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Calderon's Reproducing Formula for Bessel Wavelet Transforms

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ABSTRACT: In this paper the inversion Bessel wavelet transform is investigated, the Calderon reproducing formula of Bessel wavelet transform is obtained by generalizing the result of [7]. Some applications associated with Calderon's reproducing formula of Hankel convolution are given.

Key Words: Calderon's reproducing formula, Hankel convolution, Hankel transform, Bessel wavelet transform.

Contents

1. Introduction

Calderon's reproducing formula [7] played an important role to find the inversion formula of wavelet transform using Fourier convoluton transform. This formula can also be used to obtain several approximation results related to aforesaid transform.

Hankel convolution and Hankel transform are generalization of many integral transforms. Continuous Bessel wavelet transform is defined in [10] and found several properties using the result of [10]. Our main objective of this paper is to investigate the Calderon's reproducing formula associated with Hankel transform and Bessel wavelet transform.

Let γ be a positive real number. Set

$$d\sigma(x) = \frac{x^{2\gamma}}{2^{\gamma+\frac{1}{2}}\Gamma(\gamma+\frac{3}{2})}dx \tag{1.1}$$

and

$$\begin{aligned} j(x) &= C_\gamma x^{\frac{1}{2}-\gamma} J_{\gamma-\frac{1}{2}} \\ C_\gamma &= 2^{\gamma-\frac{1}{2}}\Gamma(\gamma+\frac{1}{2}) \end{aligned}$$

where $J_{\gamma-\frac{1}{2}}(x)$ denotes the Bessel function of order $\gamma-\frac{1}{2}$.

We define $L_{p,\sigma}(0,\infty), 1 \le p \le \infty$, as the space of those real measurable functions ϕ on $(0,\infty)$ for which

$$\begin{aligned} \|\phi\|_{p,\sigma} &= \left[\int_0^\infty |\phi(x)|^p d\sigma(x)\right]^{\frac{1}{p}} < \infty, 1 \le p \le \infty. \\ \|\phi\|_{\infty,\sigma} &= esssup_{0<x<\infty}|\phi(x)| < \infty. \end{aligned}$$

Theorem 1.1. *Let $\psi \in L_{1,\sigma}(0,\infty), \psi_a(t) = a^{-2\gamma-1}\psi\left(\frac{1}{a}\right)$, for $a > 0$ and $f \in L_{2,\sigma}(0,\infty)$ then*

$$f(x) = \int_0^\infty (\psi_a * \psi_a * f)\frac{d\sigma(a)}{a^{2\gamma}+1} \tag{1.2}$$

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and the above expression can be converted into the following

$$f(x) = \int_0^\infty \int_0^\infty \left(B_\psi f\right)(y,a)\psi\left(\frac{x-y}{a}\right)\frac{d\sigma(y)d\sigma(a)}{a^{2\gamma+1}} \tag{1.3}$$

where $\left(B_\psi f\right)(y,a) = \int_0^\infty \psi_{y,a}(t)f(t)d\sigma(t)$ *and* $\psi_{y,a}(t) = a^{-2\gamma-1}\psi\left(\frac{y}{a}\frac{x}{a}\right)$.

Now we restate the definition of Mellin transform and inverse Mellin transform and following theorem, which is given in [3].

Definition 1.2. *Let $f(t)$ be a function defined on the positive real axis $0 < t < \infty$. The Mellin transformation M is the operation mapping of the function f into the function F defined on the complex plane by the relation*

$$M\left[f : S\right] = F(S) = \int_0^\infty t^{s-1}f(t)dt \tag{1.4}$$

The function $F(S)$ is called the Mellin transform of $f(t)$. In general the integral exist only for complex values of $S = a+ib$ such that $a < a_1 < a_2$, where a_1 and a_2 depend on the function $f(t)$. This introduces the strip of definition of Mellin transform that will be denoted by $S\left(a_1, a_2\right)$.

Definition 1.3. *The inverse formula for Mellin transform is given by*

$$f(t) = \frac{1}{2\pi j}\int_{a-i\infty}^{a+i\infty} t^{-s}F(S)dS \tag{1.5}$$

where the integration is along a vertical line through $Re(S) = a$.

