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A Note on a Kirchhoff type Boundary Value Problem Involving Riemann-Liouville Fractional Derivative

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ABSTRACT: In this paper, we study some nonlinear Kirchhoff boundary value problems of fractional differential equations involving Riemann Liouville operator. Under appropriate assumptions on the functions in the given problem, we establish the existence of solutions using variational methods combined with the mountain pass theorem. Moreover, an illustrative example is presented to prove the validity of the main result.

Key Words: Fractional Riemann Liouville operator, variational methods, mountain pass theorem, boundary value problems.

Contents

1. Introduction

Fractional order models can be found to be more adequate than integer order models in some real world problems. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. Fractional calculus has attracted the attention of many researchers and has been successfully applied in various fields, such as physics, chemistry, aerodynamics, electro dynamics of complex medium. we refer the readers to [37,30] and references therein.
Recently, there has been a significant development in boundary value problems involving fractional derivatives. For example, one can see the monographs [22,29,30] and the papers [1,2,3,4,7,11,24,28,38] . Precisely, Some recent contributions on the existence of solutions for fractional boundary value problems have been made, we refer the interrest readers to [5,7,9,10,16,22,30,38]. Also, problemss including both left and right fractional derivatives are discussed. In this topic, many existence results are obtained by using different techniques, such as fixed point theory [19], critical points theory [24,25] and comparison method [5]. It should be noted that Variational medhods have also turned out to be very effective tools in determining the existence and multiplicity of solutions for fractional boundary value problems, like Nehari manifold combined with fibering maps [13,17,18,33,34] and Mountain pass theorem [15,35]. Precisely, Torres [35], studied the following problem

$$\begin{cases} {}_tD_T^{\alpha}\ ({}_0D_t^{\alpha}u(t)) = f(t,u(t)),\ t\in(0,T) \\ u(0)=u(T)=0, \end{cases} \tag{1.1}$$

where $T>0$ and $tD_T^{\alpha}, {}_0D_t^{\alpha}$ are the right and left Riemann–Liouville fractional derivatives, respectively. Under suitable assymptions on the function f and using mountain pass theorem, the author obtained the existence of at least one nontrivial solution for problem (1.1). After that, Chen and Liu [12], concidered the following Kirchoff type problem

$$\begin{cases} \left(a+b\int_0^T |{}_0D_t^{\alpha}u(t)|^2dt\right) {}_tD_T^{\alpha}\ ({}_0D_t^{\alpha}u(t)) + \lambda V(t)u(t) = f(t,u(t)),\ t\in(0,T) \\ u(0)=u(T)=0, \end{cases} \tag{1.2}$$

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where a, b, λ are positive numbers and f is a continuous function on $[0, T] \times \mathbb{R}$. By using the mountain pass theorem, the authors established some existence results for problem (1.2). Very recently, and using the Nehari manifold method, Chen and Liu [12] proved the multiplicity of solution for the following problem

$$\begin{cases} \left(a + b\int_0^T |{}_0D_t^\alpha u(t)|^p dt\right)^{p-1} {}_tD_T^\alpha \left(\Phi_p\left({}_0D_t^\alpha u(t)\right)\right) = f(t, u(t)),\ t \in (0, T) \\ u(0) = u(T) = 0, \end{cases}$$

where $\Phi_p(s) = |s|^{p-2}s$.
Motivated by the above mentioned papers, we want to contribute to the development of this new area on fractional differential equations theory. More precisely, in this paper, we study the following fractional boundary value problem

$$\begin{cases} M(u(t))\left({}_tD_T^\alpha \Phi_p\left({}_0D_t^\alpha u(t)\right) + V(t)\Phi_p\left(u(t)\right)\right) = f(t, u(t)) + \lambda g(t)|u(t)|^{q-2}u(t),\ t \in (0, T) \\ u(0) = u(T) = 0, \end{cases} \tag{1.3}$$

where λ is a positive parameter, $p, q > 1$, $\frac{1}{p} < \alpha \leq 1$, the functions V and g are continuous on $[0, T]$, $f \in C^1([0, T], \mathbb{R}, \mathbb{R})$ and $M(u(t))$ is defined by:

$$M(u(t)) = \left(a + b\int_0^T |{}_0D_t^\alpha u(t)|^p dt\right)^{p-1}, \quad a \geq 1.$$

