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Nonlocal Boundary Value Problems for Functional Hybrid Differential Equations Involving Generalized $\omega-$Caputo Fractional Operator

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ABSTRACT: In this manuscript , we establish the existence and uniqueness of solutions for boundary value problems of nonlinear hybrid fractional differential equations involving generalized $\omega-$Caputo fractional derivatives. The proofs are based on Krasnoselskii fixed point theorem and some basic concept of $\omega-$Caputo fractional analysis. As application, an nontrivial example is given in the last part of this paper to illustrate our theoretical results.

Key Words: $\omega-$fractional integral, $\omega-$Caputo fractional derivative, fixed point, Carathéodory function.

Contents

1. Introduction

Fractional calculus is a useful tool for modeling and characterizing both discrete and continuous processes in several domains of science and engineering, including mechanics, electricity, biology, control theory, signal and image processing, and so on,(see [3,9,13,23,24]). Basic issues related to the different fractional derivatives such as Riemann-Liouville [21], Caputo [2], Hilfer [19], Erdelyi-Kober [22] and Hadamard [1]),where the study of fractional differential equations have been of great object of extensive study during recent years such as functional hybrid fractional differential equations which can be employed in modeling and describing non-homogeneous physical phenomena that take place in their form. The authors in [10] proved the existence and uniqueness results for hybrid differential equations by using the theory of inequalities, hybrid fixed point theorems and operator theory. The importance of fractional hybrid differential equations lies in the fact that they include various dynamical systems as particular cases. Furthermore, hybrid differential equations arise from a variety of different areas of applied mathematics and physics, in the deflection of a curved beam having a constant or varying cross section and electromagnetic waves or gravity driven flows. We refer the readers to the articles [7,6,12] and references therein for many works on this theory.

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Dhage and Lakshmikantham [12] discussed the following first order hybrid differential equation

$$\begin{cases} \frac{d}{dt}\left(\frac{u(t)}{f(t,u(t))}\right) = g(t,u(t)), \qquad t \in J = [0,T], \\ \\ u(t_0) = u_0 \in \mathbb{R}. \end{cases}$$

results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparison results.

Zhao, Sun, Han and Li [26] are discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operators

$$\begin{cases} D^{\beta}\left(\frac{u(t)}{f(t,u(t))}\right) = g(t,u(t)), \qquad t \in J = [0,T], \\ \\ u(0) = 0. \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus 0)$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$. and f satisfied some appropriate assumptions.

In [15], Ali El Mfadel, Said Melliani, M'hamed Elomari considered boundary value problems for hybrid differential equations with fractional order involving Caputo differential operators of order $0 < \beta < 1$

$$\begin{cases} {}^{C}D^{\beta,\omega}_{0^+}\left(\frac{u(t)}{f(t,u(t))}\right) = g(t,u(t)), \qquad t \in J = [0,T], \\ \\ u(0) = 0. \end{cases}$$

Where $T > 0$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathbf{C_c}(J \times \mathbb{R}, \mathbb{R})$. They established the existence theorem for fractional hybrid differential equation, some fundamental differential inequalities are also established and the existence of extremal solutions.

On the other hand, Khalid Hilal, Ahmed Kajouni [18] investigated some existence of positive solutions of boundary value problems of fractional differential equations

$$\begin{cases} D^{\beta}\left(\frac{u(t)}{f(t,u(t))}\right) = g(t,u(t)), \qquad t \in J = [0,T], \\ \\ a\frac{u(0)}{f(t,u(0))} + b\frac{u(T)}{f(T,u(T))} = c. \end{cases}$$

Motivated by the above works, in this paper, we discuss the existence and uniqueness of maximal and minimal solution of the following nonlinear $\omega-$ Caputo fractional differential equations with Lipschitz and Carathodory conditions

$$\begin{cases} {}^{C}D^{\beta,\omega}_{0^+}\left(\frac{u(t)}{f(t,u(t))}\right) = g(t,u(t)), \qquad t \in J = [0,T], \\ \\ \theta\frac{u(0)}{f(t,u(0))} + \sigma\frac{u(T)}{f(T,u(T))} = c. \end{cases} \tag{1.1}$$

Where $T > 0$,θ, σ, c are real constants with $\theta + \sigma \neq 0$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathbf{C_c}(J \times \mathbb{R}, \mathbb{R})$.

Our paper is organized as follows. In Section 2, we give some basic definitions and properties of $\omega-$fractional integral and $\omega-$Caputo fractional derivative which will be used in the rest of this paper. In Section 3, we establish the existence result of solutions for $\omega-$Caputo fractional hybrid differential by using some Lipschitz and Carathéodory conditions. In section 4, We discuss a fundamental result relative to strict inequalities. In section 5, demonstrates the existence of maximal and minimal solutions. In Section 6, we prove some comparison theorems of maximal and minimal solutions and we give an illustrative example in section 7 followed by Conclusion in Secetion 8.

2. Preliminaries

In this section, we give some notations, definitions and results on ω-fractional derivatives and ω-fractional integrals, for more details we refer the reader to works [4,8,20].

Notations

Let $J=[0,T]\subset\mathbb{R}^+$, we also consider $C_c(J\times\mathbb{R},\mathbb{R})$ the Carathéodory class of functions on $J\times\mathbb{R}$ i.e.$h\in C_c(J\times\mathbb{R},\mathbb{R})$ if and only if

1. the map $t\mapsto g(t,x(t))$ is measurable for each $x\in\mathbb{R}$, and
2. the map $u\mapsto g(t,x(t))$ is continuous for each $t\in J$.

We denote by $C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$ the space of continuous real-valued functions defined on J provided with the topology of the supremum norm

$$\| x \| = \sup_{t\in J} | x(t) | .$$

• We denote by $L^1(J,\mathbb{R})$ the space of Lebesgue integrable real-valued functions on J equipped with the norm

$$\| x \|_{L^1} = \int_0^T | x(t) | \, dt.$$

Definition 2.1. *[5] Let $q>0$, $g\in L^1([J,\mathbb{R})$ and $\omega\in C^n(J,\mathbb{R})$ such that $\omega'(t)>0$ for all $t\in J$. The ω-Riemann-Liouville fractional integral at order q of the function g is given by*

$$I_{0^+}^{q,\omega}g(t)=\frac{1}{\Gamma(q)}\int_0^t\omega'(s)(\omega(t)-\omega(s))^{q-1}g(s)ds.$$

Definition 2.2. *[5] Let $q>0$, $g\in C^{n-1}(J,\mathbb{R})$ and $\omega\in C^n(J,\mathbb{R})$ such that $\omega'(t)>0$ for all $t\in J$. The $\omega-$Caputo fractional derivative at order q of the function g is given by*

$${}^C D_{0^+}^{q,\omega}g(t)=\frac{1}{\Gamma(n-q)}\int_0^t\omega'(s)(\omega(t)-\omega(s))^{n-q-1}g_\omega^{[n]}(s)ds,$$

Where

$$g_\omega^{[n]}(s)=\left(\frac{1}{\omega'(s)}\frac{d}{ds}\right)^n g(s) \quad \text{and} \quad n=[q]+1.$$

And $[q]$ denotes the integer part of the real number q.

Remark 2.3. *In particular, if $q\in]0,1[$, then we have*

$${}^C D_{0^+}^{q,\omega}g(t)=\frac{1}{\Gamma(q)}\int_0^t(\omega(t)-\omega(s))^{q-1}g'(s)ds.$$

and

$$\begin{aligned} {}^C D_{0^+}^{q,\omega}g(t) &= I_{0^+}^{1-q,\omega}\left(\frac{g'(t)}{\omega'(t)}\right)\\ &= \frac{1}{\Gamma(1-q)}\left(\frac{1}{\omega'(t)}\frac{d}{dt}\right)\int_0^t\omega'(s)(\omega(t)-\omega(s))^{-q}g(s)ds. \end{aligned}$$

Proposition 2.4. *[5] Let $q>0$, if $g\in C^{n-1}(J,\mathbb{R})$, then we have*

1) ${}^C D_{0^+}^{q,\omega}I_{0^+}^{q,\omega}g(t)=g(t)$.