Theorem 1.4. *Let $(M_j)(s)$, which defined in 1.4 be regular in a strip, $\sigma_1 < \sigma < \sigma_2$, where $\sigma_1 < 0$, $\sigma_2 > 1$ except perhaps for a finite number of simple poles on the imaginary axis and let $(M_j)(S)$ be of the forms:*

$$\left(M_{j_0}\right)(s)\left\{\alpha\frac{\beta}{S} + 0\left(\frac{1}{|s|^2}\right)\right\}, \left(M_{j_0}\right)(s)\left\{\gamma\frac{\delta}{S} + 0\left(\frac{1}{|s|^2}\right)\right\}$$

for large positive and negaive $t, s = \sigma + it$ respectively, where $\left(M_{j_0}\right)(s) = \Gamma(s)cos\frac{1}{2}s\pi$ is the Mellin transform of $cosx$.

Let $\left(M_{j_0}\right)(s)$ satisfy the condition

$$\left(M_{j_0}\right)(s)\left(M_{j_0}\right)(-2\gamma - s) = 1$$

Let $x > 0$, and let $f(t)$ be in $L_{1,\infty}(0,\infty)$ and be of bounded variation near $t = x$.
Then,

$$\int_0^\infty j(xu)d\sigma(u)\int_0^\infty j(ut)f(t)d\sigma(t) = \frac{1}{2}\left[f(x+0) + f(x-0)\right]. \tag{1.6}$$

Equivalent relations are

$$F(x) = (Bf)(x) = \int_0^\infty j(xt)f(t)d\sigma(t) \tag{1.7}$$

$$f(t) = (B^{-1}f)(t) = \int_0^\infty j(xt)F(x)d\sigma(x) \tag{1.8}$$

where $j(x)$ is called kernel for Hankel transform, 1.7 is called Hankel transform of $f(t)$ and 1.8 is the corresponding inversion formula.

From [10], we define the basic function

$$D(x,y,z) = \int_0^\infty j(xt)j(yt)j(zt)d\sigma(t) \tag{1.9}$$

The above integral is convergent for all $x, y \in (0, \infty)$.
The inversion of 1.9 i given by

$$\int_0^\infty D(x,y,z)j(zt)d\sigma(z) = j(xt)j(yt); 0 < x, y < \infty; 0 \leq t < \infty \tag{1.10}$$

The inversion of $t = 0$ in 1.10, we obtain

$$\int_0^\infty D(x,y,z)d\sigma(z) = 1 \tag{1.11}$$

The Hankel transformation τ_y [10] of ψ is defined by

$$\tau_y\psi(x) = \psi(x,y) = \tau_x\psi(y) = \int_0^\infty \psi(z)D(x,y,z)d\sigma(z); 0 < x, y < \infty \tag{1.12}$$

Then the convolution [10] of ϕ and ψ is defined by

$$(\phi * \psi)(x) = \int_0^\infty \psi(x,y)\phi(y)d\sigma(y) \tag{1.13}$$

$$(\phi * \psi)(x) = \int_0^\infty \psi(z)D(x,y,z)\phi(y)d\sigma(y); 0 < x < \infty \tag{1.14}$$

Let $\phi, \psi \in L_{1,\sigma}(0,\infty)$ and let $(\phi * \psi)(x)$ be defined by the above equation. Then

$$B(\phi * \psi) = (B\phi)(B\psi) \tag{1.15}$$

From [10], Bessel wavelet is define as follows:
Let $\psi \in L_{p,\sigma}(0,\infty)$ be given, for $b \geq 0$ and $a > 0$, we have

$$\begin{aligned} \psi_{b,a}(x) &= a^{-2\gamma-1}\psi\left(\frac{b}{a}, \frac{x}{a}\right) \\ &= a^{-2\gamma-1}\int_0^\infty \psi(z)D\left(\frac{b}{a}, \frac{x}{a}, z\right)d\sigma(z). \end{aligned} \tag{1.16}$$

From [10], we define the Bessel wavelet transform as follows

$$\begin{aligned} B(b,a) &= (B_\psi\phi)(b,a) \\ &= \langle \phi(t), \psi_{b,a}(x)\rangle \\ &= \int_0^\infty \phi(x)\overline{\psi_{b,a}(x)}d\sigma(x) \\ &= a^{-2\gamma-1}\int_0^\infty\int_0^\infty \phi(x)\overline{\psi(z)}D\left(\frac{b}{a}, \frac{x}{a}, z\right)d\sigma(z)d\sigma(x) \end{aligned} \tag{1.17}$$

provide the integral is convergent.

