; e is incident with u or v; and u is a neighbor of v or v is a neighbor of u. Graphs are a common abstraction to represent data. Some examples include: road networks, where the vertices are cities and there is an edge between any two cities that share a highway; protein interaction networks, where the vertices are proteins and the edges represent interactions between proteins; and social networks, where the nodes are people and the edges represent friends. Let v be a vertex in V. Then the *open neighborhood* of v is the set $N(v) = \{u \in V/uv \in E\}$, and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$. The *degree* of a vertex v is $deg(v) = |N(v)|$. The minimum and maximum degrees of a graph G are denoted by $\delta(G)$ and $\triangle(G)$ respectively. If $e = uv$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then e is called a *pendant edge* or *end edge*, u is a *leaf* or *end vertex* and v is a *support vertex* of u. For $S \subseteq V$, $N(S) = \cup_{v \in S} N(v)$. The *subgraph* induced by a set S of vertices of a graph G is denoted by $\langle S \rangle$ with $V(\langle S \rangle) = S$ and $E(\langle S \rangle) = \{uv \in E : u, v \in S\}$. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u, v)$ is called an *u-v geodesic*. Let H be a subgraph of G and $v \in V$. Then the distance $d(v, H) = min\{d(v, u)/u \in V(H)\}$.

A set $D \subseteq V$ is a *dominating* set of G if for every $v \in V \setminus D$ is adjacent to some vertex in D. A dominating set D is said to be *minimal* if no subset of D is a dominating set of G. The minimum cardinality of a minimal dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. The domination number of a graph was studied in [1,8,11]. A set $S \subseteq V$ of a graph G is a *hop*

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dominating set of G if for every $v \in V \setminus S$, there exists $u \in S$ such that $d(u,v) = 2$. The minimum cardinality of a hop dominating set of G is called the *hop domination number* and is denoted by $\gamma_h(G)$. Any hop dominating set of order $\gamma_h(G)$ is called γ_h*-set* of G. The hop domination number of a graph was studied in [4,7,9,10,14,15,17]. The hop dominating set is defined in another way. The *open hop neighborhood* of a vertex v is the set $N_2(v) = \{u \in V/d(u,v) = 2\}$. A set $S \subseteq V$ is a hop dominating set of G if for every $v \in V \setminus S$ is hop neighbor of some vertex in S. Obviously $2 \leq \gamma_h(G) \leq n$, for any connected graph G of order n. Among graphs on n vertices only complete graph attained the upper bond, while the family of graphs that attain the lower bound is much richer (see Theorem 1.2 and Corollary 2.11 in [4]). Applications of domination in graphs are known in serveral areas such as wireless sensor networks [2], mobile ad hoc networks [3,13], ware house and station placement [12], viral marketing in social networks [5], etc. Another notion of graph notion is called hop domination by applying hop domination concepts there is a effectiveness in networks. The following theorems are used in sequel.

Theorem 1.1. [4] *For a connected graph G order $n \geq 2$, $\gamma_h(G) = n$ if and only if $G = K_n$.*

Theorem 1.2. [4] *Let G be a connected graph G order $n \geq 3$. Then $\gamma_h(G) = n-1$ if and only if $G = P_3$ or $G = K_n - \{e\}$.*

2. The Upper Hop Domination Number of a Graph

Definition 2.1. *A hop dominating set S in a connected graph G is called a minimal hop dominating set if no proper subset of S is a hop dominating set of G. The upper hop domination number $\gamma_h^+(G)$ of G is the maximum cardinality of a minimal hop dominating set of G.*

Example 2.2. *For the graph G given in Figure 2.1, $S_1 = \{v_2, v_3\}$, $S_2 = \{v_3, v_4\}$, $S_3 = \{v_1, v_2, v_5\}$ and $S_4 = \{v_1, v_4, v_5\}$ are the only four minimal hop dominating sets of G so that $\gamma_h(G) = 2$ and $\gamma_h^+(G) = 3$.*

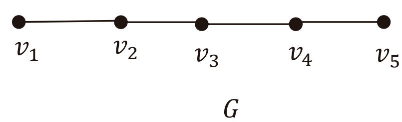


Figure 2.1

Observation 2.1. *(a) For the path $G = P_n$ $(n \geq 3)$,*

$$\gamma_h^+(G) = \begin{cases} 2 & \text{if } n = 3 \text{ or } 4 \\ 3 & \text{if } n = 5 \\ 2r+1 & \text{if } n = 6r \\ 2r+2 & \text{if } n = 6r+1 \text{ or } 6r+2 \\ 2r+3 & \text{if } n = 6r+3 \text{ or } 6r+4 \text{ or } 6r+5 \text{ where } r \geq 1 \end{cases}$$

(b)For the cycle $G = C_n$ $(n \geq 4)$,

$$\gamma_h^+(G) = \begin{cases} 2 & \text{if } n = 4 \text{ or } 5 \\ 2r & \text{if } n = 6r \\ 2r+1 & \text{if } n = 6r+1 \\ 2r+2 & \text{if } n = 6r+2 \text{ or } 6r+3 \text{ or } 6r+4 \\ 2r+3 & \text{if } n = 6r+5 \text{ where } r \geq 1 \end{cases}$$

Observation 2.2. *For a connected graph G, $2 \leq \gamma_h(G) \leq \gamma_h^+(G) \leq n$.*

The following theorems shows that the bounds in Observation 2.2 are strict and sharp. For this purpose, we define the following.