2) $I_{0^+}^{q,\omega}\,{}^C D_{0^+}^{q,\omega}g(t)=g(t)-\sum_{k=0}^{n-1}\frac{g_\omega^{[k]}(0)}{k!}(\omega(t)-\omega(0))^k$.

3) $I_{a^+}^{q,\omega}$ *is linear and bounded from $C(J,\mathbb{R})$ to $C(J,\mathbb{R})$.*

We define the multiplication in X by

$$(xy)(t) = x(t)y(t) \quad for \quad x, y \in X.$$

Clearly, $X = C(J;\mathbb{R})$ is a Banach algebra with respect to the above norm and multiplication in it.

Lemma 2.5. *[11]Let $\boldsymbol{S}$ be a non-empty, closed convex and bounded subset of a Banach algebra X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

1. *A is Lipschitzian with a Lipschitz constant λ,*
2. *B is completely continuous,*
3. *$x = AxBy \Rightarrow x \in \boldsymbol{S}$ for all $y \in \boldsymbol{S}$, and*
4. *$\lambda M < 1$ where $M = \| B(S) \| = Sup\{B(v) \mid v \in S\}$.*
 Then the equation $AxBx = x$ has a solution in S.

We assume the following assumptions throughout the rest of our paper.

(H_1) The function $u \mapsto \frac{x(t)}{f(t,x(t))}$ is increasing on $\mathbb{R}$ almost every where fort $t \in J$.

(H_2) There exists a constant $L > 0$ such that

$$\mid f(t,x) - f(t,y) \mid \leq L \mid x - y \mid \quad for \quad all \quad t \in J \quad and \quad x, y \in \mathbb{R}.$$

(H_3) There exists a function $h \in L^\infty(J,\mathbb{R})$

$$\mid g(t,x) \mid \leq h(t) \quad a.e.\ t \in J, \quad for\ all\ x \in \mathbb{R}.$$

3. Existence and uniqueness results

In this section, before we give the main result of our paper, we should prove the following fundamental lemma.

Lemma 3.1. *Suppose that hypothesis (H_1) holds and a,b,c are real canstants with $\theta + \sigma \neq 0$. Then, for any $h \in L^\infty(J,\mathbb{R})$, then functionx $\in C(J,\mathbb{R})$ is a soulution of the*

$$\begin{cases} {}^C D_{0^+}^{\beta,\omega}\left(\frac{x(t)}{f(t,x(t))}\right) = h(t), & t \in J = [0,T], \\ \theta\frac{x(0)}{f(0,x(0))} + \sigma\frac{x(T)}{f(T,x(T))} = c. \end{cases} \tag{3.1}$$

if and anly if x satisfies the hybrid integral equation

$$\begin{aligned} x(t) = [f(t,x(t))]\Bigg(&\frac{1}{\Gamma(\beta)}\int_0^t \omega'(s)(\omega(t)-\omega(s))^{\beta-1}h(s)ds \\ &-\frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds - c\right)\Bigg). \end{aligned} \tag{3.2}$$

Proof. let $x(t)$ satisfies the equation (3.2), then we divide by $f(t,x(t))$ and we apply the $\omega-$Caputo fractional derivative ${}^C D_{0^+}^{\beta,\omega}$ to both sides of equation (3.2) and we use Proposition 2.4, we obtain

$$\begin{aligned} {}^C D_{0^+}^{\beta,\omega}\left(\frac{x(t)}{f(t,x(t))}\right) = {}^C D_{0^+}^{\beta,\omega}\Bigg(&\frac{1}{\Gamma(\beta)}\int_0^t \omega'(s)(\omega(t)-\omega(s))^{\beta-1}h(s)ds \\ &-\frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds\right)\Bigg). \end{aligned} \tag{3.3}$$

$$ {}^{C}D_{0^+}^{\beta,\omega}\left(\frac{x(t)}{f(t,x)}\right) = {}^{C} D_{0^+}^{q,\omega} I_{0^+}^{q,\omega} h(t) = h(t), $$

thus

$$ {}^{C}D_{0^+}^{\beta,\omega}\left(\frac{x(t)}{f(t,x(t))}\right) = h(t). $$

We need to verify that the condition $a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c$ in the problem 3.3. by definition, $\frac{x(t)}{f(t,x(t)}$ is continuous. Appliying the $\omega-$ fractional, we obtain the first equation 3.2. Again,substituting t=0 and t=T.

$$ \frac{x(0)}{f(0,x(0))} = -\frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds - c\right) $$

$$ \frac{x(T)}{f(T,x(T))} = \frac{1}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds $$
$$ -\frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds - c\right). $$

then

$$ \begin{aligned} \theta\frac{x(0)}{f(0,x(0)} + \sigma\frac{x(T)}{f(T,x(T)} &= -\frac{\theta\sigma}{(\theta+\sigma)\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds + \frac{c\theta}{\theta+\sigma} \\ &+ \frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds \\ &- \frac{\sigma^2}{(\theta+\sigma)\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds + \frac{c\sigma}{\theta+\sigma} \\ &= c. \end{aligned} $$

conversely, ${}^{C}D_{0^+}^{\beta,\omega}\left(\frac{x(t)}{f(t,x(t))}\right) = h(t)$, by Proposition 1, we get

$$ \frac{x(t)}{f(t,x(t))} = \frac{x(0)}{f(0,x(0))} + I_{0^+}^{\beta,\omega}h(t), $$

then

$$ \sigma\frac{x(T)}{f(T,x(T))} = \sigma\frac{x(0)}{f(0,x(0))} + \frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds, $$

thus

$$ \sigma\frac{x(T)}{f(T,x(T))} + \theta\frac{x(0)}{f(0,x(0))} = (\theta+\sigma)\frac{x(0)}{f(0,x(0))} + \frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds $$

implies that

$$ \frac{x(0)}{f(0,x(0))} = -\frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds - c\right) $$

Conseauently

$$ x(t) = [f(t,x(t))]\Bigg[\frac{1}{\Gamma(\beta)}\int_0^t \omega'(s)(\omega(t)-\omega(s))^{\beta-1}h(s)ds $$
$$ -\frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds - c\right)\Bigg]. $$

Hence equation (3.2) holds.
This completes the proof. □

Theorem 3.2. *Assume that hypotheses $(H_1)-(H_3)$ hold and $\theta+\sigma\neq 0$. If*

$$L\left(\frac{(\omega(T)-\omega(0))^{\beta}\parallel h\parallel_{L^\infty}}{\Gamma(\beta+1)}\left(1+\frac{\mid\sigma\mid}{\mid\theta+\sigma\mid}\right)+\frac{\mid c\mid}{\mid\theta+\sigma\mid}\right)<1, \tag{3.4}$$

then the fractional hybrid differential equation (1) has a solution defined on J.

