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Solution of Infinite System of Fourth Order Differential Equations in ℓ_p Space

Tanweer Jalal and Asif Hussain Jan

ABSTRACT: The main objective of this paper is to investigate the solution of infinite system of fourth order differential equations in ℓ_p space by using measures of noncompactness. The result is demonstrated with an example.

Key Words: Sequence spaces, measure of noncompactness, infinite system of differential equations, fixed point theory.

Contents

1. Introduction

Fixed point theory, differential equations, functional equations, integral and integro-differential equations all use measures of noncompactness (MNC). The most frequent approaches for examining the existence of solutions to functional equations (Cauchy initial value problems, systems of infinite linear equations, and so on) involve fixed point arguments. The Banach contraction principle was utilised by several authors to establish existence results for a variety of problems [1,12,21,23].

Kuratowski [13] was the first to introduce this concept in 1930. Darbo [8] developed a fixed point theorem which assures the presence of a fixed point for the so-called condensing operators in 1955, using Kuratowski's MNC notion. This is a generalized form of the classical Schauder fixed point theorem and Banach contraction principle. In recent years, the idea of measure of noncompactness has been successfully employed in sequence spaces (see [3,5,6,17,18,20]). Several authors investigated the solvability of infinite systems of second and third order differential equations in different sequence spaces [3,5,6,11,15,16,20,22,24,25,26].

2. Preliminaries and Background

Let Ω represent the space of all complex sequences $x = (x_i)_{i=1}^{\infty}$. A sequence space is a vector subspace of Ω. The set of natural, real, and positive real numbers are denoted by $\mathbb{N}, \mathbb{R}$ and $\mathbb{R}^+$ respectively.

The Kuratowski measure of noncompactness for a bounded subset P of a metric space X is defined as

$$\alpha(P) = \inf\left\{\delta > 0 : P \subset \cup_{i=1}^{n} P_i, \mathrm{diam}(P_i) \le \delta, \text{ for } 1 \le i \le m \le \infty\right\},$$

where $\mathrm{diam}(P_i)$ denotes diameter of the set P_i.

Another important measure of non-compactness is the Hausdorff non-compactness measure, which is defined as

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$$\chi(P) = \inf \left\{ \epsilon > 0 : P \text{ has a finite } \epsilon\text{-net in } X \right\}.$$

Let $(X, ||.||)$ be a Banach space, $\mathbb{R}^+ = [0, \infty)$, the symbols $\bar{X}$ and $\text{Conv}(X)$ denote closure of X and convex closure of X respectively. Let M_E denote the family of non-empty bounded subsets of E and N_E denote the family of non-empty and relatively compact subsets of E. We now define (MNC) axiomatically given by Banas and Goebel [4].

Definition 2.1. [4] Let X be a Banach space and E be the bounded subset of X. A function $\nu : M_X \to [0, +\infty)$ is said to be measure of non-compactnes in X if it satisfies the following axioms:

1. The family ker $\nu = \{A \in M_X : \nu(E) = 0\}$ is a nonempty and ker $\nu \subset N_X$.
2. $E_1 \subset E_2 \Rightarrow \nu(E_1) \leq \nu(E_2)$.
3. $\nu(\text{Conv}(E)) = \nu(E)$.
4. $\nu(\lambda E_1 + (1 - \lambda E_2) \leq \lambda\nu(E_1). + (1 - \lambda)\nu(E_2)$ for all $\lambda \in (0, 1)$.
5. If (E_m) is a sequence of closed sets from M_X such that $E_{n+1} \subset A_m$ and $\lim\limits_{m\to\infty} \nu(E_m) = 0$, then the intersection set $E_\infty = \bigcap\limits_{m=1}^{\infty} E_m$ is non-empty.

Theorem 2.1. [8] Let Ω be a nonempty, closed, bounded and convex subset of a Banach space X and let $T : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ with the property $\nu(T(\Omega)) \leq k\nu(\Omega)$. Then T has a fixed point in Ω.

Definition 2.2.(Equicontinuous) Let (G_1, d) and (G_2, d) be two metric spaces, and T the family of functions from G_1 to G_2. The family T is equicontinuous at point $l_0 \in G_1$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d(g(l), g(l_0)) < \epsilon$ for all $g \in T$ and all $l \in G_1$ such that $d(l, l_0) < \delta$. The family is pointwise equicontinuous if it is equicontinuous at every point of G_1.