Definition 2.3. *Let $V(K_4 - \{e\}) = \{v_1, v_2, v_3, v_4\}$, where $e = v_1v_3$. Let H be the graph obtained from $K_4 - \{e\}$ by attaching an edge xv_4. Let $V(K_{1,a-6}) = \{y, u_1, u_2, u_3, ..., u_{a-6}\}$ for $a \geq 7$, where y is the central vertex of $V(K_{1,a-6})$ and $\{u_1, u_2, ..., u_{a-6}\}$ is the set of all end vertices of $K_{1,a-6}$. Let G_a be the graph obtained from H and $K_{1,a-6}$ $(a \geq 7)$ by joining x with each u_i $(1 \leq i \leq a-6)$. The graph G_a is given in Figure 2.2(a).*

Definition 2.4. *Let $V(K_{a-1}) = \{v_1, v_2, ..., v_{a-1}\}$, where $a \geq 3$ and $V(\overline{K}_2) = \{x, y\}$. Let H_a be the graph obtained from K_{a-1} and $\overline{K}_2$ by joining x and y with each v_i $(1 \leq i \leq a-1)$. The graph H_a is given in Figure 2.2(b).*

Definition 2.5. *Let $V(K_1) = \{x\}$, $V(K_2) = \{x, y\}$ and $V(K_{a-1}) = \{u_1, u_2, ..., u_{a-1}\}$, where $a \geq 3$. Let Q_a be the graph obtained from K_1, K_2 and K_{a-1} by joining x,y and z with each u_i $(1 \leq i \leq a-1)$. The graph Q_a is given in Figure 2.2(c).*

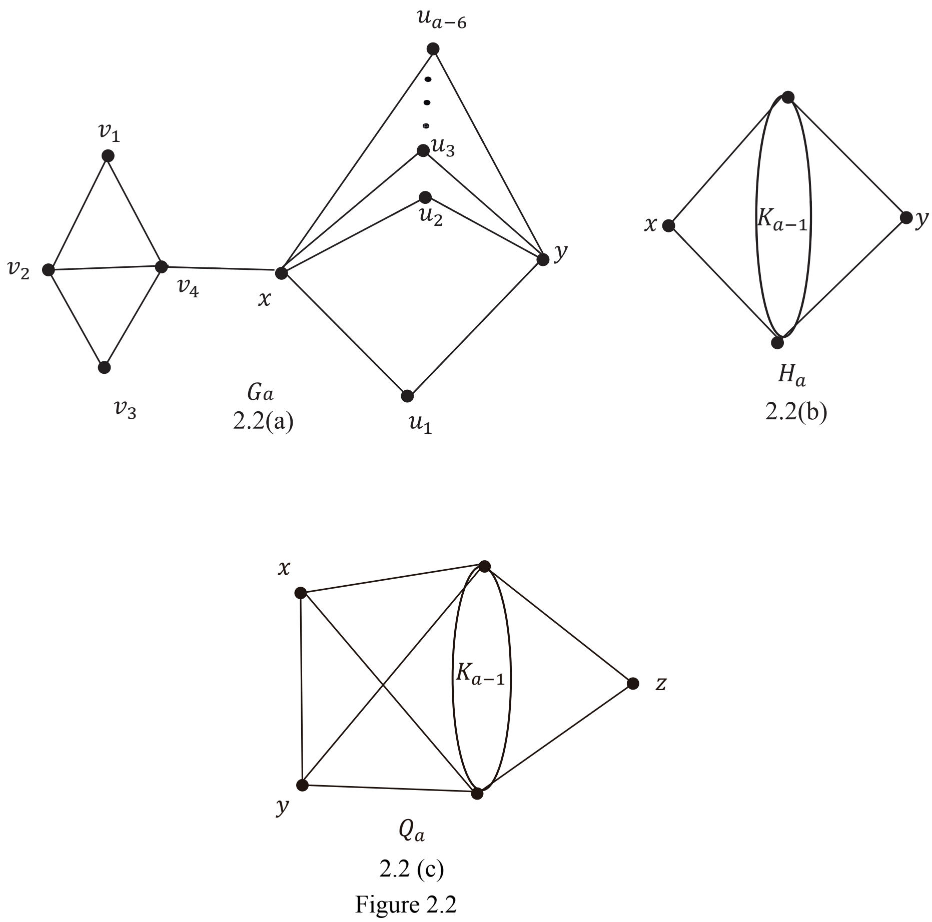


Figure 2.2

Theorem 2.6. *For the graph G_a $(a \geq 7)$, $\gamma_h(G_a) = 2$.*

Proof. Let $S = \{x, v_4\}$. Then $d(x, y) = 2$, $d(x, v_i) = 2$ for all i, $1 \leq i \leq 3$ and $d(v_4, u_i) = 2$ for all i, $1 \leq i \leq a-6$. Hence S is a hop dominating set of G_a so that $\gamma_h(G_a) = 2$. $\square$

Theorem 2.7. *For the graph H_a $(a \geq 3)$, $\gamma_h(H_a) = \gamma_h^+(H_a)$.*

Proof. Let $S = \{x\} \cup V(K_{a-1})$. Then S is a hop dominating set of H_a so that $\gamma_h(H_a) \leq a$. We prove that $\gamma_h(H_a) = a$. On the contrary, suppose that $\gamma_h(H_a) < a$. Then there exists a γ_h-set $S^{'}$ such that $|S^{'}| < a$. Also there exists a vertex $z \in H_a$ such that $z \notin S^{'}$. If $z = y$, then it follows that $x \in S^{'}$. Hence $V(H_a) = S'$ and so $\gamma_h(H_a) \geq a$, which is a contradiction. If $z = v_i$ for some i $(1 \leq i \leq a-1)$. Then $S^{'}$ is not a γ_h-set of H_a, which is a contradiction. Therefore $\gamma_h(H_a) = a$. Next we prove that $\gamma_h^+(H_a) = a$. Since $V(H_a) = a+1$, by the definition of the upper hop dominating set of G, $\gamma_h^+(H_a) = a$. □

Theorem 2.8. *For the complete graph $G = K_n$ $(n \geq 2)$, $\gamma_h^+(G) = n$.*

Proof. Since $d = 1$,we have $\gamma_h^+(G) = n$. □

Theorem 2.9. *For the graph $G = Q_a$ $(a \geq 3)$, $\gamma_h(G) = a$ and $\gamma_h^+(G) = a+1$.*