Proof. We define a subset S of $X=C(J,\mathbb{R})$ defined by

$$S=\{x\in X/\parallel x\parallel\leq N\}.$$

where

$$N=\frac{f_0\left((\omega(T)-\omega(0))^{\beta}\parallel h\parallel_{L^\infty}\left(1+\frac{|\sigma|}{|\theta+\sigma|}\right)+\frac{|c|}{|\theta+\sigma|}\right)}{\Gamma(\beta+1)-L\left((\omega(T)-\omega(0))^{\beta}\parallel h\parallel_{L^\infty}\left(1+\frac{|\sigma|}{|\theta+\sigma|}\right)+\frac{|c|}{|\theta+\sigma|}\right)},$$

and

$$f_0=\sup_{t\in J}f(t,0).$$

It is easy te see that S is a closed, convex and bounded subset of the Banach space X. By using Lemma 3.1, the fractional hybrid differential equation (1.1) is equivalent to the following nonlinear fractional hybrid integral equation

$$x(t)=[f(t,x(t))]\Bigg(\frac{1}{\Gamma(\beta)}\int_0^t\omega'(s)(\omega(t)-\omega(s))^{\beta-1}h(s)ds$$

$$-\frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T\omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds-c\right)\Bigg). \tag{3.5}$$

Define two operators $A:X\to X$ and $B:S\to X$ be two operators defined by

$$Ax(t)=f(t,x(t)),$$

and

$$Bx(t)=\frac{1}{\Gamma(\beta)}\int_0^t\omega'(s)(\omega(t)-\omega(s))^{\beta-1}h(s)ds-\frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T\omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)ds-c\right).$$

We can transforme the nonlinear fractional hybrid integral equation (3.5) into the operator equation as

$$Ax(t)Bx(t)=x(t),\quad t\in J. \tag{3.6}$$

Now, we will show that the operators A and B satisfy all the conditions of Lemma 2.5.
First, we prove that A is a Lipschitz operator on X with the Lipschitz constant λ.
Let $x,y\in X$, then by hypothesis (H_2)

$$\mid Ax(t)-By(t)\mid=\mid f(t,x(t))-f(t,y(t))\mid\leq\lambda\mid x(t)-y(t)\mid\quad for\quad all\quad t\in J,$$

Taking supremum over t, we obtain

$$\parallel Ax-By\parallel\leq\lambda\parallel x-y\parallel,\quad for\ all\ x,y\in X.$$

Secondly, we show the operator B is completely continuous.
For this purpose, it is enough to prove that the operator B is continuous and $B(S)$ is uniformly bounded and equicontinuous.
Let us show that the operator B is continuous.

Let x_n be a sequence in S converging to $x \in S$, then by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\mapsto+\infty} Bx_n(t) &= \lim_{n\mapsto+\infty}\left[\frac{1}{\Gamma(\beta)}\int_0^t \omega'(s)(\omega(t)-\omega(s))^{\beta-1}g(s,x_n(s)ds\right.\\
&- \left.\frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}g(s,x_n(s))ds-c\right)\right]\\
&= \lim_{n\mapsto+\infty}\frac{1}{\Gamma(\beta)}\int_0^t \omega'(s)(\omega(t)-\omega(s))^{\beta-1}g(s,x_n(s)ds\\
&- \lim_{n\mapsto+\infty}\frac{1}{\theta+\sigma}\left(\frac{b}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}g(s,x_n(s))ds-c\right),\\
&= ,\left[\frac{1}{\Gamma(\beta)}\int_0^t \omega'(s)(\omega(t)-\omega(s))^{\beta-1}g(s,x(s)ds\right.\\
&- \left.\frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}g(s,x(s))ds-c\right)\right]\\
&= Bx(t), \quad for \quad all \quad t\in J.
\end{aligned}$$

Wich shows that B is a continuous operator on S.
Next we show that $B(S)$ is a uniformly bounded set in X.

Let $x \in S$, then, by hypothesis (H_2), for all $t \in J$

$$|Bx(t)| = \left|\frac{1}{\Gamma(\beta)}\int_0^t \omega'(s)(\omega(t)-\omega(s))^{\beta-1}g(s,x(s)ds\right.$$

$$\left.-\frac{1}{\theta+\sigma}\left(\frac{b}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}g(s,x(s))ds-c\right)\right|,$$

$$\begin{aligned}
\mid B(t)\mid &\leq \frac{1}{\Gamma(\beta)}\int_0^t \left|\omega'(s)(\omega(t)-\omega(s))^{\beta-1}g(s,x(s))\right|ds\\
&+ \frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T \left|\omega'(s)(\omega(T)-\omega(s))^{\beta-1}g(s,x(s))\right|ds+|c|\right)\\
&\leq \frac{1}{\Gamma(\beta)}\int_0^t \left|\omega'(s)(\omega(t)-\omega(s))^{\beta-1}\right|h(s)ds\\
&+ \frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^T \left|\omega'(s)(\omega(T)-\omega(s))^{\beta-1}h(s)\right|ds+|c|\right)\\
&\leq \frac{(\omega(T)-\omega(s))^{\beta}}{\Gamma(\beta+1)}\parallel h\parallel_{L^\infty}\left(1+\frac{\mid b\mid}{\mid\theta+\sigma\mid}\right)+\frac{\mid c\mid}{\mid\theta+\sigma\mid}
\end{aligned}$$

Thus $\parallel Bx(t)\parallel\leq \frac{(\omega(T)-\omega(s))^{\beta}}{\Gamma(\beta+1)}\parallel h\parallel_{L^\infty}\left(1+\frac{|b|}{|\theta+\sigma|}\right)+\frac{|c|}{|\theta+\sigma|}$ for all $x \in S$ This shows that B is uniformly bounded on S Now, let us also show that $B(S)$ is equicontinuous on J.

Let $x \in B(S)$ and $t_1, t_2 \in J$ such that $t_1 < t_2$, then we have

$$\begin{aligned}
|Bx(t_1) - Bx(t_2)| &= \bigg| \frac{1}{\Gamma(\beta)} \int_0^{t_1} \omega'(s)(\omega(t_1) - \omega(s))^{\beta-1} g(s, x(s))ds, \\
&- \frac{1}{\Gamma(\beta)} \int_0^{t_2} \omega'(s)(\omega(t_2) - \omega(s))^{\beta-1} g(s, x(s))ds \bigg|, \\
&\leq \frac{\| g \|_{L^\infty}}{\Gamma(\beta+1)} \Big(\mid \omega^\beta(t_2) - \omega^\beta(t_1) - (\omega(t_2) - \omega(t_1))^\beta \mid + (\omega(t_2) - \omega(t_1))^\beta \Big)
\end{aligned}$$

Since ω is a continuous function, then we obatin

$$\lim_{t_1 \to t_2} |Bx(t_1) - Bx(t_2)| = 0.$$

which shows that $B(S)$ is equicontinuous.
Now the set $B(S)$ is uniformly bounded and equicontinuous and by using Arzelà–Ascoli Theorem [14] we deduce that $B(S)$ is compact, wich implies that the operator B is completely continuous.
Now it remains to show that the third assumption in Proposition 2.4 is verified.
Let $x \in X$ and $y \in S$ be arbitrary such that $x = AxBy$, then by hypothesis (H_2), we have

$$\begin{aligned}
|x(t)| &= \bigg| [f(t, x(t))] \Big(\frac{1}{\Gamma(\beta)} \int_0^t \omega'(s)(\omega(t) - \omega(s))^{\beta-1} h(s)ds \\
&- \frac{1}{\theta+\sigma} \left(\frac{\sigma}{\Gamma(\beta)} \int_0^T \omega'(s)(\omega(T) - \omega(s))^{\beta-1} h(s)ds \right) ds - c \Big). \bigg| \\
&\leq [\mid f(t, x(t)) - f(t, 0) \mid + \mid f(t, 0) \mid] \Big(\frac{(\omega(T) - \omega(0))^\beta}{\Gamma(\beta+1)} \| h \|_{L^\infty} \\
&+ \frac{\mid \sigma \mid (\omega(T) - \omega(0))^\beta}{\mid \theta + \sigma \mid \Gamma(\beta+1)} \| h \|_L^\infty + \frac{\| c \|}{\mid \theta + \sigma \mid} \Big) \\
&\leq (L \mid x(t) \mid + f_0) \Big(\frac{(\omega(T) - \omega(0))^\beta}{\Gamma(\beta+1)} \| h \|_{L^\infty} \left(1 + \frac{\mid \sigma \mid}{\mid \theta + \sigma \mid} \right) + \frac{\mid c \mid}{\mid \theta + \sigma \mid} \Big) \\
&\leq L \mid x(t) \mid \left(\frac{(\omega(T) - \omega(0))^\beta}{\Gamma(\beta+1)} \| h \|_{L^\infty} \left(1 + \frac{\mid \sigma \mid}{\mid \theta + \sigma \mid} \right) + \frac{\mid c \mid}{\mid \theta + \sigma \mid} \right) \\
&+ f_0 \Big(\frac{(\omega(T) - \omega(0))^\beta}{\Gamma(\beta+1)} \| h \|_{L^\infty} \left(1 + \frac{\mid \sigma \mid}{\mid \theta + \sigma \mid} \right) + \frac{\mid c \mid}{\mid \theta + \sigma \mid} \Big),
\end{aligned}$$