Definition 2.3. [7]

1. A sequence space X with a linear topology is said to be a K-space if each of the maps $p_n : X \to C$ defined by $p_n(x) = x_n$ is continuous foreach $n \in N$.
2. A K-space is said to be an FK-space if X is a complete linear metric space , that is, X is an FK-space if X is Frechet space with continuous coordinates
3. A normed FK-space is called a BK-space, that is, a BK-space is a Banach sequence space with continuous coordinates.
4. A sequence $(b^{(k)})_{k=1}^{\infty}$ in a linear metric space X is called a Schauder basis for X if for every $x \in X$, there exists a unique sequence $(\lambda_m)_{m=1}^{\infty}$ of scalars such that $x = \sum_{m=1}^{\infty} \lambda_n b^{(m)}$.
5. An FK-space X is said to have AK if every sequence $x = x_m \in X$ has a unique representation $x = \sum_{k=1}^{\infty} e^{(k)}$, that is, $x = \lim\limits_{n\to\infty} x^{[m]}$. An FK-space with AK property is also called an AK-space.

Theorem 2.2. [7] Let X be a BK space with AK and monotone norm, $Q \in M_X$, $P_n : X \to X, (n \in N)$ be the operator (projection) defined by $P_n(x_1, X_2, ...) \in X$. Then

$$\chi(Q) = \lim_{n\to\infty} \left(\sup_{x \in Q} ||(I - P_n)(x)|| \right).$$

In a number of important nonlinear analytic disciplines, infinite systems of differential equations emerge naturally. For example numerical methods for solving partial differential equations, frequently lead to the investigation of infinite system of ordinary differential equations. In recent years, the idea of measure of noncompactness has been successfully employed in sequence spaces for investigating the solution of infinite system of second and third order differential equations (see [3,5,6,20,24,25]).

The purpose of this paper is to investigate the existence of the solution of the fourth order differential equation

$$D_i^{(4)}(w) = k_i(w, v(w));\ \ i = 1, 2, 3, ... \tag{2.1}$$

with boundary conditions, $v_i(0) = v_i^{'}(0) = v_i^{'}(W) = v_i^{'''}(W) = 0, w \in [0, W], D^{(4)}(w) = \dfrac{d^4 v_i}{dw^4}$ in the sequence space ℓ_p. The solution is investigated by using the infinite system of integral equations and the Green's function [9].

We denote ℓ_p for $p \geq 1$, the Banach sequence space with $||.||_p$ norm defined as:

$$||x||_p = ||(x_m)||_p = \left(\sum_{m=1}^{\infty} |x_m|^p\right)^{1/p}$$

for $x = (x_m) \in \ell_p$.

In view of the Theorem (2.2), in the Banach sequence space $(\ell_p, ||.||_{\ell_p})$, the Hausdorff measure of noncompactness χ can be formulated as

$$\chi(B) = \lim_{n\to\infty} \left\{ \sup_{u\in B} \left(\sum_{k\geq n} |e_k|^p \right)^{1/p} \right\} \tag{2.2}$$

where, $v(w) = (v_i(w))_{i=1}^{\infty} \in \ell_p$ for each $w \in [0, W]$.

In this study, we consider the following infinite system of fourth order differetial equations

$$D_i^{(4)}(w) = k_i(w, v(w));\ \ i = 1, 2, 3, ... \tag{2.3}$$

with boundary conditions, $v_i(0) = v_i^{'}(0) = v_i^{'}(W) = v_i^{'''}(W) = 0, w \in [0, W]$.

Let $C([0, W], \mathbb{R})$ be the space of all real valued continuous functions over $[0, W]$ and $C^4([0, W], \mathbb{R})$ be the set of all functions with the fourth continuous derivative on $[0, W]$. A function $v \in C^4([0, W], \mathbb{R})$ is a solution of (2.3) if and only if $v \in C([0, W], \mathbb{R})$ is a solution of infinite system of integral equation

$$v_i(w) = \int_0^W Y(w, s) k_i(s, v(s))\, ds, \text{ for } w \in [0, W] \tag{2.4}$$

where, $k_i(w, v) \in C([0, W], \mathbb{R})$, $i = 1, 2, 3, ...$ The Green's function associated with the system is given by

$$Y(w, s) = \begin{cases} \dfrac{-s^2(2Ws - 6Ww + 3w^2)}{12W}, & 0 \leq w \leq s \leq W \\[2ex] \dfrac{-w^2(3s^2 - 6Ws + 2Ww)}{12W}, & 0 \leq s < w \leq W. \end{cases} \tag{2.5}$$