Now we restate the lemma 2.3 from [10].

Let $\phi, \psi \in L_{1,\sigma}(0,\infty)$ and $(B_\psi\phi)(b,a)$ be the continuous Bessel wavelet transform. Then

$$(B_\psi\phi)(b,a) = \left(\phi * \overline{\psi_a}\right)(b) \tag{1.18}$$

2. Calderon's formula

In this section we obtain Calderon's reproducing indentity using the properties of Hankel transform and Hankel convolution.

Theorem 2.1. *If $f \in L^1(0,\infty) \cap L^2(0,\infty)$ then f can be reconstructed by the formula*

$$f(x) = \frac{1}{C_\psi} \int_0^\infty \int_0^\infty (B_\psi f)(y,a) \psi\left(\frac{y}{a}, \frac{x}{a}\right) \frac{d\sigma(y) d\sigma(a)}{a^{4\gamma+2}} \tag{2.1}$$

where $C_\psi = \int_0^\infty \omega^{-2\gamma-1} |\hat{\psi}(\omega)|^2 d\omega > 0$ and $(B_\psi f)(y,a)$ is Bessel wavelet transform of the function f with respect to Bessel wavelet ψ.

Proof. Let $g \in L^1(0,\infty) \cap L^2(0,\infty)$, then by the Parseval formula for the Bessel wavelet transform, we have

$$\begin{aligned}
C_\psi \langle f, g\rangle &= \int_0^\infty \int_0^\infty (B_\psi f)(y,a) \overline{(B_\psi g)(y,a)} a^{-2\gamma-1} d\sigma(a) d\sigma(y) \\
&= \int_0^\infty \int_0^\infty (B_\psi f)(y,a) \overline{\langle g(x), \psi_{y,a}(x)\rangle} a^{-2\gamma-1} d\sigma(a) d\sigma(y) \\
&= \int_0^\infty \int_0^\infty (B_\psi f)(y,a) \left[\int_0^\infty \overline{g(x)\overline{\psi_{y,a}}} d\sigma(x)\right] a^{-2\gamma-1} d\sigma(a) d\sigma(y) \\
&= \int_0^\infty \int_0^\infty (B_\psi f)(y,a) \left[\int_0^\infty \overline{g(x)} \psi_{y,a} d\sigma(x)\right] a^{-2\gamma-1} d\sigma(a) d\sigma(y) \\
&= \int_0^\infty \int_0^\infty (B_\psi f)(y,a) \left[\int_0^\infty \overline{g(x)} a^{-2\gamma-1} \left(\frac{y}{a}, \frac{x}{a}\right) d\sigma(x)\right] a^{-2\gamma-1} d\sigma(a) d\sigma(y) \\
&= \int_0^\infty \left(\int_0^\infty \int_0^\infty (B_\psi f)(y,a) \psi\left(\frac{y}{a}, \frac{x}{a}\right) \frac{d\sigma(a) d\sigma(y)}{a^{4\gamma+2}}\right) \overline{g(x)} d\sigma(x) \\
&= \left\langle \int_0^\infty \int_0^\infty (B_\psi f)(y,a) \psi\left(\frac{y}{a}, \frac{x}{a}\right) \frac{d\sigma(a) d\sigma(y)}{a^{4\gamma+2}}, g(x) \right\rangle \qquad (2.2)
\end{aligned}$$

Therefore

$$C_\psi f(x) = \int_0^\infty \int_0^\infty (B_\psi f)(y,a) \psi\left(\frac{y}{a}, \frac{x}{a}\right) \frac{d\sigma(a) d\sigma(y)}{a^{4\gamma+2}} \tag{2.3}$$

If we put $f = g$ in 2.3, then

$$C_\psi \|f\|^2 = \int_0^\infty \int_0^\infty |(B_\psi f)(y,a)|^2 \frac{d\sigma(a) d\sigma(y)}{a^{4\gamma+2}}$$