taking supremum over t, we obtain

$$\| x \| \leq \frac{f_0 \Big(\frac{(\omega(T)-\omega(0))^\beta}{\Gamma(\beta+1)} \| h \|_{L^\infty} \left(1 + \frac{|\sigma|}{|\theta+\sigma|} \right) + \frac{|c|}{|\theta+\sigma|} \Big)}{1 - L \Big(\frac{(\omega(T)-\omega(0))^\beta}{\Gamma(\beta+1)} \| h \|_{L^\infty} \left(1 + \frac{|\sigma|}{|\theta+\sigma|} \right) + \frac{|c|}{|\theta+\sigma|} \Big)} = N,$$

Since

$$M = \| \mathcal{B}(S) \| = Sup\{B(x) : x \in S\} \leq \frac{(\omega(T) - \omega(0))^\beta}{\Gamma(\beta+1)} \| h \|_{L^\infty} \left(1 + \frac{\mid \sigma \mid}{\mid \theta + \sigma \mid} \right) + \frac{\mid c \mid}{\mid \theta + \sigma \mid}$$

then we get

$$\lambda M \leq k \Big(\frac{(\omega(T) - \omega(0))^\beta}{\Gamma(\beta+1)} \| h \|_{L^\infty} \left(1 + \frac{\mid \sigma \mid}{\mid \theta + \sigma \mid} \right) + \frac{\mid c \mid}{\mid \theta + \sigma \mid} \Big) < 1.$$

Finally, all conditions of Lemma 2.5 are satisfied for the operators A and B. Hence the fractional hybrid differential equation (1.1) has a solution defined on J. $\square$

4. Fractional hybrid differental inequalities

We discuss a fundamental result relative to strict inequalities
We begin with the definition of the class $C_p([0,T],\mathbb{R})$.

Definition 4.1. *$m \in C_p([0,T],\mathbb{R})$ means that $m \in C([0,T],\mathbb{R})$ and $t^p m(t) \in C_p([0,T],\mathbb{R})$*

Lemma 4.2. *Let $m \in C_p([0,T],\mathbb{R})$. suppose That for any $t_1 \in (0,+\infty)$ we have $m(t_1)=0$ and $m(t) \leq 0$ for $0 \leq t \leq t_1$. Then it follows that*

$$^{C}D_{0^+}^{\beta,\omega} m(t_1) \geq 0$$

Proof. Consider $m \in C_p([0,T],\mathbb{R})$, such that $m(t_1)=0$ and $m(t)<0$ for $0<t\leq t_1$. Then, m(t) is continuous on $[0,T]$ and $t^p m(t)$ is continuous on $[0,T]$.
since m(t) ant $\omega(t)$ is continuous on $[0,T]$, given any t_1 such that $0<t_1<T$, there exists a $k_1(t_1)>0, k_2(t_1)>0$ and $h>0$ such that $-k_1(t_1)(t_1-s) \leq m(t)-m(s) \leq k_1(t_1)(t_1-s)$ and $-k_2(t_1)(t_1-s) \leq \omega(t)-\omega(s) \leq k_2(t_1)(t_1-s)$
for $0<t_1-h \leq s \leq t_1+h<T$. Because we have

$$^{C}D_{0^+}^{\beta,\omega} m(t_1) = \frac{1}{\Gamma(1-\beta)}\left(\frac{1}{\omega'(t)}\frac{d}{dt}\right)\int_0^t \omega'(s)(\omega(t)-\omega(s))^{-\beta} m(s)ds$$

set $H(t)=\int_0^t \omega'(s)(\omega(t)-\omega(s))^{-\beta} m(s)ds$ and consider

$$\begin{aligned}
H(t_1)-H(t_1-h) &= \int_0^{t_1} \omega'(s)(\omega(t_1)-\omega(s))^{-\beta} m(s)ds - \int_0^{t_1-h} \omega'(s)(\omega(t_1-h)-\omega(s))^{-\beta} m(s)ds. \\
&= \int_{t_1-h}^{t_1} \omega'(s)(\omega(t_1)-\omega(s))^{-\beta} m(s)ds + \int_0^{t_1-h} \omega'(s)(\omega(t_1)-\omega(s))^{-\beta} m(s)ds \\
&- \int_0^{t_1-h} \omega'(s)(\omega(t_1-h)-\omega(s))^{-\beta} m(s)ds. \\
&= \int_0^{t_1-h} \omega'(s)[(\omega(t_1)-\omega(s))^{-\beta} - (\omega(t_1-h)-\omega(s))^{-\beta}] m(s)ds \\
&+ \int_{t_1-h}^{t_1} \omega'(s)(\omega(t_1)-\omega(s))^{-\beta} m(s)ds
\end{aligned}$$

Let

$$I_1 = \int_0^{t_1-h} \omega'(s)[(\omega(t_1)-\omega(s))^{-\beta} - (\omega(t_1-h)-\omega(s))^{-\beta}] m(s)ds$$

and

$$I_2 = \int_{t_1-h}^{t_1} \omega'(s)(\omega(t_1-h)-\omega(s))^{-\beta} m(s)ds.$$

Since $t_1 > t_1-h$ and $-\beta<0$ and $\omega(t)$ is increasing, we have

$$(\omega(t_1)-\omega(s))^{-\beta} < (\omega(t_1-h)-\omega(s))^{-\beta}.$$

This, coupled with the fact that $m(t) \leq 0$, $0<t\leq t_1$, implies that $I_1 \geq 0$.
Now, consider

$$I_2 = \int_{t_1-h}^{t_1} \omega'(s)(\omega(t_1-h)-\omega(s))^{-\beta} m(s)ds.$$

Using

$$-k_1(t_1)(t_1-s) \leq m(t)-m(s) \leq k_1(t_1)(t_1-s)$$

and

$$-k_2(t_1)(t_1-s) \leq \omega(t)-\omega(s) \leq k_2(t_1)(t_1-s)$$

and the fact that $m(t_1) = 0$, for $s \in (t_1 - h, t_1 + h)$ we obtain, $m(s) \geq +k_1(t_1)(t_1 - s)$, and

$$\begin{aligned} I_2 &\geq -k_1(t_1)\int_{t_1-h}^{t_1} \omega'(s)(\omega(t_1)-\omega(s))^{-\beta}(t_1-s)ds \\ &\geq -k_1(t_1)\frac{(k_2(t_1))^{-\beta+1}h^{-\beta+2}}{-\beta+1} \end{aligned}$$

Thus, we have

$$H(t_1) - H(t_1 - h) \geq -k_1(t_1)\frac{(k_2(t_1))^{-\beta+1}h^{-\beta+2}}{-\beta+1}$$

Then dividing through by h and taking limits as $h \longrightarrow 0$, we have

$$\lim_{h\longleftrightarrow 0}\left[\frac{H(t_1)-H(t_1-h)}{h} - k(t_1)\frac{(k_2(t_1))^{-\beta+1}h^{-\beta+1}}{-\beta+1}\right] \geq 0$$

Since $\beta \in (0,1)$, we conclude that $\frac{dH}{dt} \geq 0$, which implies that ${}^C D_{0^+}^{\beta,\omega} m(t_1) \geq 0$. □

Theorem 4.3. *Assume that hypothesis (H_1) holds. Suppose that there exist functions $y, z \in C_p([0,T],\mathbb{R})$ such that*

$${}^C D_{0^+}^{\beta,\omega}\left(\frac{y(t)}{f(t,y(t))}\right) \leq g(t,y(t)) \quad a.e.\ t \in J \tag{4.1}$$

and

$${}^C D_{0^+}^{\beta,\omega}\left(\frac{z(t)}{f(t,z(t))}\right) \geq g(t,z(t)) \quad a.e.\ t \in J \tag{4.2}$$

$0 < t \leq T$*, with one of the inequalities being strict. Then*

$$y^0 < z^0,$$

where$y^0 = t^{1-\beta}y(t)|_{t=0}$ *and* $z_0 = t^{1-\beta}z(t)|_{t=0}$ *implies*

$$y(t) < z(t)$$

for all $t \in J$.