Throughout this paper, we assume that $f \in C\left([0, T] \times \mathbb{R}, \mathbb{R}\right)$ is positively homogeneous of degree $r - 1$ that is $f(x, tu) = t^{r-1}f(x, u)$ hold for all $(x, u) \in [0, T] \times \mathbb{R}$. Also, we assume the following hypotheses:

($\mathbf{H}_1$) The functions g and V are nonnegative continuous on $[0, T]$, moreover

$$V(t) \geq \min_{s \in [0,T]} V(s) > 0, \ \ \forall\, t \in [0, T].$$

($\mathbf{H}_2$) $F : [0, T] \times \mathbb{R} \longrightarrow [0, \infty)$ is a C^1 function such that

$$F(x, tu) = t^r F(x, u)(t > 0) \ \text{ for all } \ x \in [0, T],\ u \in \mathbb{R}.$$

where $F(x, s) := \int_0^s f(x, t)dt$,

Note that, from **($\mathbf{H}_2$)**, f leads to the so-called Euler identity

$$uf(x, u) = rF(x, u)$$

and

$$|F(x, u)| \leq K|u|^r \quad \text{for some constant } K > 0. \tag{1.4}$$

Definition 1.1. *We say that $u \in E_0^{\alpha,p}$ is a solution for problem (1.3) if $\forall v \in E_0^{\alpha,p}$ we have :*

$$\begin{aligned} (a + b\|u\|_V)^{p-1} \int_0^T & |D_t^\alpha u(t)|^{p-2} D_t^\alpha u(t) D_t^\alpha v(t) + V(t)|u(t)|^{p-2}u(t)V(t)dt \\ &= \int_0^T f(t, u(t))V(t)dt + \lambda \int_0^T g(t)|u(t)|^{q-2}u(t)V(t)dt, \end{aligned}$$

where $E_0^{\alpha,p}$ and $\|u\|_V$, will be introduced later in Section 3.

Now, we can state our main result of this paper.

Theorem 1.2. *Assume that the hypotheses ($\mathbf{H}_1$)-($\mathbf{H}_2$) hold. If $\min(q, r) > p^2$, then, there exists λ_0 such that for all $\lambda \in (0, \lambda_0)$, problem (1.3) has a nontrivial solution.*

The rest of this paper is organized as follows. In Section 2, some preliminaries and usefull results on the fractional calculus are presented. In Section 3, we set up the variational framework of problem (1.3) and prove our main result (Theorem 1.2). In Section 4, we present an important example in order to illustrate the validity of our main result.

2. Preliminaries

In this section, we recall from [21,22,26], some necessary definitions and results which will be used throughout this paper.

Definition 2.1. *Let $\alpha > 0$ and u be a function defined a.e. on $(a, b) \subset \mathbb{R}$ with values in $\mathbb{R}$. The Left (resp. right) fractional integral in the sense of Riemann-Liouville with inferior limit a (resp. superior limit b) of order α of u is given by*

$$_aI_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s)ds, \;\; t \in (a, b]$$

respectively

$$_tI_b^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-s)^{\alpha-1} u(s)ds, \;\; t \in [a, b),$$

provided the right side is point-wise defined on $[a, b]$, where Γ denotes Euler's Gamma function. If $u \in L^1(a, 1b)$, then $_aI_t^\alpha u$ and $_tI_b^\alpha u$ are defined a.e. on (a, b)

Definition 2.2. *Let $0 < \alpha < 1$. Then, the Left (resp. right) fractional derivative in the sense of Riemann-Liouville with inferior limit a (resp. superior limit b) of order α of u is given by*

$$_aD_t^\alpha u(t) = \frac{d}{dt}\left({_aI_t^{1-\alpha}} u\right)(t), \; \forall\, t \in (a, b],$$

respectively

$$_tD_b^\alpha u(t) = \frac{d}{dt}\left({_tI_b^{1-\alpha}} u\right)(t), \; \forall\, t \in [a, b]),$$

provided that the right-hand side is point-wise defined.