□

Lemma 2.2. *Let $\psi \in L^2_{2,\sigma}(0,\infty)$ be a basic Bessel wavelet which satisfies the following admissibility condition*

$$C_\psi = \int_0^\infty \omega^{-2\gamma-1} |\hat{\psi}(\omega)|^2 d\sigma(\omega) = 1 \tag{2.4}$$

then

$$\int_0^\infty \int_0^\infty (B_\psi f)(y,a) \psi\left(\frac{y}{a}, \frac{x}{a}\right) \frac{d\sigma(a) d\sigma(y)}{a^{4\gamma+2}} = \int_0^\infty \left(f * \overline{\psi_a} * \psi_a\right)(x) \frac{d\sigma(a)}{a^{2\gamma+1}} \tag{2.5}$$

for $f \in L^1(0,\infty) \cap L^2(0,\infty)$.

Proof. From 1.18, we have

$$\int_0^\infty \int_0^\infty (B_\psi f)(y,a) \psi\left(\frac{y}{a}, \frac{x}{a}\right) \frac{d\sigma(a) d\sigma(y)}{a^{4\gamma+2}} = \int_0^\infty \int_0^\infty \left(f * \overline{\psi_a}\right)(y) \psi\left(\frac{y}{a}, \frac{x}{a}\right) \frac{d\sigma(a) d\sigma(y)}{a^{4\gamma+2}} \tag{2.6}$$

Using the symmetry of $D(x,y,z)$ in 2.6, we get

$$\begin{aligned}
\int_0^\infty \int_0^\infty (B_\psi f)(y,a) \psi_{y,a}\left(\frac{y}{a}, \frac{x}{a}\right) \frac{d\sigma(a) d\sigma(y)}{a^{4\gamma+2}} &= \int_0^\infty \int_0^\infty \left(f * \overline{\psi_a}\right)(y) \psi_a(x,y) a^{2\gamma+1} \frac{d\sigma(a) d\sigma(y)}{a^{4\gamma+2}} \\
&= \int_0^\infty \int_0^\infty \left(f * \overline{\psi_a}\right)(y) \psi_a(x,y) a^{2\gamma+1} \frac{d\sigma(a) d\sigma(y)}{a^{2\gamma+1}}
\end{aligned}$$

for $\psi\left(\frac{y}{a},\frac{x}{a}\right)=\psi_a(x,y)$.

Hence from 1.14, we obtain

$$\int_0^\infty\int_0^\infty\left(B_\psi f\right)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\frac{d\sigma(a)d\sigma(y)}{a^{4\gamma+2}}=\int_0^\infty\left(f*\overline{\psi_a}*\psi_a\right)(x)\frac{d\sigma(a)}{a^{2\gamma+1}}$$

□

Theorem 2.3. *Let $\phi,\psi\in L_{1,\sigma}(0,\infty)$ and $(B\phi)(B\psi)\in L_{1,\sigma}$ be such that the following admissibility condition holds:*

$$\int_0^\infty(B\phi)(\omega)(B\psi)(\omega)\frac{d\sigma(\omega)}{w^{2\gamma+1}}=1 \tag{2.7}$$

Then the following Calderon's reproducing identit holds

$$f(x)=\int_0^\infty\left(f*\overline{\psi_a}*\psi_a\right)(x)\frac{d\sigma(a)}{a^{2\gamma+1}},forall f\in L_{1,\sigma}(0,\infty) \tag{2.8}$$

Proof. If we put $\phi=\psi$ in 2.2, then we can find the theorem 2.3. □

3. Application

In this section we give some application related to Bessel wavelet transform by using the theory of Hankel convolution and Mellin transform.