Proof. Suppose that inequality (4.2) holds. Assume that the claim is false. Then, since $y^0 < z^0$ and $t^{1-\beta}y(t)$ and $t^{1-\beta}z(t)$ are continuous functions, there exists t_1 such that $0 < t_1 \leq T$ with $y(t_1) = z(t_1)$ and $y(t) < z(t)$, $0 \leq t < t_1$. we pose

$$Y(t) = \frac{y(t)}{f(t,y(t))} \quad and \quad Z(t) = \frac{z(t)}{f(t,z(t))}$$

We have $Y(t_1) = Z(t_1)$, and by virtue of hypothesis (H_1), we get $Y(t) < Z(t)$ for all $0 \leq t < t_1$.
Let $m(t) = Y(t) - Z(t)$, $0 \leq t \leq t_1$, We find that $m(t) < 0$ and $m(t_1) = 0$ with $m \in C_p([0,1],\mathbb{R})$. Then by lemma 3. We have ${}^C D_{0^+}^{\beta,\omega} m(t_1) \geq 0$, bay (4.1) and (4.2)

$$g(t_1, y(t_1)) \geq^C D_{0^+}^{\beta,\omega}(Y(t_1) \geq^C D_{0^+}^{\beta,\omega} Z(t_1) > g(t_1, z(t_1)).$$

This is acontradiction with $y(t_1) = z(t_1)$.Thus the conclusion of the theorem and the proof is complet. □

Theorem 4.4. *Asusum that hypothesis (H_1) holds and a, b, c are real constans with $\theta + \sigma \neq 0$. Suppose that there exist functions $y, z \in C_p([0,T],\mathbb{R})$ such that*

$${}^C D_{0^+}^{\beta,\omega}\left(\frac{z(t)}{f(t,z(t))}\right) \leq g(t,z(t)) \quad a.e.\ t \in J \tag{4.3}$$

and

$$
{}^{C}D_{0^+}^{\beta,\omega}\Big(\frac{z(t)}{f(t,z(t))}\Big) \geq g(t,z(t)) \quad a.e.\ t\in J \tag{4.4}
$$

one The inequalities being strict, and if $\theta>0,\sigma<0$ *and* $y(T)<z(T)$, *then*

$$
\theta\frac{y(0)}{f(0,y(0))}+\sigma\frac{y(T)}{f(T,y(T))}<\theta\frac{z(0)}{f(0,z(0))}+\sigma\frac{z(T)}{f(T,z(T))} \tag{4.5}
$$

implies

$$
y(t)<z(t) \ \ for \quad all \quad t\in J \tag{4.6}
$$

Proof. We have

$$
\theta\frac{y(0)}{f(0,y(0))}+\sigma\frac{y(T)}{f(T,y(T))}<\theta\frac{z(0)}{f(0,z(0))}+\sigma\frac{z(T)}{f(T,z(T))},
$$

this implies

$$
\theta\Big(\frac{y(0)}{f(0,y(0))}-\frac{z(0)}{f(0,z(0))}\Big)<\sigma\Big(\frac{z(T)}{f(T,z(T))}-\frac{y(T)}{f(T,y(T))}\Big).
$$

Since $\sigma<0$ and $y(T)<z(T)$ and by hypothesis (H_1), then $\frac{z(T)}{f(T,z(T))}-\frac{y(T)}{f(T,y(T))}>0$.
This shows that $\frac{y(0)}{f(0,y(0))}-\frac{z(0)}{f(0,z(0))}<0$ since $\theta>0$, and by hypothesis (H_1) we have $y(0)<z(0)$. hence the application of Theoreme 4.3. yields that $y(t)<z(t)$ □

Theorem 4.5. *Assume that the conditions of theorem 4.4 hold with inequalities 4.1 and 4.2. Suppose that there existes a real number* $M>0$ *such that*

$$
g(t,x_1)-g(t,x_2)\leq\frac{M}{1+(\omega(t)-\omega(0))^{\beta+1}}\Big(\frac{x_1}{f(t,x_1)}-\frac{x_2}{f(t,x_2)}\Big) \ \ a.e\ t\in J \tag{4.7}
$$

for all $x_1,x_2\in\mathbb{R}$ *with* $x_1\geq x_2$. *Then*

$$
\theta\frac{y(0)}{f(0,y(0))}+\sigma\frac{y(T)}{f(T,y(T))}<\theta\frac{z(0)}{f(0,z(0))}+\sigma\frac{z(T)}{f(T,z(T))},
$$

provided that $M\leq\Gamma(1+\beta)(\omega(t)-\omega(0))^{2\beta}$, *then*

$$
y(t)<z(t),\ \ for\ all\ t\in J.
$$

Proof. We set $\frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))}=\frac{z(t)}{f(t,z(t))}+\varepsilon(1+(\omega(t)-\omega(0))^{\beta+1}$ for small $\varepsilon>0$ and let $Z_\varepsilon(t)=\frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))}$ and $Z(t)=\frac{z(t)}{f(t,z(t))}$ for $t\in J$. So that we have

$$
Z_\varepsilon(t)>Z(t)\Rightarrow z_\varepsilon(t)>z(t).
$$

Since

$$
g(t,x_1)-g(t,x_2)\leq\frac{M}{1+(\omega(t)-\omega(0))^{\beta+1}}\Big(\frac{x_1}{f(t,x_1)}-\frac{x_2}{f(t,x_2)}\Big)
$$

and

$$
{}^{C}D_{0^+}^{\beta,\omega}\Big(\frac{z(t)}{f(t,z(t))}\Big)\geq g(t,z(t))
$$

for all $t\in J$, one has

$$
\begin{aligned}
{}^{C}D_{0^+}^{\beta,\omega}Z_\varepsilon(t) &= {}^{C}D_{0^+}^{\beta,\omega}Z(t)+{}^{C}D_{0^+}^{\beta,\omega}\varepsilon(1+(\omega(t)-\omega(0))^{\beta+1})\\
&\geq g(t,z(t))+\varepsilon\Gamma(1+\beta)(\omega(t)-\omega(0))^{2\beta}\\
&\geq g(t,z_\varepsilon(t))-\frac{M}{1+(\omega(t)-\omega(0))^{\beta+1}}(Z_\varepsilon-Z)+\varepsilon\Gamma(1+\beta)(\omega(t)-\omega(0))^{2\beta}\\
&\geq g(t,z_\varepsilon(t))+\varepsilon(\Gamma(1+\beta)(\omega(t)-\omega(0))^{2\beta}-M)\\
&> g(t,z_\varepsilon(t))
\end{aligned}
$$

provided $M \leq \Gamma(1+\beta)(\omega(t)-\omega(0))^{2\beta}$
Also, we have $z_\varepsilon(0) > z(0) \geq y(0)$. Hence, the application of theorem 3.2 yields that $y(t) < z_\varepsilon(t)$ for all $t \in J$.
By the arbitrariness of $\varepsilon > 0$, taking teh limits as $\varepsilon \longrightarrow 0$, we have $y(t) \leq z(t)$ for all $t \in J$.
□

5. Existence of maximal and minimal solutions

In the section, we prove the existence of maximal and minimal solution for 1.1 on $J=[0,T]$. We need the following fundamental defintion in what follows.