Remark 2.3. *From [22], if u is an absolutely continuous function in $[a, b]$, then $_aD_t^\alpha u$ and $_tD_b^\alpha u$ are defined a.e. on (a, b) and satisfy*

$$_aD_t^\alpha u(t) = \;{_aI_t^{1-\alpha}} u'(t) + \frac{u(a)}{(t-a)^\alpha \Gamma(1-\alpha)} \tag{2.1}$$

and

$$_tD_b^\alpha u(t) = \; -{_tI_b^{1-\alpha}} u'(t) + \frac{u(b)}{(b-t)^\alpha \Gamma(1-\alpha)}. \tag{2.2}$$

Moreover, if $u(a) = u(b) = 0$, then $_aD_t^\alpha u(t) ={_a} \, I_t^{1-\alpha} u'(t)$ and $_tD_b^\alpha u(t) = -{_tI_b^{1-\alpha}} u'(t)$. Hence in this case we have the equality of Riemann-Liouville fractional derivative and Caputo derivative defined by

$$^c_aD_t^\alpha u(t) = \;{_aI_t^{1-\alpha}} u'(t)$$

and

$$^c_tD_b^\alpha u(t) = -\;{_tI_b^{1-\alpha}} u'(t).$$

Consequently, one gets

$$_aD_t^\alpha u(t) = \;{^c_aD_t^\alpha} u(t) + \frac{u(a)}{(t-a)^\alpha \Gamma(1-\alpha)},$$

and

$$_tD_b^\alpha u(t) = \;{^c_tD_b^\alpha} u(t) + \frac{u(b)}{(b-t)^\alpha \Gamma(1-\alpha)}.$$

Next, we provide some properties concerning the left fractional operators of Riemann-Liouville. One can easily derive the analgous version for the right one. For more details, we refer the reader to [26]. The first result yields the semi-group property of the left Riemann-Liouville fractional integral:

Proposition 2.4. *for any $\alpha,\ \beta > 0$ and any $u \in L^1(a,b)$, the following equality holds*

$$ {}_aI_t^{\alpha} \circ {}_aI_t^{\beta} u = {}_aI_t^{\alpha+\beta}. $$

From Property 2.4 and the equations (2.1) and (2.2), one can easily deduce the following results concerning the composition of the fractional integral and the fractional derivative. That is, for any $0 < \alpha < 1$, if $u \in L^1(a,b)$ then

$$ {}_aD_t^{\alpha} \circ {}_aI_t^{\alpha} u = u, $$

and if u is absolutely continous such that $u(a) = 0$, then one has

$$ {}_aI_t^{\alpha} \circ {}_aD_t^{\alpha} u = u. $$

Next, we give a classical result on the boundness of the left fractional integral from $L^p(a,b)$ to $L^p(a,b)$:

Proposition 2.5. *for any $\alpha > 0$ and any $p \geq 1$, ${}_aI_t^{\alpha}$ is linear and contineuous from $L^p(a,b)$ to $L^p(a,b)$. Moreover, for all $u \in L^p(a,b)$, we have*

$$ \|{}_aI_t^{\alpha} u\|_p \leq \frac{(b-a)^{\alpha}}{\Gamma(1+\alpha)} \|u\|_p. $$

Now, we will need the following formulae for integration by parts:

Proposition 2.6. *Let $0 < \alpha < 1$ and p, q are such that*

$$ p \geq 1,\ q \geq 1 \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \alpha \quad \text{or} \quad p \neq 1,\ q \neq 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1 + \alpha, $$

then, for all $u \in L^p(a,b)$ and all $v \in L^q(a,b)$, one has

$$ \int_a^b v(t)\ {}_aI_t^{\alpha} u(t) dt = \int_a^b u(t)\ {}_aI_t^{\alpha} v(t) dt, \tag{2.3} $$

and

$$ \int_a^b u(t)\ {}_a^cD_t^{\alpha} v(t) dt = v(t) {}_tI_b^{1-\alpha} u(t)\Big|_{t=a}^{t=b} + \int_a^b v(t)\ {}_aD_t^{\alpha} u(t) dt. \tag{2.4} $$

Moreover, if $v(a) = v(b) = 0$, then, one gets

$$ \int_a^b u(t)\ {}_aD_t^{\alpha} v(t) dt = \int_a^b v(t)\ {}_a^cD_t^{\alpha} u(t) dt. \tag{2.5} $$

Lemma 2.7. *Let $0 < \alpha \leq 1$, and $1 < p < \infty$. Then, for all $u \in E_0^{\alpha,p}$, one has*

$$ \|u\|_p \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|{}_0D_t^{\alpha} u\|_p. \tag{2.6} $$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{\widetilde{p}} = 1$, we have

$$ \|u\|_{\infty} \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)\left((\alpha-1)\widetilde{p}+1\right)^{\frac{1}{\widetilde{p}}}} \|{}_0D_t^{\alpha} u\|_p. \tag{2.7} $$

Lemma 2.8. *Let $0 < \alpha \leq 1$, and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence $\{u_n\} \rightharpoonup u$ weakly in $E_0^{\alpha,p}$. Then, $\{u_n\} \to u$ in $C([0,T])$, that is*

$$ \|u_n - u\|_{\infty} \to 0 \quad \text{as } n \to \infty. $$

In order to prove Theorem 1.1, we need the following result.

Definition 2.9. *Let X be a Banach space and $\varphi \in C^1(X,\mathbb{R})$, $c \in \mathbb{R}$. We say that φ satisfies the (PS) condition at level c if any sequence $u_n \subset X$, such that*

$$\varphi(u_n) \to c, \text{ and } \varphi'(u_n) \to 0, \text{ in } X^*, \text{ as } n \to \infty,$$

contains a convergent subsequence.

Now, we recall the mountain pass theorem due to Ambrosetti-Rabinowitz [6].

Theorem 2.10. (*Mountain pass theorem*). *Let X be a Banach space, $\varphi \in C^1(X,\mathbb{R})$ and $e \in X$ with $||e|| > r$ for some $r > 0$. Assume that*

$$\inf_{||u||=r} \varphi(u) > \varphi(0) \geq \varphi(e).$$

If φ satisfies the (PS) condition at level c, then, c is a critical value of φ, where

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)), \text{ and } \Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = e\}.$$

3. Proof of the main result

To prove the main result of this paper, we will use the mountain pass theorem. For this purpose, we introduce some basic notations and results.
The set of all functions $u \in C^\infty([0,T],\mathbb{R})$ with $u(0) = u(T) = 0$ is denoted by $C_0^\infty([0,T],\mathbb{R})$. We define the space $E_0^{\alpha,p}$ as the closure of $C_0^\infty([0,T],\mathbb{R})$ under the norm

$$\|u\|_{\alpha,p} = \left(\|u\|_p^p + \|{}_0D_t^\alpha u\|_p^p\right)^{\frac{1}{p}}. \tag{3.1}$$

According to (2.6), we can consider $E_0^{\alpha,p}$ with respect to the equivalent norm

$$\|u\| = \|{}_0D_t^\alpha u\|_p.$$

Remark 3.1. *Concerning the space $E_0^{\alpha,p}$, we have the following properties.*

(i) $E_0^{\alpha,p}$ is the space of functions $u \in L^p([0,T])$ having an α-order Riemann fractional derivative ${}_0D_t^\alpha u \in L^p([0,T])$ and $u(0) = u(T) = 0$.