The function satisfies the inequality

$$Y(w,s) \leq \frac{W^3}{12}. \tag{2.6}$$

From (2.4) and (2.5), we obtain

$$v_i(w) = \int_0^w \frac{-w^2(3s^2 - 6Ws + 2Ww)}{12W} k_i(s,v(s))\,ds + \int_w^W \frac{-s^2(2Ws - 6Ww + 3w^2)}{12W} k_i(s,v(s))\,ds.$$

Differentiating this expression three times, we get

$$\begin{aligned}\frac{d}{dw}\left(\frac{d^2 v_i}{dw^2}\right) &= \int_0^w \frac{-12W}{12W} k_i(s,v(s))\,ds + 0.\\ &= \int_0^w -k_i(s,v(s))\,ds.\end{aligned}$$

Further, differentiation gives

$$\begin{aligned}\frac{d}{dw}\left(\frac{d^3 v_i}{dw^3}\right) &= \frac{d}{dw}\int_0^w -k_i(s,v(s))\,ds\\ &= k_i(w,v(w)).\end{aligned}$$

Since equation (2.3) can be written as $\frac{d}{dw}\left(\frac{dv_i^3}{dw^3}\right) = k_i(w,v(w))$. So, $v_i(w)$ as given in equation (2.4) satisfies equation (2.3). Hence finding existence of solution for the system with boundary conditions is equivalent to find the existence of solution for the infinite system of integral equations.

Remark: let χ_X be the Hausdorff measure of noncompactness in the Banach space X, and A_o be an arbitrary subset of $C([0,W],X)$ the Banach space of continuous functions, which is equicontinuous on the interval $[0,W]$. Then, Hausdorff measure of noncompactness of A_o is given by [7,15]

$$\chi(A_o) = \sup\{\chi_X(A_o(w)) : w \in [0,W]\}.$$

3. Solution of the system (2.3) in ℓ_p space

The following assumptions are made in order to identify the condition under which the system (2.3) has a solution in ℓ_p:

(Q1) The functions k_i are defined on the set $[0,W] \times \mathbb{R}^\infty$ and take real values $(i = 1,2,3,...)$.

(Q2) The operator k defined on the space $[0,W] \times \ell_p$ as

$$(w,v) \to (kv) = (k_1(w,v), k_2(w,v), k_3(w,v), ...)$$

is such that the class of all functions $((kv)(w)), w\in [0,W]$ is equicontinuous at every point of the space ℓ_p.

(Q3) The following inequality holds:

$$|k_i(w, v_1, v_2, v_3, ...)|^p \leq g_i(w) + h_i(w)|v_i(w)|^p, \tag{3.1}$$

where $g_i(w)$ and $h_i(w)$ are real functions defined and continuous on $[0, W]$, such that $\sum\limits_{k=1}^{\infty} g_k(w)$ converges uniformly on $[0, W]$ and the sequence $(h_i(w))$ is equibounded on $[0, W]$.

To prove the result, we set

$$g(w) = \sum_{k=1}^{\infty} g_k(w),$$

$$G_o = \sup\{g(w) : w \in [0, W]\},$$

$$H_o = \sup\{h_n(w) : n \in N, w \in [0, W]\}.$$

Theorem 3.1. Under the hypotheses $(Q_1) - (Q_3)$, the infinite system of differential equations (2.3) has at least one solution $v(w) = (v_i(w))$, whenever $\dfrac{H_0^{1/p} W^4}{12} < 1$ such that $v(w) \in \ell_p$ space, $p \geq 1$, for all $w \in [0, W]$.