Definition 5.1. *A solution r of 1.1 is said to be maximal if for any other solution x to 1.1 one has $x(t) \leq r(t)$ for all $t \in J$. Similarly, a solution ρ of 1.1 is said to be minimal if $\rho(t) \leq x(t)$ for all $t \in J$, where x is any solution of 1.1 on J.*

We discuss the case of maximal solution only, as the case of minimal solution is similar and can be obtained with the same arguments with appropriate modifications. Given an arbitrarily small real number $\varepsilon > 0$, consider the following boundary value problem of order $0 < \beta < 1$:

$$\begin{cases} {}^C D_{0^+}^{\beta,\omega}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) + \varepsilon, \qquad t \in J=[0,T], \\ \\ \theta \frac{x(0)}{f(0,x(0))} + \sigma \frac{x(T)}{f(T,x(T))} = c + \varepsilon. \end{cases} \tag{5.1}$$

An existence theorem for (5.1) can be stated as follows.

Theorem 5.2. *Assume that hypotheses $(H_1)-(H_3)$ hold and a, b, c are real constants with $a+b \neq 0$. Suppose that inequality (3.4) holds. Then, for every small number $\varepsilon > 0$, (5.1) has a solution defined on J*

Proof. By inequality (3.4), there exists $\varepsilon_0 > 0$ such that

$$L\left(\frac{(\omega(T)-\omega(0))^\beta \parallel h \parallel_{L^\infty} + \varepsilon\frac{(\omega(T)-\omega(0))}{\beta}}{\Gamma(\beta+1)}\left(1+\frac{\mid \sigma \mid}{\mid \theta+\sigma \mid}\right) + \frac{\mid c \mid + \varepsilon}{\mid \theta+\sigma \mid}\right) < 1,$$

for all $0 < \varepsilon \leq \varepsilon_0$. Now the rest of the proof is similar to Theorem 3.2 □

Our main existence theorem for maximal solution for (1.1) is the following.

Theorem 5.3. *Assume that hypotheses $(H_1)-(H_3)$ hold with the conditions of Theorem 4.4 and θ, σ, c are real constants with $\theta+\sigma \neq 0$. Furthermore, if condition (3.4) holds, then (1.1) has a maximal solution defined on J*

Proof. Let $\{\varepsilon_n\}_0^\infty$ be be a decreasing sequence of positive real numbers such that

$$\lim_{n\to\infty} \varepsilon_n = 0,$$

where ε_n is a positive real number satisfying the inequality

$$L\left(\frac{(\omega(T)-\omega(0))^\beta \parallel h \parallel_{L^\infty} + \varepsilon_0\frac{(\omega(T)-\omega(0))}{\beta}}{\Gamma(\beta+1)}\left(1+\frac{\mid \sigma \mid}{\mid \theta+\sigma \mid}\right) + \frac{\mid c \mid + \varepsilon_0}{\mid \theta+\sigma \mid}\right) < 1,$$

The number ε_0 exists in view of inequality (3.4). By Theorem 5.2 , there exists a solution $r(t,\varepsilon_n)$ defined on J of the 1.1

$$\begin{cases} {}^C D_{0^+}^{\beta,\omega}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) + \varepsilon_n, \qquad t \in J=[0,T], \\ \\ \theta \frac{x(0)}{f(0,x(0))} + \sigma \frac{x(T)}{f(T,x(T))} = c + \varepsilon_n. \end{cases} \tag{5.2}$$

Then any solution u of (1.1)

$$ {}^{C}D_{0^+}^{\beta,\omega}\left(\frac{u(t)}{f(t,u(t))}\right) \leq g(t,u(t)), $$

□

and any of auxiliary probleme (5.2) satisfies

$$ {}^{C}D_{0^+}^{\beta,\omega}\left(\frac{r(t,\varepsilon_n)}{f(t,r(t,\varepsilon_n))}\right) = g(t,r(t,\varepsilon_n)) + \varepsilon_n > g(t,r(t,\varepsilon_n)), $$

where

$$ \theta\frac{u(0)}{f(0,u(0))} + \sigma\frac{u(T)}{f(T,u(T))} \leq \theta\frac{r(0,\varepsilon_n)}{f(0,r(0,\varepsilon_n))} + \sigma\frac{r(T,\varepsilon_n)}{f(T,r(T,\varepsilon_n))}. $$

By using Thorem 4.4 we have

$$ u(t) \leq r(t,\varepsilon_n). \tag{5.3} $$

for all $t \in J$ and $t \in \mathbb{N}$, since $\{\varepsilon_n\}$ be decreasing sequence, then

$$ \begin{aligned} c+\varepsilon_2 &= \theta\frac{r(0,\varepsilon_2)}{f(0,r(0,\varepsilon_2))} + \sigma\frac{r(T,\varepsilon_2)}{f(T,r(T,\varepsilon_2))} \\ &\leq \theta\frac{r(0,\varepsilon_1)}{f(0,r(0,\varepsilon_1))} + \sigma\frac{r(T,\varepsilon_1)}{f(T,r(T,\varepsilon_1))} = c+\varepsilon_1 \end{aligned} $$

then by theorem 5.2, we have $r(t,\varepsilon_2) \leq r(t,\varepsilon_1)$, Therefore, $r(t,\varepsilon_n)$ is a decreasing sequence of positive real numbers, and the limit

$$ r(t) = \lim_{n\longrightarrow\infty} r(t,\varepsilon_n) \tag{5.4} $$

exists. We show that the convergence in (5.4) is uniform on J. To finish, it is enough to prove that the sequence $r(t,\varepsilon_n$ is equicontinuous in $C(J,\mathbb{R})$. Let $t_1,t_2\in J$ with $t_1<t_2$ be arbitrary. Then

$$\begin{aligned}
\mid r(t_1,\varepsilon_n)-r(t_2,\varepsilon_n)\mid &= \Bigg|[f(t_1,r(t_1,\varepsilon_n))]\Bigg(\frac{1}{\Gamma(\beta)}\int_0^{t_1}\omega'(s)(\omega(t_1)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)ds\\
&- \frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^{T}\omega'(s)(\omega(T)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)ds\right)-c-\varepsilon_n\Bigg)\\
&- [f(t_2,r(t_2,\varepsilon_n))]\Bigg(\frac{1}{\Gamma(\beta)}\int_0^{t_2}\omega'(s)(\omega(t_2)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)ds\\
&- \frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^{T}\omega'(s)(\omega(T)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)ds\right)-c-\varepsilon_n\Bigg)\Bigg|\\
&= \Bigg|[f(t_1,r(t_1,\varepsilon_n))]\Bigg(\frac{1}{\Gamma(\beta)}\int_0^{t_1}\omega'(s)(\omega(t_1)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\Bigg)ds\\
&- [f(t_2,r(t_2,\varepsilon_n))]\Bigg(\frac{1}{\Gamma(\beta)}\int_0^{t_2}\omega'(s)(\omega(t_2)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\Bigg)ds\\
&- [f(t_1,r(t_1,\varepsilon_n))-f(t_2,r(t_2,\varepsilon_n))]\\
&\times \frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^{T}\omega'(s)(\omega(T)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)ds\right)-c-\varepsilon_n\Bigg)\Bigg|\\
&\le \Bigg|[f(t_1,r(t_1,\varepsilon_n))]\Bigg(\frac{1}{\Gamma(\beta)}\int_0^{t_1}\omega'(s)(\omega(t_1)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\Bigg)ds\\
&- [f(t_2,r(t_2,\varepsilon_n))]\Bigg(\frac{1}{\Gamma(\beta)}\int_0^{t_1}\omega'(s)(\omega(t_1)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\Bigg)ds\Bigg|\\
&+ \Bigg|[f(t_1,r(t_1,\varepsilon_n))-f(t_2,r(t_2,\varepsilon_n))]\\
&\times \frac{1}{\theta+\sigma}\left(\frac{\sigma}{\Gamma(\beta)}\int_0^{T}\omega'(s)(\omega(T)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)ds\right)-c-\varepsilon_n\Bigg)\Bigg|\\
&+ \Bigg|[f(t_2,r(t_2,\varepsilon_n))]\Bigg(\frac{1}{\Gamma(\beta)}\int_0^{t_1}\omega'(s)(\omega(t_1)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\Bigg)ds\\
&- [f(t_2,r(t_2,\varepsilon_n))]\Bigg(\frac{1}{\Gamma(\beta)}\int_0^{t_2}\omega'(s)(\omega(t_2)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\Bigg)ds\Bigg|\\
&\le f_0\frac{(\| h\|_{L^\infty}+\varepsilon_n)}{\Gamma(\beta+1)}\Bigg(\mid \omega^\beta(t_2)-\omega^\beta(t_1)-(\omega(t_2)-\omega(t_1))^\beta\mid +\omega(t_2)-\omega(t_1))^\beta\Bigg)\\
&+ \mid f(t_1,r(t_1,\varepsilon_n))-f(t_2,r(t_2,\varepsilon_n))\mid\\
&\times \left[\frac{(\| h\|_{L^\infty}+\varepsilon_n)(\omega(T))^\beta}{\Gamma(\beta+1)}+\frac{\mid b\mid(\| h\|_{L^\infty}+\varepsilon_n)(\omega(T))^\beta}{\mid\theta+\sigma\mid\Gamma(\beta+1)}+\frac{\mid c\mid+\varepsilon_n}{\mid\theta+\sigma\mid}\right].
\end{aligned}$$