(ii) The fractional space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

(iii) For $u \in$, we set

$$\|u\|_V = \left(\int_0^T |D_t^\alpha u(t)|^p dt + \int_0^T V(t)|u(t)|^p dt\right)^{\frac{1}{p}}.$$

Then, from Definition 1.1, we see that the norms $\|.\|_V$ and $\|u\|_{\alpha,p}$ are equivalent.

Associated to the problem (1.3), we define the functional $J_\lambda : E_0^{\alpha,p} \longrightarrow \mathbb{R}$ defined by :

$$J_\lambda(u) = \frac{1}{bp^2}\left(a + b\int_0^T |{}_0D_t^\alpha u(t)|^p + V(t)|u(t)|^p dt)\right)^p - \int_0^T F(t,u(t))dt - \frac{\lambda}{q}\int_0^T g(t)|u(t)|^q dt - \frac{a^p}{bp^2}.$$

It is not difficult note that J_λ is of class C^1, moreover, for any $\phi \in E_0^{\alpha,p}$, we have

$$\begin{aligned} < J_\lambda'(u), \varphi > &= M(u(t))\int_0^T \phi_p\, {}_0D_t^\alpha(u(t))_0D_t^\alpha\varphi(t) + V(t)\phi_p(u(t))\varphi(t)dt \\ &\quad - \int_0^T f(t,u(t))\varphi(t)dt - \lambda\int_0^T g(t)|u(t)|^{q-2}u(t)\varphi(t)dt. \end{aligned}$$

Hence, from the definition 1.1, one can see that critical points of J_λ are solutions for problem (1.3).

Definition 3.2. *We say that J_λ satisfy a palais-smale condition, if any palais-smale sequence has a convergent subsequence. That is, if $J_\lambda(u_n) \longrightarrow c$ and $J_\lambda(u_n) \longrightarrow 0$, then, $\{u_n\}$ has a convergent subsequence.*

Proposition 3.3. *Assume that the hypotheses* $(\mathbf{H}_1)$-$(\mathbf{H}_2)$ *hold. If $\min(q,r) > p^2$, the functional J_λ is of class C^1, moreover , J_λ satisfies palais-smale condition.*

Proof. It is easy to prove that J_λ is of class C^1. So, let as prove that J_λ satisfies the palais-smale condition. To this aim, let $\{u_n\}$ be a sequence in $E_0^{\alpha,p}$ such that

$$J_\lambda(u_n) \to c, \text{ and } J_\lambda^{'}(u_n) \to 0. \tag{3.2}$$

From (3.2), there exist two positive constants M_1 and M_2 , such that :

$$|J_\lambda(u_n)| \leqslant M_1 \text{ and } |< J_\lambda^{'}(u_n), u_n >| \leqslant M_2. \tag{3.3}$$

To finish the proof of Proposition 3.3, we must prove that u_n is bounded. If not, there exists a subsequence still denoted by $\{u_n\}$ such that

$$\|u_n\| \geqslant 1 \ \forall\, n \in \mathbb{N} \text{ and } \|u_n\| \longrightarrow \infty \text{ as } n \to \infty.$$

Put $\theta = \min(q,r)$. Then, from (3.3), we obtain

$$\begin{aligned}
\theta M_1 + M_2 &\geqslant \theta J_\lambda(u_n) - < J_\lambda^{'}(u_n), u_n > \\
&\geqslant (a + b\|u_n\|_V^p)^{p-1}\left[\frac{\theta a}{bp^2} + (\frac{\theta}{p^2} - 1)\|u_n\|_V^p\right] \\
&\quad + \int_0^T f(t,u_n)u_n - \theta F(t,u_n(t))dt + \lambda \int_0^T g(t)|u_n(t)|^q dt(1 - \frac{\theta}{q}) - \frac{\theta a^p}{bp^2} \\
&\geqslant a^{p-1}(\frac{\theta}{p^2} - 1)\|u\|_V^p + (r-\theta)\int_0^T F(t,u_n(t))dt + \lambda(1 - \frac{\theta}{q})\int_0^T g(t)|u_n(t)|^q dt \\
&\geqslant a^{p-1}(\frac{\theta}{p^2} - 1)\|u_n\|_V^p
\end{aligned}$$