Theorem 3.1. *Let $\psi\in L_{2,\sigma}(0,\infty)$ be a basic Bessel wavelet and $\left(B_\psi f\right)(y,a)$ be continuous Bessel wavelet transform, then*

$$\int_0^\infty\int_0^\infty\left(B_\psi f\right)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\frac{d\sigma(a)d\sigma(y)}{a^{4\gamma+2}}=C_\psi\int_0^\infty j(xu)(Bf)(u)d\sigma(u). \tag{3.1}$$

Proof. From 1.16, we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\left(B_\psi f\right)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\frac{d\sigma(a)d\sigma(y)}{a^{4\gamma+2}}\\
=&\int_0^\infty\int_0^\infty\left(B_\psi f\right)(y,a)\left(\int_0^\infty\psi(z)D\left(\frac{y}{a},\frac{x}{a},z\right)d\sigma(z)\right)\frac{d\sigma(a)d\sigma(y)}{a^{4\gamma+2}}\\
=&\int_0^\infty\int_0^\infty\left(B_\psi f\right)(y,a)\left\{\psi(z)\left(\int_0^\infty j\left(\frac{x\omega}{a}\right)j\left(\frac{y\omega}{a}\right)j(z\omega)d\sigma(\omega)\right)d\sigma(z)\right\}\frac{d\sigma(a)d\sigma(y)}{a^{4\gamma+2}}\\
=&\int_0^\infty\int_0^\infty\left(B_\psi f\right)(y,a)\left\{\int_0^\infty j\left(\frac{x\omega}{a}\right)j\left(\frac{y\omega}{a}\right)\left(\int_0^\infty j(z\omega)\psi(z)d\sigma(z)\right)d\sigma(\omega)\right\}\frac{d\sigma(a)d\sigma(y)}{a^{4\gamma+2}}\\
=&\int_0^\infty\int_0^\infty\left(B_\psi f\right)(y,a)\left\{\int_0^\infty j\left(\frac{x\omega}{a}\right)j\left(\frac{y\omega}{a}\right)(B\psi)(\omega)d\sigma(\omega)\right\}\frac{d\sigma(a)d\sigma(y)}{a^{4\gamma+2}}\\
=&\int_0^\infty\int_0^\infty j\left(\frac{x\omega}{a}\right)(B\psi)(\omega)\left[B\left\{\left(B_\psi f\right)(y,a)\right\}\right]\left(\frac{\omega}{a}\right)\frac{d\sigma(a)d\sigma(y)}{a^{4\gamma+2}}\\
=&\int_0^\infty\int_0^\infty j\left(\frac{x\omega}{a}\right)(B\psi)(\omega)\left[B\left\{\left(B_\psi f\right)(y,a)\right\}\right]\left(\frac{\omega}{a}\right)\frac{\omega^{2\gamma}d(\omega)d\sigma(a)}{2^{\gamma+\frac{1}{2}}\Gamma\left(\gamma+\frac{3}{2}\right)a^{4\gamma+2}}
\end{aligned}$$

□

Putting $\left(\frac{\omega}{a}\right) = u$ in the above expression

$$\begin{aligned}
&\int_0^\infty \int_0^\infty (B_\psi f)(y,a)\psi\left(\frac{y}{a},\frac{x}{a}\right)\frac{d\sigma(a)d\sigma(y)}{a^{4\gamma+2}} \\
= &\int_0^\infty \int_0^\infty j(xu)(B\psi)(au)\left[B\left\{(B_\psi f)(y,a)\right\}\right](u)\frac{(au)^{2\gamma} adud\sigma(a)}{a^{4\gamma+2}2^{\gamma+\frac{1}{2}}\Gamma\left(\gamma+\frac{3}{2}\right)} \\
= &\int_0^\infty \int_0^\infty j(xu)(B\psi)(au)B\left[(B_\psi f)(y,a)\right](u)\frac{u^{2\gamma} dud\sigma(a)}{a^{2\gamma+1}2^{\gamma+\frac{1}{2}}\Gamma\left(\gamma+\frac{3}{2}\right)} \\
= &\int_0^\infty \int_0^\infty j(xu)\overline{(B\psi)(au)}(Bf)(u)(B\psi)(au)\frac{d\sigma(u)d\sigma(a)}{a^{2\gamma+1}} \\
= &\int_0^\infty j(xu)(Bf)(u)\left(\int_0^\infty \frac{|(B\psi)(au)|^2}{a^{2\gamma+1}}d\sigma(a)\right)d\sigma(u) \\
= &\ C_\psi \int_0^\infty j(xu)(Bf)(u)d\sigma(u).
\end{aligned}$$

Theorem 3.2. *Let $f \in L_{1,\sigma}(0,\infty)$ and $\psi \in L_{1,\sigma}(0,\infty)$. Then*

$$\begin{aligned}
\left(f * \overline{\psi_a}\right)(\omega) &= M^{-1}\left[\left(f * \overline{\psi_a}\right)(-2\gamma - s)(Mj)(s)\right](w) \\
&= \int_0^\infty j(\omega y)B\left(f * \overline{\psi_a}\right)(y)d\sigma(y).
\end{aligned}$$

where $(Mj)(s)(-2\gamma - s) = 1$.