Where $f_0=\sup_{(t,x)\in J\times[-N,N]}|f(t,x)|$.
Since f is continuous on a compact set $J\times[-N,N]$, it is uniformly continuous there.
Hence,

$$f(t_1,r(t_1,\varepsilon_n))-f(t_2,r(t_2,\varepsilon_n))\longrightarrow 0\quad as\quad t_1\longrightarrow t_2$$

uniformly for all $n\in\mathbb{N}$. Therefore, from the above inequality, it follows that

$$r(t_1,\varepsilon_n)-r(t_2,\varepsilon_n)\longrightarrow 0\ \ as\ \ t_1\longrightarrow t_2$$

uniformly for all $n\in\mathbb{N}$. Therefore,

$$r(t,\varepsilon_n)\overset{n\longrightarrow\infty}{\longrightarrow} r(t)\quad for\quad all\quad t\in J$$

Next, we show that the function r(t) is a solution of (1.1) defined on J. Now, since $r(t,\varepsilon_n)$ is a solution of (5.2), we have

$$\begin{aligned} r(t,\varepsilon_n) &= [f(t,r(t,\varepsilon_n))]\Big(\frac{1}{\Gamma(\beta)}\int_0^t \omega'(s)(\omega(t)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)ds \\ &- \frac{1}{\theta+\sigma}\Big(\frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)ds-\varepsilon_n-c\Big)\Big), \end{aligned}$$

for all $t\in J$. Taking the limit as $n\longrightarrow\infty$ in the above equation yields

$$\begin{aligned} r(t) &= [f(t,r(t))]\Big(\frac{1}{\Gamma(\beta)}\int_0^t \omega'(s)(\omega(t)-\omega(s))^{\beta-1}g(s,r(s))ds \\ &- \frac{1}{\theta+\sigma}\Big(\frac{\sigma}{\Gamma(\beta)}\int_0^T \omega'(s)(\omega(T)-\omega(s))^{\beta-1}g(s,r(s))ds-c\Big)\Big), \end{aligned}$$

for all $t\in J$. Thus, the function r is a solution of (1.1) on J. Finally, from inequality (5.4) it follows that $u(t)\le r(t)$ for all $t\in J$. Hence, (1.1) has a maximal solution on J

6. Comparison theorems

The main problem of the differential inequalities is to estimate a bound for the solution set for the differential inequality related to the (1.1). In this section we prove that the maximal and minimal solutions serve as the bounds for the solutions of the related differential inequality to (1.1) on .

Theorem 6.1. *Assume that hypotheses $(H_1)-(H_3)$ and condition (3.4) hold and θ,σ,c are real constants with $\theta+\sigma\neq 0$. Suppose that there exists a real number $M>0$ such that*

$$g(t,x_1)-g(t,x_2)\le \frac{M}{1+(\omega(t)-\omega(0))^{\beta+1}}\Big(\frac{x_1}{f(t,x_1)}-\frac{x_2}{f(t,x_2)}\Big)\ a.e\ t\in J$$

for all $x_1,x_2\in\mathbb{R}$ with $x_2\ge x_1$, where $M\le\Gamma(1+\beta)$. Furthermore, if there exists a function $u\in C(J,\mathbb{R})$ such that

$$\begin{cases} {}^C D_{0^+}^{\beta,\omega}\left(\frac{u(t)}{f(t,u(t))}\right)\le g(t,u(t)), \qquad t\in J=[0,T], \\ \theta\frac{u(0)}{f(0,u(0))}+\sigma\frac{u(T)}{f(T,u(T))}\le c. \end{cases} \tag{6.1}$$

then

$$u(t)\le r(t) \tag{6.2}$$

for all $t\in J$, where r is a maximal solution of (1.1) on J.

Proof. Let $\varepsilon>0$ be arbitrarily small. By Theorem 5.3, $r(t,\varepsilon)$ is a maximal solution of (5.1) so that the limit

$$r(t)=\lim_{\varepsilon\longrightarrow 0} r(t,\varepsilon) \tag{6.3}$$

is uniform on J and the function r is a maximal solution of (1.1) on J. Hence, we obtain

$$\begin{cases} {}^C D_{0^+}^{\beta,\omega}\left(\frac{r(t,\varepsilon)}{f(t,r(t,\varepsilon))}\right)=g(t,r(t,\varepsilon))+\varepsilon, \qquad t\in J=[0,T], \\ \theta\frac{r(0,\varepsilon)}{f(0,r(0,\varepsilon))}+\sigma\frac{r(T,\varepsilon)}{f(T,r(T,\varepsilon))}=c+\varepsilon. \end{cases} \tag{6.4}$$

From the above inequality it follows that

$$\begin{cases} {}^C D_{0^+}^{\beta,\omega}\left(\dfrac{r(t,\varepsilon)}{f(t,r(t,\varepsilon))}\right)> g(t,r(t,\varepsilon)), \qquad t\in J=[0,T], \\ \theta\dfrac{r(0,\varepsilon)}{f(0,r(0,\varepsilon))}+\sigma\dfrac{r(T,\varepsilon)}{f(T,r(T,\varepsilon))}=c+\varepsilon. \end{cases} \tag{6.5}$$

Now we apply Theorem 4.5 to inequalities (6.1) and (6.5) and conclude that $u(t) < r(t, \varepsilon)$ for all $t \in J$. This, in view of limit (6.3), further implies that inequality (6.2) holds on J. This completes the proof. □

Theorem 6.2. *Assume that hypotheses $(H_1)-(H_3)$ and condition (3.4) hold and θ, σ, c are real constants with $\theta + \sigma \neq 0$. Suppose that there exists a real number $M > 0$ such that*

$$g(t, x_1) - g(t, x_2) \leq \frac{M}{1 + (\omega(t) - \omega(0))^{\beta+1}} \left(\frac{x_1}{f(t, x_1)} - \frac{x_2}{f(t, x_2)} \right) \ a.e\ t \in J$$

for all $x_1, x_2 \in \mathbb{R}$ with $x_2 \geq x_1$, where $M \leq \Gamma(1 + \beta)(\omega(t) - \omega(0))^{2\beta}$. Furthermore, if there exists a function $u \in C(J, \mathbb{R})$ such that

$$\begin{cases} {}^{C}D^{\beta,\omega}_{0^+} \left(\dfrac{v(t)}{f(t, v(t))} \right) \geq g(t, v(t)), \qquad t \in J = [0, T], \\ \\ \theta \dfrac{v(0)}{f(0, v(0))} + \sigma \dfrac{v(T)}{f(T, v(T))} > c. \end{cases} \tag{6.6}$$

then

$$\rho(t) \leq v(t) \tag{6.7}$$

for all $t \in J$, where ρ is a minimal solution of (1.1) on J.