Proof. Let S be a hop dominating set of G. We prove that $V(K_{a-1}) \subset S$. On the contrary suppose that $V(K_{a-1}) \nsubseteq S$. Then there exists $u_i \in V(K_{a-1})$ such that $v_i \notin S$ for some i $(1 \leq i \leq a-1)$. Since $d(x, v_i) = d(y, v_i) = d(z, v_i) = 1$ for all i $(1 \leq i \leq a-1)$, S is not a hop dominating set of G. Therefore $V(K_{a-1}) \subset S$. Since $V(K_{a-1})$ is not hop dominating set of G, $\gamma_h(G) \geq a$. Let $S = \{z\} \cup V(K_{a-1})$. Then S is a hop dominating set of G so that $\gamma_h(G) = a$. Next we prove that $\gamma_h^+(G) = a+1$. Let $S^{'} = V(K_{a-1}) \cup \{x, y\}$. Then $S^{'}$ is a hop dominating set of G. We prove $S^{'}$ is a minimal hop dominating set of G. On the contrary, suppose that $S^{'}$ is not a minimal hop dominating set of G. Then there exists a hop dominating set $S^{''}$ of G such that $S^{''} \subset S^{'}$. Hence there exists $u \in S^{'}$ such that $u \notin S^{''}$. Then $u \notin V(K_{a-1})$. Therefore u is either x or y. If $u = x$, then $d(x, S^{''}) = 1$ and if $u = y$, then $d(y, S^{''}) = 1$. Hence it follows that $S^{''}$ is not a hop dominating set of G, which is a contradiction. Therefore $S^{'}$ is a minimal hop dominating set of G and so $\gamma_h^+(G) \geq a+1$. We prove that $\gamma_h^+(G) = a+1$. Since $V(G) = a+2$, by the definition of the upper hop dominating set of G, $\gamma_h^+(G) = a+1$. Since $a \geq 3$, $2 < \gamma_h(G) < \gamma_h^+(G) < n$. Hence it follows that $\gamma_h^+(G) \geq a-1$. □

Theorem 2.10. *For a connected graph G order $n \geq 2$, $\gamma_h(G) = n$ if and only if $\gamma_h^+(G) = n$.*

Proof. Let $\gamma_h^+(G) = n$. Then $S = V(G)$ is the unique minimal hop dominating set of G. Since no proper subset of S is a hop dominating set, it follows that S is the unique hop dominating set of G and so $\gamma_h(G) = n$. The converse follows from Observation 2.2. □

Corollary 2.11. *Let G be a connected graph of order $n \geq 2$. Then the following are equivalent.*
(1) $\gamma_h(G) = n$. (2)$\gamma_h^+(G) = n$. (3) $G = K_n$.

Proof. This follows from Theorems 1.1 and 2.10. □

Theorem 2.12. *If G is a a connected graph of order $n \geq 3$ with $\gamma_h(G) = n-1$, then $\gamma_h^+(G) = n-1$.*

Proof. Since $\gamma_h(G) = n-1$, it follows from Observation 2.2, we get $\gamma_h^+(G) = n$ or $n-1$. If $\gamma_h^+(G) = n$, then by Theorem 2.10, $\gamma_h(G) = n$, which is a contradiction. Therefore $\gamma_h^+(G) = n-1$. □

Theorem 2.13. *For any connected graph G of order $n \geq 3$, $\gamma_h(G) = n-1$ if and only if $\gamma_h^+(G) = n-1$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$. First assume that $\gamma_h^+(G) = n-1$. Let $S = \{v_1, v_2, ..., v_{n-1}\}$ be a minimal hop dominating set of G with maximum cardinality. We claim that v_n is adjacent to at most $n-2$ elements of S. On the contrary, suppose that v_n is adjacent to each elements of S. Then $d(v_n, x) = 1$, for every $x \in S$. Hence it follows that S is not a hop dominating set of G, which is a contradiction. Therefore v_n is adjacent to at most $n-2$ elements of S. We prove that S is a γ_h-set of G. If $G = K_n - \{e\}$, then the result is obvious. So we assume that $G \neq K_n - \{e\}$. On the contrary, suppose that S is not a γ_h-set of G. Since $G \neq K_n - \{e\}$, v_n is adjacent to at most $n-3$ elements of S. Then there exist $y, z \in S$ such that $yz \notin E(G)$. Hence it follows that $y \neq v_n$, $z \neq v_n$. Then $S_1 = S - \{y\}$ is a hop dominating set of G, which is a contradiction. Therefore S is a γ_h-set of G. Hence $\gamma_h(G) = n-1$. The converse follows from Theorem 2.12. □

Corollary 2.14. *Let G be a connected graph of order $n \geq 3$. Then the following are equivalent.*
(1) $\gamma_h(G) = n-1$. (2)$\gamma_h^+(G) = n-1$. (3) $G = K_n - \{e\}$ or P_3.

Proof. This follows from Theorems 1.2 and 2.13. □

Theorem 2.15. *Let G be a connected graph of order n and $u \in V(G)$. Then $\gamma_h^+(G-u) \leq \gamma_h^+(G)$.*

Proof. Let $u \in V(G)$ and S be a minimal hop dominating set of $G-u$ with maximum cardinality of G. Therefore $\gamma_h^+(G-u) = |S|$

Case 1: Let $uv \in E(G)$. We have the following two cases.

Case 1a: $v \in S$. Since S is a hop dominating set of $G-u$, there exists a vertex $w \in V(G-u) \setminus S$ such that $d_{G-u}(v,w) = 2$.

If $d_G(u,w) = 2$, then $S^{'} = S - \{v\} \cup \{u,w\}$ is a minimal hop dominating set of G so that $\gamma_h^+(G-u) \leq |S| \leq |S^{'}| \leq \gamma_h^+(G)$. Therefore $\gamma_h^+(G-u) \leq \gamma_h^+(G)$.

If $d_G(u,w) = 1$, then $S^{'} = S - \{v\} \cup \{u,w\}$ is a minimal hop dominating set of G so that $\gamma_h^+(G-u) \leq |S| \leq |S^{'}| \leq \gamma_h^+(G)$. Therefore $\gamma_h^+(G-u) \leq \gamma_h^+(G)$.

If $d_G(u,w) \geq 3$, then $S^{'} = S - \{v\} \cup \{u,w\}$ is a minimal hop dominating set of G so that $\gamma_h^+(G-u) \leq |S| \leq |S^{'}| \leq \gamma_h^+(G)$. Therefore $\gamma_h^+(G-u) \leq \gamma_h^+(G)$.

Case 1b: $v \notin S$. Then consider $S^{'} = S \cup \{u\}$. It is straight forward to verify that $S^{'}$ is a minimal hop dominating set of G so that $\gamma_h^+(G-u) \leq |S| \leq |S^{'}| \leq \gamma_h^+(G)$. Therefore $\gamma_h^+(G-u) \leq \gamma_h^+(G)$.

Case 2: Let $uv \notin E(G)$.