Proof: On the space $C([0, W], \ell_p)$ define the operator γ as:

$$(\gamma v)(w) = ((\gamma v)_n(w)) = \left(\int_0^W Y(w, s) k_n(s, v(s))\, ds \right) \tag{3.1}$$
$$= \left(\int_0^W Y(w, s) k_1(s, v(s))\, ds, \int_0^W Y(w, s) k_2(s, v(s))\, ds, \ldots \right).$$

We first show γ maps $C([0, W], \ell_p)$ into itself. Fixing $v(w) = (v_m(w)) \in C([0, W], \ell_p)$. Then from the relation (2.6), the hypothesis $(Q2)$ and Hölder's inequality, we have for an arbitrary $w \in [0, W]$

$$\begin{aligned}
(||(\gamma v)(w)||_p)^p &= \sum_{i=1}^{\infty} \left| \int_0^W Y(w, s) k_i(s, v(s))\, ds \right|^p \\
&\leq \sum_{i=1}^{\infty} \left| \int_0^W Y(w, s) k_i(s, v(s))\, ds \right|^{1/p} \left(\int_0^W ds \right)^{1/p} \\
&\leq \sum_{i=1}^{\infty} \int_0^W |Y(w, s)|^p |k_i(s, v(s))|^p\, ds \left(\int_0^W ds \right)^{p/q} \\
&\leq (W)^{p/q} \sum_{i=1}^{\infty} \int_0^W |Y(w, s)|^p [(g_i(s) + h_i(s)|v_j(s)|^p]\, ds \\
&\leq \left(\frac{W^3}{12} \right)^p (W)^{p/q} \sum_{i=1}^{\infty} \left[\int_0^W g_i(s)\, ds + \int_0^W h_i(s)|v_i(s)|^p\, ds \right] \\
&= \left(\frac{W^{\frac{3q+1}{q}}}{12} \right)^p \sum_{i=1}^{\infty} \left[\int_0^W g_i(s)\, ds + \int_0^W h_i(s)|v_i(s)|^p\, ds \right].
\end{aligned}$$

Now, by Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
||(\gamma v)(t)||_{d(\ell_1)} &\leq \left(\frac{W^{\frac{3q+1}{q}}}{12} \right)^p \left[\int_0^W g_(s)\, ds + H_0 \int_0^W \sum_{i=1}^{\infty} |v_i(s)|^p\, ds \right] \\
&\leq \left(\frac{W^{3+1/p+1/q}}{12} \right)^p \left[G_0 + H_0(||v||_p)^p \right].
\end{aligned}$$

Therefore

$$(||(\gamma v)(t)||_p) \leq \left(\frac{W^{3+1/p+1/q}}{12}\right)\left(G_0 + H_0(||v||_p)^p\right)^{1/p}. \tag{3.2}$$

Hence γv is bounded on the interval $[0, W]$. Thus γ transforms the space $C([0, W], \ell_p)$ into itself.

Now, using (3.2) and above procedure, we get

$$||v||_p \leq \frac{G_0^{1/p}\left[\frac{W^{3+1/p+1/q}}{12}\right]}{\left\{1 - H_0\left[\frac{W^{3+1/p+1/q}}{12}\right]\right\}^{1/p}} = r. \tag{3.3}$$

Where, the positive number r is the optimal solution of the inequality

$$\left(\frac{W^{3+1/p+1/q}}{12}\right)\left(G_0 + H_0 R^p\right)^{1/p} \leq R.$$

Hence, by (3.2) the operator γ transforms the ball $B_r \subset C([0, W], \ell_p)$ into itself.

We than show that on B_r, γ is continuous. Let $t \in [0, W]$ and $\epsilon > 0$ be arbitrarily fixed then, for any $u = u(t), v = v(t) \in B_r$ with $||u - v|| < \epsilon$, we have

$$\begin{aligned}
\left(||(\gamma u)(w) - (\gamma v)(w)||_{\ell_p}\right)^p &= \sum_{i=1}^{\infty}\left|\int_0^W Y(w,s)[k_i(s,u(s)) - k_i(s,v(s))]\,ds\right|^p \\
&\leq \sum_{i=1}^{\infty}\int_0^W |Y(w,s)||k_i(s,u(s)) - k_i(s,v(s))|\,ds\left(\int_0^W ds\right)^{p/q} \\
&\leq (W)^{p/q}\sum_{i=1}^{\infty}\int_0^W |Y(w,s)|^p|k_i(s,u(s)) - k_i(s,v(s))|^p\,ds.
\end{aligned}$$