since $\theta > p^2$, then, by letting n tends to infinity and using the fact that $\|u_n\| \longrightarrow \infty$ as $n \to \infty$., we obtain a contradiction. Hence, $\{u_n\}$ is bounded. Finally, since $E_0^{\alpha,p}$ is a separable Banach space, then, $\{u_n\}$ has a convergent subsequence. □

Lemma 3.4. *Under the hypotheses* $(\mathbf{H}_1)$-$(\mathbf{H}_2)$. *If $\min(q,r) > p$, then there exist $\rho > 0$, $\sigma > 0$ and $\lambda_0 > 0$, such that for all $\lambda \in (0, \lambda_0)$, we have*

$$J_\lambda(u) \geqslant \sigma \ \ \forall\, u \in E_0^{\alpha,p} : \|u\| = \rho,$$

for some positive constant λ_0.

Proof. Let $u \in E_0^{\alpha,p}$, then, from $(\mathbf{H}_1)$-$(\mathbf{H}_2)$, we have

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{bp^2}(a + b\|u\|_V^p)^p - \int_0^T F(t,u(t))dt - \frac{\lambda}{q}\int_0^T g(t)|u(t)|^q dt - \frac{a^p}{bp^2} \\
&\geqslant \frac{1}{bp^2}(a + b\|u\|_V^p)^p - K\int_0^T |u|^r dt - \frac{\lambda\|g\|_\infty}{q}\int_0^T |u(t)|^q dt - \frac{a^p}{bp^2} \\
&\geqslant \frac{1}{bp^2}(a + b\|u\|_V^p)^p - \frac{k}{r}\|u\|_{L^r}^r - \frac{\lambda\|g\|_\infty}{q}\|u\|_{L^q}^q - \frac{a^p}{bp^2} \\
&\geqslant \frac{1}{bp^2}\left(a^p + ba^{p-1}\|u\|_V^p\right) - \frac{kc_1}{r}\|u\|_V^r - \frac{\lambda c_2}{q}\|g\|_\infty\|u\|_V^q - \frac{a^p}{bp^2} \\
&\geqslant \frac{a^{p-1}}{p^2}\|u\|_V^p - \frac{kc_1}{r}\|u\|_V^r - \frac{\lambda c_2}{q}\|g\|_\infty\|u\|_V^q \\
&\geqslant \|u\|_V^p\left(\frac{a^{p-1}}{p^2} - \frac{kc_1}{r}\|u\|_V^{r-p} - \frac{\lambda c_2}{q}\|g\|_\infty\|u\|_V^{q-p}\right).
\end{aligned} \tag{3.4}$$

Fix $\rho \in (0,1)$, such that

$$\rho < \left(\frac{ra^{p-1}}{Kc_1p^2}\right)^{\frac{1}{r-p}}.$$

Put

$$\lambda_0 = \frac{q}{c_2\|g\|_\infty \rho^{q-p}}\left(\frac{a^{p-1}}{p^2} - \frac{kc_1}{r}\rho^{r-p}\right).$$

Finally, using (3.4), for all $\lambda \in (0,\lambda_0)$, if $\|u\| = \rho$, then we obtain

$$J_\lambda(u) \geq \rho^p\left(\frac{a^{p-1}}{p^2} - \frac{kc_1}{r}\rho^{r-p} - \frac{\lambda c_2}{q}\|g\|_\infty\rho^{q-p}\right) := \sigma > 0.$$

□

Lemma 3.5. *Under the hypothesis of Lemma 3.4, there exists $e \in E_0^{\alpha,p}$ with $\|e\| > \sigma$ and $J_\lambda(e) < 0$, where σ is given in Lemma 3.4.*