Case 2a: Let $v \in S$. Let x be an internal vertex in u-v path such that $x \notin S^{'}$. Let $d_G(u,x) = 2$. Then $S^{'} = S - \{v\} \cup \{x\}$ is a minimal hop dominating set of G so that $\gamma_h^+(G-u) \leq |S| \leq |S^{'}| \leq \gamma_h^+(G)$. Therefore $\gamma_h^+(G-u) \leq \gamma_h^+(G)$.

Case 2b: Let $v \notin S$. Let w be a vertex in u-v geodesic such that $d(w,v) = 2$. If $w \in S$, then $S^{'} = S$ is a minimal hop dominating set of G so that $\gamma_h^+(G-u) \leq |S| \leq |S^{'}| \leq \gamma_h^+(G)$. Therefore $\gamma_h^+(G-u) \leq \gamma_h^+(G)$. If $w \notin S$ then $S^{'} = S \cup \{w\}$ is a minimal hop dominating set of G so that $\gamma_h^+(G-u) \leq |S| \leq |S^{'}| \leq \gamma_h^+(G)$. Therefore $\gamma_h^+(G-u) \leq \gamma_h^+(G)$. □

In view of Observation 2.2, we have the following realization theorem.

Theorem 2.16. *For every two integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $\gamma_h(G) = a$ and $\gamma_h^+(G) = b$.*

Proof. Let $V(K_{a-1}) = \{v_1, v_2, ..., v_{a-1}\}$ and $V(K_{b-a+1}) = \{u_1, u_2, ..., u_{b-a+1}\}$. Let H be the graph obtained from K_{a-1} and K_{b-a+1} by joining each vertex of K_{a-1} with each vertex of K_{b-a+1}. Let G be the graph obtained from H by introducing the vertex x and joining x with each vertex of K_{a-1}. The graph G is shown in Figure 2.3.

First we prove that $\gamma_h(G) = a$. Let S be a hop dominating set of G. We prove that $V(K_{a-1}) \subset S$. On the contrary, suppose that there exists $v_i \in V(K_{a-1})$ such that $v_i \notin S$ for some i $(1 \leq i \leq a-1)$. Since $d(x,v_i) = d(v_i,u_j) = 1$ for all i $(1 \leq i \leq a-1)$ and j $(1 \leq j \leq b-a+1)$, S is not a hop dominating set of G, which is a contradiction. Therefore $V(K_{a-1}) \subset S$ and so $\gamma_h(G) \geq a-1$. Since $V(K_{a-1})$ is not a hop dominating set of G, $\gamma_h(G) \geq a$. Let $S = V(K_{a-1}) \cup \{x\}$. Then S is a hop dominating set of G so that $\gamma_h(G) = a$.

Next we prove that $\gamma_h^+(G) = b$. Let $S_1 = V(K_{a-1}) \cup V(K_{b-a+1})$. Then S_1 is a hop dominating set of G. We prove that S_1 is a minimal hop dominating set of G. On the contrary, suppose that S_1 is not a minimal hop dominating set of G. Then there exists a hop dominating set $S^{'}$ such that $S^{'} \subset S_1$. Hence it follows that $S^{'}$ contains no elements of $V(K_{a-1})$. Since $d(v_i,u_j) = 1$ for all i $(1 \leq i \leq a-1)$ and j $(1 \leq j \leq b-a+1)$, $S^{'}$ is not a hop dominating set of G, which is a contradiction. Therefore S_1 is a minimal hop dominating set of G and so $\gamma_h^+(G) \geq b-a+1+a+1 = b$. Since $|V(G)| = b+1$, it follows that $\gamma_h^+(G) = b$. □

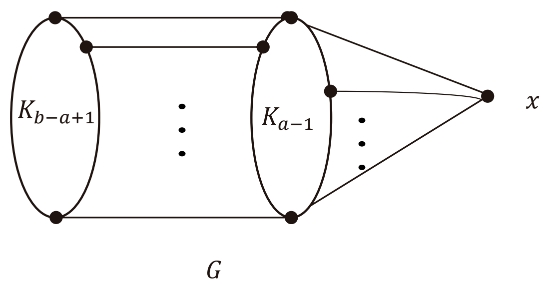


Figure 2.3

Remark 2.17. *The graph G in Figure 2.3 contains two minimal hop dominating sets viz. $V(K_{a-1}) \cup \{x\}$ and $T = V(K_{a-1}) \cup V(K_{b-a+1})$. Hence this example shows that there is no "Intermediate value Theorem" for minimal hop dominating sets, i.e, if k is an integer such that $\gamma_h(G) < k < \gamma_h^+(G)$, then there need not exist a minimal hop dominating set of cardinality k in G. Using the structure of the graph G constructed in the proof of Theorem 2.18, we can obtain a graph G_n of order $n \geq 5$ with $\gamma_h(G_n) = 3$ and $\gamma_h^+(G_n) = n-1$ for all $n \geq 5$. Thus we have the following theorem.*

Theorem 2.18. *There is an infinite sequence $\{G_n\}$ of connected graphs G_n of order $n \geq 5$ such that $\gamma_h(G_n) = 3$, $\gamma_h^+(G_n) = n-1$, $lim_{n\to\infty} \dfrac{\gamma_h(G_n)}{n} = 0$ and $lim_{n\to\infty} \dfrac{\gamma_h(G_n)}{n} = 1$.*

Proof. Let G_n be the graph obtained from the complete graph K_2 with vertex set $\{v_1, v_2\}$ and K_{n-3} with vertex set $\{u_1, u_2, ..., u_{n-3}\}$ by adding x with v_i $(1 \leq i \leq 2)$ and also joining v_i $(1 \leq i \leq 2)$ with each u_j $(1 \leq j \leq n-3)$. The graph G_n is shown in Figure 2.4. Let $S_1 = \{x, v_1, v_2\}$ and $S_2 = \{v_1, v_2, u_1, u_2, ..., u_{n-3}\}$. It is clear from the proof of Theorem 2.16, that G_n contains exactly 2 minimal hop dominating sets S_1 and S_2 so that $\gamma_h(G_n) = 3$ and $\gamma_h^+(G_n) = n-1$. Hence the result follows. □

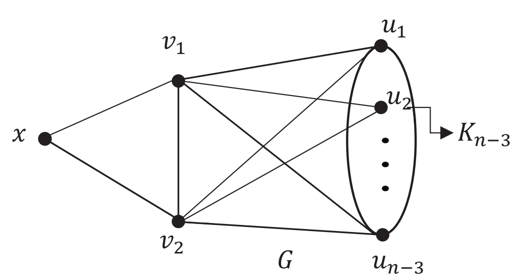


Figure 2.4

The following corollary gives the smallest order of a graph satisfying the hypothesis of Theorem 2.18 consequence of Theorems 2.13 and 2.16.