Now, by using (2.6) and the assumption $(Q2)$ of equicontinuity, we get

$$\begin{aligned}
\left(||(\gamma v)(w) - (\gamma v)(w)||_{\ell_p}\right)^p &\leq \left(\frac{W^{\frac{3q+1}{q}}}{12}\right)^p \sum_{i=1}^{\infty}\int_0^W |k_i(s,u(s)) - k_i(s,v(s))|^p\,ds \\
&\leq \left(\frac{W^{\frac{3q+1}{q}}}{12}\right)^p \lim_{m\to\infty}\sum_{i=1}^{m}\int_0^W |k_i(s,u(s)) - k_i(s,v(s))|^p\,ds \\
&= \left(\frac{W^{\frac{3q+1}{q}}}{12}\right)^p \lim_{m\to\infty}\int_0^W \left(\sum_{i=1}^{m}|k_i(s,u(s)) - k_i(s,v(s))|^p\right) ds. \qquad (3.4)
\end{aligned}$$

Further, let us define the function $\delta(\epsilon)$ as:

$$\delta(\epsilon) = \sup\{|k_i(s,u(s)) - k_i(s,v(s))| : u, v \in \ell_p, ||u - v|| \leq \epsilon, w \in [0, W], i \in N\}.$$

Then clearly $\delta(\epsilon) \to 0$ as $\epsilon \to 0$, since the family $\{(kv)(w) : w \in [O, W]\}$ is equicontinuous at every point $v \in \ell_p$.

Therefore, by (3.4) and using Lebesgue dominant convergence theorem, we obtain

$$\begin{aligned}\Big(||(\gamma v)(w)-(\gamma v)(w)||_{\ell_p}\Big)^p &\leq \Big(\frac{W^{\frac{3q+1}{q}}}{12}\Big)^p\int_0^W \delta(\epsilon)\,ds\\ &= \Big(\frac{W^{3+1/p+1/q}}{12}\Big)^p(\delta(\epsilon))^p.\end{aligned}$$

This implies that the operator γ is continuous on the ball B_r.

Further since $Y(w,s)$ as defined in (2.5) is uniformly continuous on $[0,W]^4$, so by definition of operator γ, it is easy to show that $\{\gamma_u : u \in B_r\}$ is equicontinuous on $[0,W]$. Let $B_{r_1} = \mathrm{Conv}(\gamma B_r)$, then $B_{r_1} \subset B_r$ and the functions from the set B_{r_1} are equicontinuous on $[0,W]$.

Let $E \subset B_{r_1}$, then E is equicontinuous on $[0,W]$. If $v \in E$ is a function then for arbitrarily fixed $w \in [0,W]$, we have by assumption (Q_3)

$$\begin{aligned}\sum_{i=f}^{\infty}\Big|(\gamma v)_i(w)\Big|^p &= \sum_{i=k}^{\infty}\Big|\int_0^W Y(w,s)k_i(s,v(s))\,ds\Big|^p\\ &\leq \sum_{i=f}^{\infty}\Big(\int_0^W |Y(w,s)||k_i(s,v(s))|\,ds\Big)^p.\end{aligned}$$

Applying Hölder's inequality and (2.5), we get

$$\begin{aligned}\sum_{i=f}^{\infty}\Big|(\gamma v)_i(w)\Big|^p &\leq \sum_{i=f}^{\infty}\Big(\int_0^W |Y(w,s)|^p|k_i(s,v(s))|^p\,ds\Big)\Big(\int_0^W ds\Big)^{p/q}\\ &\leq \Big(\frac{W^{\frac{3q+1}{q}}}{12}\Big)^p\sum_{i=k}^{\infty}\Big(\int_0^W |k_i(s,v(s))|^p\,ds\Big).\end{aligned}$$

Again, using the Lebesgue dominant convergence theorem and the assumption (Q_2), we derive the following inequality

$$\begin{aligned}\sum_{i=f}^{\infty}\Big|(\gamma v)_i(w)\Big|^p &\leq \Big(\frac{W^{\frac{3q+1}{q}}}{12}\Big)^p\sum_{i=f}^{\infty}\Big(\int_0^W \big[g_i(s)+h_i(s)|v_i(s)|^p\big]\,ds\Big)\\ &= \Big(\frac{W^{\frac{3q+1}{q}}}{12}\Big)^p\Big(\int_0^W\Big(\sum_{i=f}^{\infty}g_i(s)ds\Big)+\int_0^W\Big(\sum_{i=f}^{\infty}h_i(s)|v_i(s)|^p\,ds\Big)\Big)\\ &\leq \Big(\frac{W^{\frac{3q+1}{q}}}{12}\Big)^p\Big(\int_0^W\Big(\sum_{i=f}^{\infty}g_i(s)ds\Big)+H_0\int_0^W\Big(\sum_{i=f}^{\infty}|v_i(s)|^p\,ds\Big)\Big).\end{aligned}$$