Note that Theorem 6.1 is useful to prove the boundedness and uniqueness of the solutions for (1.1) on J. A result in this direction is as follows.

Theorem 6.3. *Assume that hypotheses $(H_1)-(H_3)$ and condition (3.4) hold and a, b, c are real constants with $a + b \neq 0$. Suppose that there exists a real number $M > 0$ such that*

$$g(t, x_1) - g(t, x_2) \leq \frac{M}{1 + (\omega(t) - \omega(0))^{\beta+1}} \left(\frac{x_1}{f(t, x_1)} - \frac{x_2}{f(t, x_2)} \right) \ a.e\ t \in J.$$

for all $x_1, x_2 \in \mathbb{R}$ with $x_2 \geq x_1$, where $M \leq \Gamma(1 + \beta)(\omega(t) - \omega(0))^{2\beta}$. If an identically zero function is the only solution of the differential equation

$$\begin{cases} {}^{C}D^{\beta,\omega}_{0^+} m(t) = \dfrac{M}{1 + (\omega(t) - \omega(0))^{\beta+1}} m(t), \qquad t \in J = [0, 1], \\ \\ \theta m(0) + \sigma m(T) = 0. \end{cases} \tag{6.8}$$

then (1.1) has a unique solution on J.

Proof. By Theorem 3.2, (1.1) has a solution defined on J. Suppose that there are two solutions u_1 and u_2 of (1.1) existing on J with $u_1 > u_2$. Define a function $m : J \longrightarrow \mathbb{R}$ by

$$m(t) = \frac{u_1(t)}{f(t, u_1(t))} - \frac{u_2(t)}{f(t, u_2(t))}$$

In view of hypothesis (H_1), we conclude that $m(t) > 0$. Then we have

$$\begin{aligned} {}^{C}D^{\beta,\omega}_{0^+} m(t) &= {}^{C}D^{\beta,\omega}_{0^+} \left(\frac{u_1(t)}{f(t, u_1(t))} - \frac{u_2(t)}{f(t, u_2(t))} \right) \\ &= {}^{C}D^{\beta,\omega}_{0^+} \left(\frac{u_1(t)}{f(t, u_1(t))} \right) - {}^{C}D^{\beta,\omega}_{0^+} \left(\frac{u_2(t)}{f(t, u_2(t))} \right), \\ &= g(t, u_1(t)) - g(t, u_2(t)), \\ &\leq \frac{M}{1 + (\omega(t) - \omega(0))^{\beta+1}} \left(\frac{x_1}{f(t, x_1)} - \frac{x_2}{f(t, x_2)} \right), \\ &= \frac{M}{1 + (\omega(t) - \omega(0))^{\beta+1}} m(t), \end{aligned}$$

for almost everywhere $t \in J$. Since

$$m(0) = \frac{u_1(0)}{f(0, u_1(0))} - \frac{u_2(0)}{f(0, u_2(0))}$$

and

$$m(T) = \frac{u_1(T)}{f(T, u_1(T))} - \frac{u_2(T)}{f(T, u_2(T))},$$

and

$$\theta \frac{u_1(0)}{f(0, u_1(0))} + \sigma \frac{u_1(T)}{f(T, u_1(T))} = \theta \frac{u_2(0)}{f(0, u_2(0))} + \sigma \frac{u_2(T)}{f(T, u_2(T))},$$

we have

$$\theta m(0) + \sigma m(T) = 0.$$

Now, we apply Theorem 6.1 with $f(t, x) = 0$ and $c = 0$ to get that $m(t) \leq 0$ for all $t \in J$, where an identically zero function is the only solution of the differential equation (6.8) $m(t) \leq 0$ is a contradiction with $m(t) > 0$. Then we can get $u_1 = u_2$. This completes the proof. $\square$

7. An illustrative example

In this section we give an example to illustrate our main result.
Consider the following hybrid fractional differential equation:

$$\begin{cases} {}^C D_{0^+}^{\frac{2}{3}, t^2} \left(\frac{u(t)}{f(t,u(t))} \right) = g(t, u(t)), \qquad t \in J = [0, 1], \\ \\ \frac{u(0)}{f(0,u(0)} + \frac{u(T)}{f(T,u(T)} = 0. \end{cases} \tag{7.1}$$

where $\beta = \frac{2}{3}$, $T = 1$, $\omega(t) = t$, $g(t, u(t)) = \dfrac{t^2}{12} sin(u(t))$ and
$f(t, u(t)) = \dfrac{e^{-t}}{9 + e^t} \left(\dfrac{|u(t)|}{1 + |u(t)|} \right)$.
It is clear that the assumption (H_1) is satisfied.
To prove the assumption (H_2), let $t \in J$ and $u, v \in \mathbb{R}$, then we have

$$|f(t, u(t)) - f(t, v(t))| = \left| \frac{e^{-t}}{9 + e^t} \left(\frac{|u(t)|}{1 + |u(t)|} \right) - \frac{e^{-t}}{9 + e^t} \left(\frac{|v(t)|}{1 + |v(t)|} \right) \right|,$$

$$|f(t, u(t)) - f(t, v(t))| \leq \left| \frac{e^{-t}}{9 + e^t} \right| \left| \left(\frac{|u(t)|}{1 + |u(t)|} \right) - \left(\frac{|v(t)|}{1 + |v(t)|} \right) \right|,$$

$$|f(t, u(t)) - f(t, v(t))| \leq \frac{1}{10} \left| \frac{u(t) - v(t)}{(1 + |u(t)|)(1 + |v(t)|)} \right|,$$

$$|f(t, u(t)) - f(t, v(t))| \leq \frac{1}{10} |u(t) - v(t)|,$$

Thus, the assumption (H_2) in holds true with $L = \frac{1}{10}$.

It remains to verify the assumption (H_3). Let $t \in J$ and $u \in \mathbb{R}$, then we have

$$|g(t, u(t))| = \left| \frac{t^2}{12} sin(u(t)) \right|,$$

$$|g(t, u(t))| \leq \frac{t^2}{12} |sin(u(t))|,$$

$$|g(t, u(t))| \leq \frac{t^2}{12}.$$

Wich implies that the assumption (H_3) is verified with $h(t) = \frac{t^2}{12}$.
Moreover, we have

$$\begin{aligned} \frac{3L}{2} \times \frac{(\omega(T) - \omega(0))^{\beta} \parallel h \parallel_{L^\infty}}{\Gamma(\beta+1)} &= \frac{3}{20} \times \left(\frac{(1-0)^{\frac{1}{2}} \times \frac{1}{36}}{\Gamma(\frac{5}{3})} \right) \\ &= \frac{1}{240} \times \frac{1}{\Gamma(\frac{5}{3})} \sim 0,027 < 1. \end{aligned}$$

Finally, all the conditions of Theorem 3.2 are satisfied, thus the hybrid fractional problem (7.1) has a solution on $[0, 1]$.

8. Conclusion

In this paper, we studied the existence results of hybrid fractional differential equations involving $\omega-$Caputo fractional derivative of order $0 < \beta < 1$. The existence theorems are proved by using some Lipschitz and Carathéodory conditions. As application, an example is presented to illustrate the applicability our main result.

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