Corollary 2.19. *For every two positive integers a and b, where $2 \leq a < b$, the smallest order of a graph G with $\gamma_h(G) = a$ and $\gamma_h^+(G) = b$ is $b+1$.*

In [16], it is proved that hop independent dominating set (HIDS) is NP-complete for planar graphs by using vertex cover problem. In the following, we show that minimal hop dominating set (MHDS) is NP-complete by using independent set (IS) problem of G.

Theorem 2.20. *Minimal hop dominating set is NP-complete.*

Proof. $MHDS \in NP$. Given an instance (G, K) for MHDS, we guess the certificate, which consists of the K-vertices that will form the MHDS. Then we verify that these vertices form a MHDS, by checking that every vertex of G is either in this set or at a distance two in this set. In this case, we output "yes" and otherwise "no" (Again if G has MHDS of size K, one of these guesses will work and we accurately characterize in G as having a MHDS of size K. Otherwise all fail and we classify G as not having such a MHDS).

$IS \subseteq_p MHDS$. We claim that given an instance of the IS problem (G, K), we can produce an equivalent instance of the MHDS problem in polynomial time. We create a new vertex u_{uv} in $G^{'}$ an add $\{u, w_{uv}\}$ and $\{v, w_{uv}\}$ in $G^{'}$. Set an vertex set with cardinality $K^{'} = K + n_s$, where K is cardinality of a IS, say S and n_s is the nearest vertex of S. Output $(G^{'}, K^{'})$. This reduction outlined in Figure 2.5. Note that every step can be performed in polynomial time.

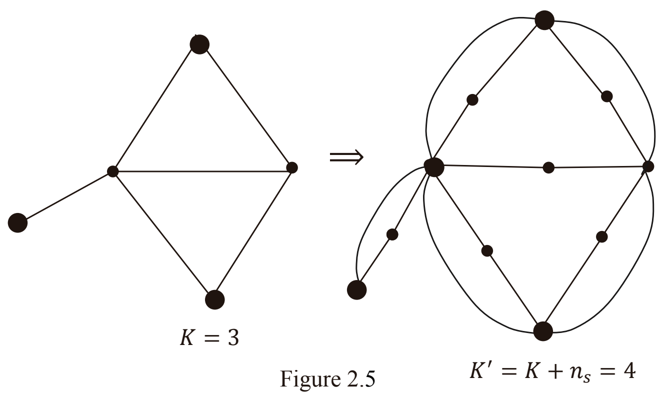


Figure 2.5

To establish the correctness of the reduction, we need to prove that G has a IS of size K if and only if $G^{'}$ has a MHDS of size $K^{'}$. First we prove that if G has a IS of size K, then $G^{'}$ has a MHDS of size $K^{'}$. Let S be a set of vertices of G and $S_1 = N(S)$. We claim that if S is a IS of G, then $S^{'} = S \cup S_1$ is a MHDS of $G^{'}$. Observate that $\mid S^{'} \mid = K + n_s = K^{'}$. To prove that $S^{'}$ is a MHDS. For each special vertex w_{uv} in $G^{'}$ corresponds to an edge uv in G implying that either u or v is in IS of $v^{'}$. Thus $d(w_{uv}, N(v)) = d(u, N(v)) = 2$. Then $S^{'}$ is a hop dominating set of $G^{'}$. Next to prove that $S^{'}$ is a minimal hop dominating set of $G^{'}$. On the contrary, suppose $S^{'}$ is not a minimal hop dominating set of $G^{'}$. Then there exists a hop dominating set S such that $S \subset S^{'}$. Then $d(u, v) = 1$, where $u \in S$ and $v \in V \setminus S$, $S^{'}$ is not a hop dominating set of $G^{'}$, which is a contradiction. Therefore $S^{'}$ is a MHDS of $G^{'}$. Conversely we prove that $S^{'}$ is a MHDS of $G^{'}$, then S is a IS of size K. On the contrary, suppose that S is not a IS of G of size K. Let $S^{''} = S^{'} \setminus S$ be the remaining K vertices. We might to claim something like $S^{''}$ is IS of G. Since $S^{''}$ have vertices that are not part of original graph. However we claim that we never need to use any of newly created vertices in $S^{''}$. In particular, if some vertex $w_{uv} \in S^{''}$ then modify $S^{''}$ by replacing w_{uv} with u. Then $d(v, x) = 1$ for every $x \in S^{'}$, which is a contradiction. Hence S is a IS of some K. □

3. On the hop domination number and the domination number of a graph

Most of the domination parameters are comparable. The following example shows that $\gamma_h(G)$ and $\gamma(G)$ are in general incomparable.

Example 3.1. *For $G = P_3$, $\gamma(G) = 1$ and $\gamma_h(G) = 2$. Thus $\gamma(G) < \gamma_h(G)$. Also for the graph G given in Figure 3.1. $S = \{v_1, v_2\}$ is a γ_h-set of G so that $\gamma_h(G) = 2$ and $D = \{v_2, v_4, v_6\}$ is a γ-set of G so that $\gamma(G) = 3$. Thus $\gamma_h(G) < \gamma(G)$.*