Proof. Let $e_0 \in C^\infty([0,T],\mathbb{R})$ with $e_0 \neq 0$. Then, for all $t > 0$, we have

$$\begin{aligned} J_\lambda(tu) &= \frac{1}{bp^2}\left(a + bt^p\|u\|_V^p\right)^p - \int_0^T F(s,tu(s))ds - \frac{\lambda}{q}\int_0^T g(s)|tu(s)|^q ds - \frac{a^p}{bp^2}r \\ &\leqslant \frac{1}{bp^2}\left(a + bt^p\|u\|_V^p\right)^p - t^r\int_0^T F(s,u(s))ds - \frac{\lambda}{q}t^q\int_0^T g(s)|u(s)|^q ds - \frac{a^p}{bp^2} \\ &\leqslant \frac{t^{p^2}}{bp^2}\left(\frac{a}{t^p} + bt^p\|u\|_V^p\right)^p - t^r\int_0^T F(s,u(s))ds - \frac{\lambda}{q}t^q\int_0^T g(s)|u(s)|^q ds - \frac{a^p}{bp^2}. \end{aligned}$$

Since the functions F, g are nonnegative and $\min(r,q) > p^2$. Then, from the above inequality, we see that $\lim\limits_{t\to\infty} J_\lambda(te_0) = -\infty$. So we can find $t > 0$ large enough such that if $e = te_0$, then, $\|e\| > \sigma$ and $J_\lambda(e) < 0$. □

Proof of Theorem1.2: It is clear that $E_0^{\alpha,p}$ is a Banch space. Moreover, from Proposition 3.3, $J_\lambda \in C^1(E_0^{\alpha,p},\mathbb{R})$, and J_λ satisfies the palais-smale condition. On the other hand, since $J_\lambda(0) = 0$, then by combining Lemma 3.4 with Lemma 3.5, there exist $\rho > 0$, $\sigma > 0$ and $e \in E_0^{\alpha,p}$, such that

$$\inf_{\|u\|=\rho} J_\lambda(u) \geq \sigma > 0 = J_\lambda(0) \geq J_\lambda(e). \tag{3.5}$$

Hence, by applying Theorem 2.10, we can deduce that J_λ has a critical point. Moreover, from Definition 1.1, we can see that this critical point is a solution for the problem (1.3). Finally, from (3.5), this solution is nontrivial. This completes the proof of Theorem 1.2.

4. Example

In order to illustrate the validity of the main result, we present the following example. Precisely, let $T > 0$ and $\mu > 0$, if we assume that $V(t) = \mu$, for all $t \in [0,T]$, then hypothesis $(\mathbf{H}_1)$ is satisfied.
Let $p > 1$ and $r > p^2$, and put $f(t,x) = h(t)|x|^{r-2}x$, where h is a continuous function on $[0,T]$. It is not difficult to see that $f \in C([0,T]\times\mathbb{R},\mathbb{R})$ and f is positively homogenuous of degree $r-1$. On the other hand, a simple calculation shows that

$$F(t,x) = \int_0^x f(t,s)ds = \frac{h(t)}{r}|x|^r,$$

which is of class C^1 and positively homogenuous of degree r. That is, hypothesis $(\mathbf{H}_2)$ is also satisfied. Therefore, if $q > p^2$ and g is a nonnegative continuous function on $[0, T]$. Then, Theorem 1.2 implies that the following problem

$$\begin{cases} \left(a + b\int_0^T |{}_0D_t^{\alpha}u(t)|^p dt\right)^{p-1} \left({}_tD_T^{\alpha}\Phi_p\left({}_0D_t^{\alpha}u(t)\right) + \mu\Phi_p\left(u(t)\right)\right) = h(t)|u(t)|^{r-2}u(t) + \lambda g(t)|u(t)|^{q-2}u(t), \ t \in (0, T) \\ u(0) = u(T) = 0, \end{cases}$$

has at least one nontrivial weak solution provided that $\lambda > 0$ is small enough.

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