Taking supremum over all $v \in E$, we obtain

$$\sup_{v\in E}\sum_{i=f}^{\infty}\Big|(\gamma v)_i(w)\Big|^p \leq \Big(\frac{W^{\frac{3q+1}{q}}}{12}\Big)^p\Big(\int_0^W\Big(\sum_{i=f}^{\infty}g_i(s)ds\Big)+H_0\sup_{v\in E}\int_0^W\Big(\sum_{i=f}^{\infty}|v_i(s)|^p\,ds\Big)\Big).$$

As E is the set of equicontinuous functions on I, so using the definition of Hausdorff measure of noncompactness in ℓ_p space and by above Remark, we get by Hölder's inequality

$$\begin{aligned}\Big(\chi(\gamma E)\Big)^p &\leq H_0\Big(\frac{W^4}{12}\Big)^p\big(\chi(E)\big)^p.\\ \implies \Big(\chi(\gamma E)\Big) &\leq H_0^{1/p}\Big(\frac{W^4}{12}\Big)\big(\chi(E)\big).\end{aligned}$$

If $H_0^{1/p}\left(\frac{W^4}{12}\right) < 1$, that is $H_0^{1/p}W^4 < 12$.
Then by Theorem (2.1), the operator γ on the set B_r has a fixed point, which completes the proof of the theorem.

Now since the system of integral equations (2.4) is equivalent to the boundary value problem (2.3), we conclude that the infinite system of fourth order differential equations

$$D_i^{(4)}(w) = k_i(w, v(w));\ \ i = 1, 2, 3, ...$$

with boundary conditions, $v_i(0) = v_i^{'}(0) = v_i^{'}(W) = v_i^{'''}(W) = 0, w \in [0, W]$,

has atleast one solution $v(w) = (v_1(w), v_2(2), ...) \in \ell_p$ such that $v_i(w) \in C^4([0, W], \ell_p)$, $(i = 1, 2, 3, ...)$ for any $w \in [0, W]$, if the assumptions of the Theorem (3.1) are satisfied.

Note: We choose value of W in such a way that $H_0^{1/p}W^4 < 12$ is satisfied.

4. Application

In this section, we demonstrate our result with the help of the following example.

Example 3.2. Consider the following infinite system of fourth order differential equations in l_2

$$\frac{d^4 v_m}{dw^4} = \frac{\sqrt{w}e^{-u_m^2(w)}}{(2m+1)^2} + \sum_{l=m}^{\infty} \frac{cosw}{l^3m^2}\frac{v_l(w)[1-\sqrt{(l-m)}v_l(w)]}{\sqrt{(l-m+1)}}, m \in N, w \in [0, W],\ \text{for } m = 1, 2, ... \tag{4.1}$$

Solution. Compare (4.1) with (2.3) we have

$$k_m(w, v) = \frac{\sqrt{W}e^{-u_m^2(w)}}{(2m+1)^2} + \sum_{l=m}^{\infty} \frac{\cos w}{l^3m^2}\frac{v_l(w)[1-\sqrt{(l-m)}v_l(w)]}{\sqrt{(l-m+1)}}. \tag{4.2}$$

Assumption (Q_1) of the Theorem (3.1) is clearly satisfied. We now show that assumption (Q_2) of the Theorem (3.1) is also satisfied that is

