Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–5.
ISSN-0037-8712 in press
doi:10.5269/bspm.62610

On Some Closed-form Evaluations for the Generalized Hypergeometric Function

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ABSTRACT: The main objective of this note is to provide eight closed-form evaluations for the generalized hypergeometric function with argument 1. This is achieved by separating a generalized hypergeometric function into even and odd components together with the use of several known sums involving ratios of binomial coefficients recently obtained by Sofo.

Key Words: Generalized hypergeometric function, central binomial coefficients, combinatorial sum.

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1. Introduction

The generalized hypergeometric function ${}_pF_q(z)$ with p numerator and q denominator parameters is defined by [7]

$$ {}_pF_q\left[\begin{array}{c} a_1, a_2, \dots, a_p \\ \\ b_1, b_2, \dots, b_q \end{array} ; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n (b_2)_n \dots (b_q)_n} \frac{z^n}{n!}, \tag{1.1} $$

where $(a)_n$ is the well-known Pochhammer's symbol defined by

$$ (a)_n = \begin{cases} a(a+1)\dots(a+n-1) & ; n \in \mathbb{N} \\ 1 & ; n = 0. \end{cases} $$

In terms of gamma function, we have

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. $$

Here, as usual the parameters a_j $(1 \le j \le p)$ and b_j $(1 \le j \le q)$ can have arbitrary complex values with zero or negative integer values of b_j excluded. The generalized hypergeometric function ${}_pF_q(z)$ converges for $|z| < \infty, (p \le q)$, $|z| < 1$ $(p = q+1)$ and $|z| = 1$ $(p = q+1$ and $Re(s) > 0)$, where s is the parametric excess defined by

$$ s = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j. $$

It is interesting to mention here that the generalized hypergeometric function occurs in many theoretical and practical applications such as mathematics, theoretical physics, engineering and statistics. For more details about this function, we refer the standard texts [1, 2, 4, 9].
On the other hand, the binomial coefficients are defined by

$$ \binom{n}{m} = \begin{cases} \dfrac{n!}{m!(n-m)!} & ; n \ge m, \\ 0 & ; n \le m, \end{cases} $$

2010 *Mathematics Subject Classification:* Primary: 05A15, 11B68, 33C20, Secondary: 30C05, 33C05, 33C15.
Submitted February 21, 2022. Published May 12, 2022

for non-negative integers n and m. It is well-known that the binomial and reciprocal of binomial coefficients play an important role in many areas of mathematics (including number theory, probability and statistics). A large number of very interesting results can be seen the research papers by Mansour [3], Pla [5], Rockett [8], Sofo [10, 11], Suri [12], Sury et al. [13], Trif [14] and Zhao and Wang [15]. However, in our present investigations, we are interested in the following results obtained by Sofo [10, 11]. These are

$$\sum_{n=0}^{\infty} \frac{\binom{n+\frac{7}{2}}{n}}{\binom{2n+9}{2n}} = \frac{9\pi}{2} - \frac{456}{35}, \tag{1.2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+\frac{7}{2}}{n}}{\binom{2n+9}{2n}} = 9\ln(\sqrt{2}+1) + \frac{2559\sqrt{2}}{35} - \frac{552}{5}, \tag{1.3}$$

$$\sum_{n=0}^{\infty} \frac{\binom{n+3}{n}}{\binom{3n+5}{3n}} = \frac{100\sqrt{3}\pi}{243} - \frac{10}{9}, \tag{1.4}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+3}{n}}{\binom{3n+5}{3n}} = \frac{10}{9} + \frac{40}{27}\ln 2 - \frac{160\sqrt{3}\pi}{729}. \tag{1.5}$$

$$\sum_{n=0}^{\infty} \frac{1}{\binom{6n+5}{5}} = \frac{40}{3}\ln 2 - \frac{15}{2}\ln 3. \tag{1.6}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{6n+5}{5}} = \frac{5\pi}{3}\left(\frac{5}{2} - \frac{4}{\sqrt{3}}\right). \tag{1.7}$$

$$\sum_{n=0}^{\infty} \frac{\binom{n+1}{1}}{\binom{\frac{3n}{2}+3}{3}} = 3\ln 3 + \frac{\sqrt{3}\pi}{9} - 2. \tag{1.8}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{n+1}{1}}{\binom{\frac{3n}{2}+3}{3}} = 2 - \frac{2\sqrt{3}\pi}{9}. \tag{1.9}$$