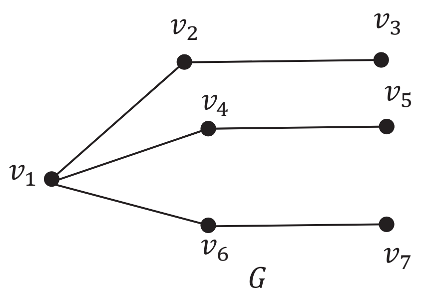
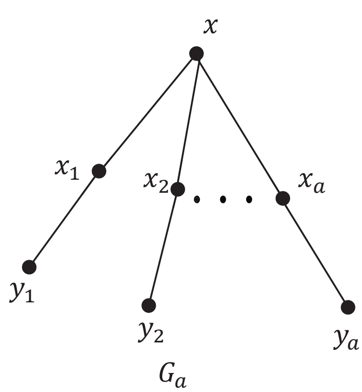
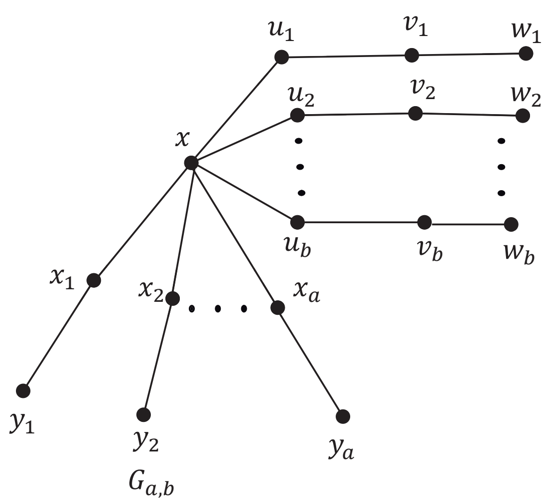


Figure 3.1

In view of Example 3.1, we have the following realization result. For this purpose, we present some graphs from which various graphs arrives in Theorem 3.3 are generated using identification. Let $H = K_{1,a}$. Let G_a be the graph obtained from H by subdividing each edge exactly once. Let $V(G_a) = \{x, x_1, x_2, ..., x_a, y_1, y_2, ..., y_a\}$, where x is the central vertex of G_a and $\{y_1, y_2, ..., y_a\}$ is the set of all end vertices of G_a. The graph G_a is shown in Figure 3.2(a).
Let $P_i : u_i, v_i, w_i$ $(1 \leq i \leq b)$ be a copy of path on three vertices. The graph $G_{a,b}$ is obtained from G_a and P_i $(1 \leq i \leq b)$ by introducing the edges xu_i $(1 \leq i \leq b)$. The graph Q_a is obtained from G_a by introducing the edges xy_i $(1 \leq i \leq a)$ and y_iy_j $(1 \leq i \leq j \leq a)$. The graph Q_a is given in Figure 3.2(c). Let $Q_{a,b}$ be the graph obtained from Q_a and P_i $(1 \leq i \leq b)$ by introducing the edges x_iu_j $(1 \leq i \leq a)$, $(1 \leq j \leq b)$ and xu_j $(1 \leq j \leq b)$. The graph $Q_{a,b}$ is given in Figure 3.2(d).



3.2 (a)



3.2 (b)

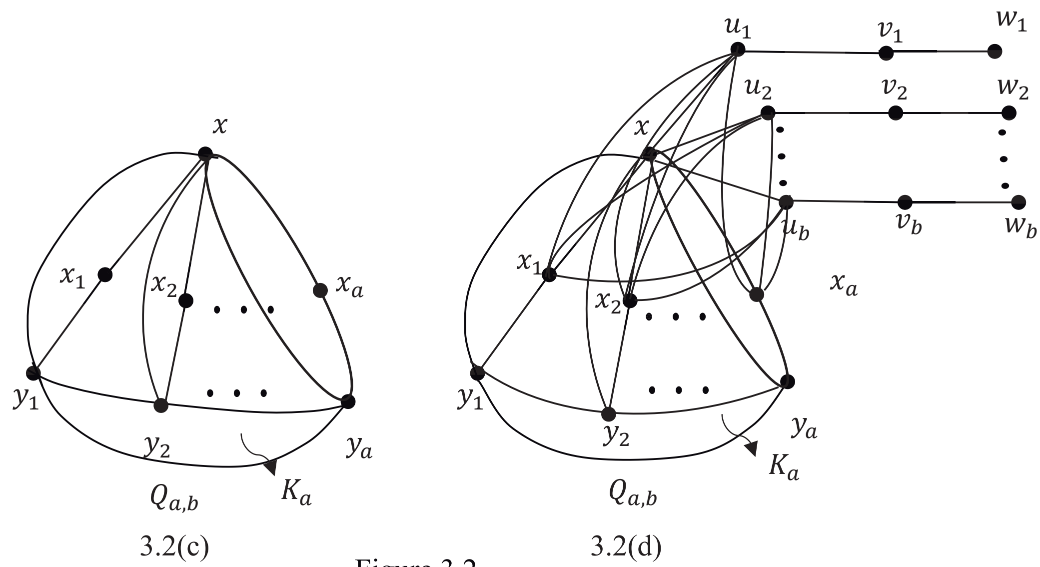


Figure 3.2

Observation 3.1. *Let G be a connected graph, x a cut vertex of G and C a component of $G - x$. If $d(x, C) \geq 2$, then every γ-set of G contains at least one element from C.*

Theorem 3.2. *For every pair of integers a and b with $a \geq 2$ and $b \geq 1$, there exists a connected graph G such that $\gamma_h(G) = a$ and $\gamma(G) = b$.*

Proof. **Case 1**: $a = b$.

Let $F_i : u_i, v_i, w_i, x_i, u_i$ $(1 \leq i \leq a)$ be a copy of C_4. Let G be the graph obtained from F_i $(1 \leq i \leq a)$ by identifying the vertex x_{i-1} of F_{i-1} and the vertex u_i of F_i $(2 \leq i \leq a)$. The graph G is shown in Figure 3.3.

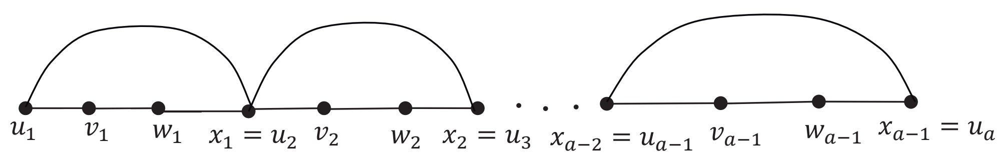


Figure 3.3

We have to prove that $\gamma_h(G) = a$. Let $S = \{u_1, u_2, u_3, ..., u_a\}$. Since $d(u_i, w_i) = 2$ for $1 \leq i \leq a-1$ and $d(u_{i+1}, v_i) = 2$ for $1 \leq i \leq a-1$, S is a subset of every hop dominating set of G and so $\gamma_h(G) \geq a$. Since for every $x \in V \setminus S$, there exists a $y \in S$ such that $d(x, y) = 2$, S is a hop dominating set of G so that $\gamma_h(G) = a$.