$$|k_m(w, v)|^2 \leq g_n(w) + h_n(w)|v_n|^2. \tag{4.3}$$

Using Cauchy-Schwarz inequality and equation (4.1), we have

$$\begin{aligned}
|k_m(w, v)|^2 &= \left|\frac{\sqrt{W}e^{-u_m^2(w)}}{(2m+1)^2} + \sum_{l=m}^{\infty} \frac{\cos w}{l^3m^2}\frac{v_l(w)[1-\sqrt{(l-m)}v_l(w)]}{\sqrt{(l-m+1)}}\right|^2 \\
&\leq 2\left\{\frac{We^{-2u_m^2(w)}}{(2m+1)^4} + \left[\sum_{l=m}^{\infty} \frac{\cos w}{l^3m^2}\frac{v_l(w)[1-\sqrt{(l-m)}v_l(w)]}{\sqrt{(l-m+1)}}\right]^2\right\} \\
&\leq \frac{2We^{-2u_m^2(w)}}{(2m+1)^4} + 2\left(\sum_{l=m}^{\infty} \frac{\cos^2 w}{l^6m^4}\right)\sum_{l=m}^{\infty}\left[\frac{v_l(w)[1-\sqrt{(l-m)}v_l(w)]}{\sqrt{(l-m+1)}}\right]^2.
\end{aligned}$$

Now, using the fact that

$$\frac{a_o(1-a_ob_o)}{b_o} \leq \frac{1}{(2b_o)^2}, b_o \neq 0 \tag{4.4}$$

for any real a_o, b_o, we have

$$\begin{aligned}
|k_m(w,v)|^2 &\leq \frac{2We^{-2u_m^2(w)}}{(2m+1)^4} + 2\left(\frac{\cos^2 w}{m^4} \times \frac{\pi^6}{945}\right)\left[v_m^2 + \sum_{l=m}^{\infty} \frac{v_l(w)[1-\sqrt{(l-m)}v_l(w)]}{\sqrt{(l-m+1)}}\right]^2 \\
&\leq \frac{2We^{-2u_m^2(t)}}{(2m+1)^4} + \frac{2\pi^6 \cos^2 w}{945m^4} v_m^2 + \frac{2\pi^6 \cos^2 w}{945m^4} \times \sum_{l=m+1}^{\infty} \frac{1}{(2\sqrt{(l-m)})^2} \\
&\leq \frac{2We^{-2u_m^2(w)}}{(2m+1)^4} + \frac{\pi^8 \cos^2 w}{15120m^4} + \frac{2\pi^6 \cos^2 w}{945m^4} v_m^2.
\end{aligned}$$

Hence, by taking

$$g_n(w) = \frac{2We^{-2u_m^2(w)}}{(2n+1)^4} + \frac{\pi^8 \cos^2 w}{15120m^4}$$

and

$$h_n(w) = \frac{2\pi^6 \cos^2 w}{945m^4},$$

it is clear that $g_n(w)$ and $h_n(w)$ are real valued continuous functions on $[0, W]$. Also

$$\begin{aligned}
|g_n(w)| &= \frac{2W}{(2m+1)^4} + \frac{\pi^8}{15120m^4} \\
&\leq \left(2W + \frac{\pi^8}{15120m^4}\right)\frac{1}{m^4} \text{ for all } w \in [0, W].
\end{aligned}$$

Therefore by Weirstrass test for uniform convergence of the function series we see that $\sum\limits_{l\geq 1} g_l(w)$ is uniformly convergent on $[0, W]$. Further we have

$$|h_n(w)| = \frac{2\pi^6}{945m^4} \text{ for all } w \in [0, W].$$

Thus the function sequence $(h_i(w))$ is equibounded on $[0, W]$. Thus (4.2) is satisfied and hence the assumption (Q_3) is satisfied.

Also

$$G_o = \sup\left\{\sum_{k\geq 1} g_k(w) : w \in [0, W]\right\} = \left(2W + \frac{\pi^8}{15120m^4}\right) \times \frac{\pi^2}{90}$$

and

$$H_o = \sup\left\{h_i(w) : w \in [0, W]\right\} = \frac{2\pi^6}{945}.$$

The assumption (Q_2) is also satisfied as for fixed $w \in [0, W]$ and $(v_i(w)) = (v_1(w), v_2(w), ...) \in \ell_2$. We have

$$\begin{aligned}
\sum_{i=1}^{\infty} |k_i(w,v)| &= \sum_{i=1}^{\infty} g_i(w) + \sum_{i=1}^{\infty} h_i(w)|v_i(w)|^2 \\
&\leq G_o + H_o \sum_{i=1}^{\infty} |v_i(w)|^2.
\end{aligned}$$