Proof. The Bessel wavelet transform 1.17 can be expressed in the following form:

$$(B_\psi f)(y,a) = \left(f * \overline{\psi_a}\right)(y) = \int_0^\infty j(\omega y)B\left(f * \overline{\psi_a}\right)(\omega)d\sigma(\omega)$$

Therefore $\int_0^\infty y^{s-1}\left(f * \overline{\psi_a}\right)(y) = \int_0^\infty y^{s-1}\left(\int_0^\infty j(\omega y)B\left(f * \overline{\psi_a}\right)(\omega)d\sigma(\omega)\right)d\sigma(y)$
From 1.4, we have

$$\begin{aligned}
M[\left(f * \overline{\psi_a}\right)(y)](s) &= \int_0^\infty B\left(f * \overline{\psi_a}\right)(\omega)\left(\int_0^\infty j(\omega y)y^{s-1}d\sigma(y)\right) \\
&= \int_0^\infty B\left(f * \overline{\psi_a}\right)(\omega)\left(\int_0^\infty j(\omega y)y^{s-1}\frac{y^{2\gamma}}{2^{\gamma+\frac{1}{2}}\Gamma\left(\gamma+\frac{3}{2}\right)}dy\right)d\sigma(\omega)
\end{aligned}$$

Replacing $\omega y = u$, we get

$$\begin{aligned}
M[\left(f * \overline{\psi_a}\right)(y)] &= \int_0^\infty B\left(f * \overline{\psi_a}\right)(\omega)\left(\int_0^\infty j(u)\left(\frac{u}{\omega}\right)^{s-1}\left(\frac{u}{\omega}\right)^{2\gamma}\frac{1}{2^{\gamma+\frac{1}{2}}\Gamma\left(\gamma+\frac{3}{2}\right)}\frac{1}{\omega}du\right)d\sigma(\omega) \\
&= \int_0^\infty B\left(f * \overline{\psi_a}\right)(\omega)\left(\int_0^\infty j(u)\left(\frac{u}{\omega}\right)^{s-1}\frac{u^{2\gamma}}{2^{\gamma+\frac{1}{2}}\Gamma\left(\gamma+\frac{3}{2}\right)}du\right)\omega^{-2\gamma-1}d\sigma(\omega) \\
&= \int_0^\infty B\left(f * \overline{\psi_a}\right)(\omega)\left(\int_0^\infty j(u)u^{s-1}d\sigma(u)\right)\omega^{-2\gamma-s}d\sigma(\omega) \\
&= \int_0^\infty \omega^{-2\gamma-s}B\left(f * \overline{\psi_a}\right)(\omega)\left(\int_0^\infty j(u)u^{s-1}d\sigma(u)\right)d\sigma(\omega) \\
&= (Mj)(s)M\left[B\left(f * \overline{\psi_a}\right)(\omega)\right](-2\gamma - s)
\end{aligned}$$

Replacing s by $-2\gamma - s$, we get

$$M\left[\left(f * \overline{\psi_a}\right)(y)\right](-2\gamma - s) = (Mj)(-2\gamma - s)M\left[B\left(f * \overline{\psi_a}\right)(\omega)\right](s)$$

Hence

$$M[B\left(f*\overline{\psi_a}\right)(\omega)](s) = M\left[\left(f*\overline{\psi_a}\right)(y)\right](-2\gamma - s)(Mj)(s)$$

where $(Mj)(s) = \frac{1}{(Mj)(-2\gamma-s)}$.

Taking inverse Mellin transform in both sides of above expression and from [6], we get

$$[B\left(f*\overline{\psi_a}\right)](\omega) = \int_0^\infty \left(f*\overline{\psi_a}\right)(y)j(\omega y)d\sigma(y).$$

□

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