Next we have to prove that $\gamma(G) = a$. Since every element of $V \setminus S$ is dominated by at least one element of S, S is a dominating set of G and so $\gamma(G) \leq a$. Suppose that $\gamma(G) < a$. Then there exists a γ-set S' such that $|S'| < a$. Let x be a vertex of S such that $x \notin S'$. Then x is not dominated by any element of S', or v_i or w_i is not dominated by any element of S' for some i, $(1 \leq i \leq a)$. Hence S' is not a dominating set of G, which is a contradiction. Therefore $\gamma(G) = a$.

Case 2: $a < b$.

Consider $G = G_{b-a+1,a-1}$. First we prove that $\gamma_h(G) = a$. Since $d(u_i, w_i) = 2$ for $1 \leq i \leq a-1$ and $d(u_i, x_j) = 2$ for $1 \leq i \leq a-1$ and $1 \leq j \leq b-a+1$, $W = \{u_1, u_2, ..., u_{a-1}\}$ is a subset of every minimum hop dominating set of G and so $\gamma_h(G) \geq a-1$. Also since $xu_i \in E(G)$ $(1 \leq i \leq a-1)$, W is not a hop dominating set of G and so $\gamma_h(G) \geq a$. Let $S = W \cup \{x\}$. Then S is a hop dominating set of G so that $\gamma_h(G) = a$.

Next we prove that $\gamma(G) = b$. Since $d(x, v_i) = 2$ for $1 \leq i \leq a-1$ and $d(x, y_i) = 2$ for $1 \leq i \leq b-a+1$, by Observation 3.1, $\gamma(G) \geq a-1+b-a+1 = b$. Let $D = \{v_1, v_2, ..., v_{a-1}\} \cup \{x_1, x_2, ..., x_{b-a+1}\}$. Then D is a dominating set of G so that $\gamma(G) = b$.

Case 3: $b < a$.
For $b = 1$ and $a = 2$, let $G = P_3$. Then $\gamma(G) = 1 = b$ and $\gamma_h(G) = 2 = a$. For $b = 1$ and $a \geq 3$, let $G = K_{a+1} - \{e\}$. Since $\triangle(G) = n-1$, $\gamma(G) = 1 = b$. Also by Theorem 1.2, $\gamma_h(G) = a$. So, let $b \geq 2$ and $a \geq 3$. Consider $G = G_{a-b+1,b-1}$. First we prove that $\gamma_h(G) = a$. Since $d(u_i, w_i) = 2$ for $1 \leq i \leq b-1$, $X = \{u_1, u_2, ..., u_{b-1}\}$ is a subset of every minimum hop dominating set of G and so $\gamma_h(G) \geq b-1$. Also since $xu_i, xx_j, u_1x_j \in E(G)$ for $1 \leq i \leq b-1$ and $1 \leq j \leq a-b+1$, X is not a hop dominating set of G and so $\gamma_h(G) \geq a$. Let $S = X \cup \{x, x_1, x_2, ..., x_{a-b+1}\}$. Then S is a hop dominating set of G so that $\gamma_h(G) = a$.

Next we prove that $\gamma(G) = b$. Since $d(x, v_i) = 2$ for $1 \leq i \leq b-1$, by Observation 3.1, $\gamma(G) \geq b$. Let $D = \{x, v_1, v_2, ..., v_{b-1}\}$. Then D is a dominating set of G and so $\gamma(G) = b$. □

Theorem 3.3. *The difference between the hop domination number and the domination number is arbitrarily large.*

Proof. First we prove that $\gamma(G) - \gamma_h(G) = a$. Consider $G = G_{a+2}$. Let $S = \{x, x_1\}$. Then S is a hop dominating set of G so that $\gamma_h(G) = 2$. Since $d(x, y_i) = 2$ for $1 \leq i \leq a+2$, every minimum dominating set of G contains at least one element from each component of $G - x$ and so $\gamma(G) \geq a+2$. Let $D = \{x_1, x_2, ..., x_{a+2}\}$. Hence D is a dominating set of G so that $\gamma(G) = a+2$. Now $\gamma(G) - \gamma_h(G) = a+2-2 = a$.

Next we prove that $\gamma_h(G) - \gamma(G) = a$. Consider $G = Q_a$. Since x is a universal vertex of G, $\gamma(G) = 1$. We have to prove that $\gamma_h(G) = a+1$. Since $d(u_i, v_{i+1}) = 2$, $1 \leq i \leq a-1$, $W = \{u_1, u_2, ..., u_a\}$ is a subset of every minimum hop dominating set of G and so $\gamma_h(G) \geq a$. Also since $xv_i, xu_i \in E(G)$ $1 \leq i \leq a$, X is not a hop dominating set of G and so $\gamma_h(G) \geq a+1$. Let $S_1 = W \cup \{x\}$. Then S_1 is a hop dominating set of G so that $\gamma_h(G) = a+1$. Now $\gamma_h(G) - \gamma(G) = a+1-1 = a$. □

4. Integer Linear Programming for hop domination

In this section hop domination number of a connected graph G can be identified by converting the problem into an integer linear programming problem of G.
Let G be a connected graph with n vertices and m edges. Decision variables x_i indicates whether i belongs to a hop dominating set S.

$$x_i = \begin{cases} 1 & \text{if } v_i \in S \\ 0 & \text{if } v_i \notin S \end{cases} \tag{4.1}$$

The integer linear programming for hop dominating set problem can be formulated as
Minimize $\sum\limits_{i=1}^{n} x_i$ ... (4.2)
subject to: $x_i + \sum\limits_{j \in N_2(i)} x_j \geq 1$, $1 \leq i \leq n$ ...(4.3)
$x_i \in \{0, 1\}$, $1 \leq i \leq n$ ...(4.4)