Hence, the operator $k=(k_i)$ transforms the space $([0,W],\ell_2)$ into ℓ_2. Also for $\epsilon>0$ and $u=(u_i), v=(v_i)$ in ℓ_2 with $||u-v||_2<\epsilon$, we have

$$\begin{aligned}\Big(||(k_u)(w)-(k_v)(w)|||_2\Big)^2 &= \sum_{m=1}^{\infty}|k_m(w,u(w))-k_m(w,v(w))|^2\\ &= \sum_{m=1}^{\infty}\Bigg\{\Bigg|\sum_{l=m}^{\infty}\frac{\cos wu_l(w)[1-\sqrt{l-m}u_l(w)]}{l^3m^2\sqrt{l-m+1}}\\ &\qquad -\frac{\cos wv_l(w)[1-\sqrt{l-m}v_l(w)]}{l^3m^2\sqrt{l-m+1}}\Bigg|^2\Bigg\}\end{aligned}$$

$$\begin{aligned}&\le\sum_{m=1}^{\infty}\Bigg\{\Big(\frac{1}{m^4}\Big)\Bigg|\sum_{l=m}^{\infty}\frac{\cos wu_l(w)[1-\sqrt{l-m}u_l(w)]-\cos wv_l(w)[1-\sqrt{l-m}v_l(w)]}{l^3\sqrt{l-m+1}}\Bigg|^2\Bigg\}\\ &\le\sum_{m=1}^{\infty}\Bigg\{\Big(\frac{1}{m^4}\Big)\Bigg|\sum_{l=m}^{\infty}\frac{(u_l(w)-v_l(w))[1-(l-m)(u_l(w)+v_l(w))]}{l^3\sqrt{l-m+1}}\Bigg|^2\Bigg\}.\end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}&\Big(||(k_u)(w)-(k_v)(w)||_2\Big)^2\\ &\quad\le\sum_{m=1}^{\infty}\Bigg\{\frac{1}{m^4}\Big(\sum_{l=m}^{\infty}\frac{1}{l^6}\Big)\sum_{l=m}^{\infty}\Bigg|\frac{(u_l(w)-v_l(w))[1-(l-m)(u_l(w)+v_l(w))]}{\sqrt{l-m+1}}\Bigg|^2\Bigg\}\\ &\quad\le\frac{\pi^6}{945}\sum_{m=1}^{\infty}\Bigg\{\frac{1}{m^4}\sum_{l=m}^{\infty}|(u_l(w)-v_l(w))|^2\Bigg|\frac{[1-(l-m)(u_l(w)+v_l(w))]}{\sqrt{l-m+1}}\Bigg|^2\Bigg\}\\ &\le\frac{\pi^6}{945}\sum_{m=1}^{\infty}\Bigg\{\frac{1}{m^4}\Bigg[|u_m(w)-v_m(w)|^2+\sum_{l=m}^{\infty}|(u_l(w)-v_l(w))|^2\Bigg|\frac{[1-(l-m)(u_l(w)+v_l(w))]}{\sqrt{l-m+1}}\Bigg|^2\Bigg]\Bigg\}.\end{aligned}$$

Using (4.4), we get

$$\begin{aligned}\Big(||(k_u)(w)-(k_v)(w)||_2\Big)^2 &\le\frac{\pi^6}{945}\sum_{m=1}^{\infty}\Bigg\{\frac{1}{m^4}\Bigg[|u_m(w)-v_m(w)|^2+\frac{\pi^2}{48}\Bigg]\Bigg\}\\ &\le\Big(\frac{\pi^8}{48}\Big)\epsilon^2.\end{aligned}$$

Thus, for any $w\in[0,W]$, we have

$$||(k_u)(w)-(k_v)(w)||_2\le\frac{\pi^4\epsilon}{48}.$$

Therefore the family $\{(k_v)(w): w\in[0,W]\}$ is equicontinuous.

Finally, it is seen that the condition $H_0^{1/p}W^4<12$ is satisfied for all $W\le 1.7030725$.

So, by Theorem (3.1), there exists at least one solution to given infinite system of differential equation (4.1) in $C([0,W],\ell_2)$.

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Tanweer Jalal,
Department of Mathematics,
National Institute of Technology Srinagar,
India.
E-mail address: tjalal@nitsri.net

and

Asif Hussain Jan ,
Department of Mathematics,
National Institute of Technology Srinagar,
India.
E-mail address: asif_06phd20@nitsri.net