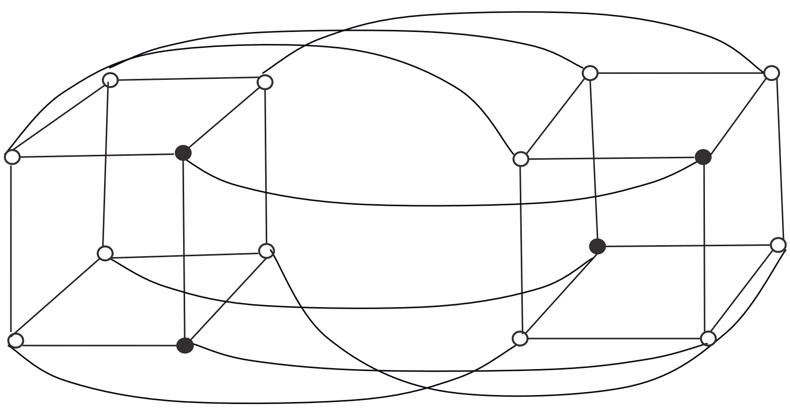
Theorem 4.1. *Set S is a hop dominating set of G if and only if (4.2) to (4.4) are satisfied.*

Proof. Let S be a hop dominating set of G and decision variables are characterized by (4.1). Constraints about binary variables x are trivially satisfied by (4.4). Since S is a hop dominating set, then $(\forall i \in V)$ $(\exists j \in N_2(i))$ $(i \in S \vee j \in S)$ imply that $(\forall i \in V)(\exists j \in N_2(i))$ $(x_i = 1$ or $x_j = 1)$, which means $(\forall i \in V)$

$x_i + \sum_{j \in N_2(i)} x_j \geq 1$. From the equation (4.3), it holds that $|S| = \sum_{i=1}^{n} x_i$. Since decision variables x represent a feasible solution has to be less or equal to $|S|$.
Conversely, let $S = \{i/x_i = 1\}$. Since variables x_i are binary, from equation (4.3), it holds that $((\forall i) x_i = 1 \vee \sum_{j \in N_2(i)} x_j \geq 1)$. Then, we get $(\forall i)$ $(x_i = 1 \vee (\exists j \in N_2(i)) x_j = 1)$. So $(\forall i)$ $(i \in S \vee (\exists j \in N_2(i))\ j \in S)$ and therefore S is a hop dominating set of G. □

5. Application

Consider a computer network modelled by a 4-cube.The vertices of the 4-cube represents computers and edges represent direct communication link between two computers. So, in this model we have 16 computers or processors and each processor can pass information to the processor to which it is directly connected. The problem is to collect information from all processors and we need to do it relatively often and relatively fast. So we identify a small set of processors called collecting processors and each processor send its information to one of a small set of collecting processors. We assume that a two–unit delay between the time a processor sends its information and the time it arrives at a nearest collector is allowed. In this case, we have to find a minimum hop dominating set of all processors. From Figure 6.1, the set of vertices marked in dark forms a minimum hop dominating set in the hypercube network.



G
Figure 5.1

6. Conclusion

It is proved that $\gamma_h(G)$ and $\gamma(G)$ are incomparable. Hence it can be investigated to find out under which condition the inequality $\gamma_h(G) \leq \gamma(G)$ or $\gamma(G) \leq \gamma_h(G)$ holds true.

References

1. H. Abdollahzadeh Ahangar, J. Amjadi, S. M. Sheikholeslami and M. Soroudi, *On The Total Roman Domination Number of Graphs*, Ars Combinatoria 150, 225-240, (2020).

2. A.D. Amis, R.Prakash, T.H.P. Vuong, D.T. Huynh, *Max-min d-cluster formation in wireless ad hoc networks*, In Proc. 19th IEEE INFOCOM 1, 32–41, (2000).

3. V.S. Anitha, M.P. Sebastian, *(k,r)-Dominating set-based, weighted and adaptive clustering algorithms for mobile ad hoc networks*, IET Communications 5, (13), 836-1853, (2011).

4. D.Anusha, J.John and S.Joseph Robin, *Graphs with small and large hop domination numbers*, Bulletin of IMVI 11(3), 483-489, (2021).

5. P. Basuchowdhuri, S. Majumder, *Finding influential nodes in social networks using minimum k-hop dominating set*, In Proc. 1st ICAA, 8321, 137–151, (2014).

6. F.Buckley and F.Harary, *Distance in Graph*, Addition-Wesly-wood city, CA (1990).

7. G. Chartrand, F. Harary and K. Hossain M.Schultz, *Exact 2-step domination in graphs*, Mathematica Bohemica 120, 125-134, (1995).

8. Doost Ali Mojdeh and Nader Jafari Rad, *On domination and its forcing in Mycielski's graphs*, Scientia Iranica 15,(2), 218-222, (2008).

9. Doost Ali Mojdeh and Azam Sadat Emadi, *Hop domination polynomial of graphs, Journal of Discrete Mathematical Sciences and Cryptography*, 23(1),1-16,(2019).

10. Gemma P. Salasalan and Sergio R. Canoy, *Jr.,Global hop domination numbers of graphs*, European Journal of Pure And Applied Mathematics 14(1),112-125, (2021).

11. T.W.Haynes, S.T.Hedetniemi and P.J.Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York, (1998).

12. C.L. Liu, *Introduction to Combinatorial Mathematics*, Computer Science Series, McGraw-Hill, (1968).

13. N. Meghanathan, *On the use of centrality measure to determine connected dominating sets for mobile ad hoc networks*, International Journal of Ad Hoc and Ubiquitous Computing 26, (4), 205–221,(2017).

14. Michael A.Henning and Nader Jafari Rad, *On 2-step and hop dominating sets in graphs*, Graphs and Combinatorics 33,(2),1-15, (2017).

15. Michael A. Henning, Saikat Pal and D. Pradhan, *Hop domination in chordal bipartite graphs*, Discussiones Mathematicae Graph Theory, DOI:10.7151/dmgt.2403.

16. Michael A.Henning, Saikat Pal and D.Pradhan, *Algorithm and hardness results on hop domination in graphs*, Information Processing Letters 153, 105872, (2020).

17. Nader Jafari Rad and Elahe Shabani, *On the complexity of some hop domination parameters*, Electronic Journal of Graph Theory and Applications 7(1), 77-89, (2019).

18. C. Natarajan and S.K.Ayyaswamy, *Hop domination sets in graphs-II*, Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica,(ASUOC) 23(2), 187-199,(2015).

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