In terms of the generalized hypergeometric function, the results (1.2)-(1.9) can be written in the following manner.

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, 1, \frac{9}{2} \\ 5, \frac{11}{2} \end{matrix}; 1\right] = \frac{9\pi}{2} - \frac{456}{35}, \tag{1.10}$$

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, 1, \frac{9}{2} \\ 5, \frac{11}{2} \end{matrix}; -1\right] = 9\ln(\sqrt{2}+1) + \frac{2559\sqrt{2}}{35} - \frac{552}{5}, \tag{1.11}$$

$$ {}_4F_3\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, 1, 4 \\ 2, \frac{7}{3}, \frac{8}{3} \end{matrix} ; 1\right] = \frac{100\sqrt{3}\pi}{243} - \frac{10}{9}, \tag{1.12} $$

$$ {}_4F_3\left[\begin{matrix} \frac{1}{3}, \frac{2}{3}, 1, 4 \\ 2, \frac{7}{3}, \frac{8}{3} \end{matrix} ; -1\right] = \frac{10}{9} + \frac{40}{27}\ln 2 - \frac{160\sqrt{3}\pi}{729}, \tag{1.13} $$

$$ {}_6F_5\left[\begin{matrix} \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \\ \frac{7}{6}, \frac{3}{2}, \frac{5}{3}, \frac{5}{3}, \frac{11}{6} \end{matrix} ; 1\right] = \frac{40}{3}\ln 2 - \frac{15}{2}\ln 3, \tag{1.14} $$

$$ {}_6F_5\left[\begin{matrix} \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \\ \frac{7}{6}, \frac{3}{2}, \frac{5}{3}, \frac{5}{3}, \frac{11}{6} \end{matrix} ; -1\right] = \frac{5\pi}{3}\left(\frac{5}{2} - \frac{4}{\sqrt{3}}\right), \tag{1.15} $$

$$ {}_4F_3\left[\begin{matrix} \frac{2}{3}, \frac{4}{3}, 2, 2 \\ \frac{5}{3}, \frac{7}{3}, 3 \end{matrix} ; 1\right] = 3\ln 3 + \frac{\sqrt{3}\pi}{9} - 2, \tag{1.16} $$

$$ {}_4F_3\left[\begin{matrix} \frac{2}{3}, \frac{4}{3}, 2, 2 \\ \frac{5}{3}, \frac{7}{3}, 3 \end{matrix} ; -1\right] = 2 - \frac{2\sqrt{3}\pi}{9}. \tag{1.17} $$

Further, it is well-known that the process of resolving a generalized hypergeometric function ${}_pF_q(z)$ into even and odd components can lead to new results. We shall employ this procedure combined with the results (1.10) to (1.17) in Section 2 to obtain eight new closed-form evaluations of the series ${}_4F_3(1)$, ${}_6F_5(1)$ and ${}_7F_6(1)$.

2. Closed-form evaluations

In this section, we shall establish the following eight new closed-form evaluations for the generalized hypergeometric function ${}_4F_3(1)$, ${}_6F_5(1)$ and ${}_7F_6(1)$.

$$ {}_4F_3\left[\begin{matrix} \frac{1}{4}, \frac{3}{4}, 1, \frac{9}{4} \\ \frac{5}{2}, 3, \frac{13}{4} \end{matrix} ; 1\right] = \frac{1}{2}\left(\frac{9\pi}{2} + 9\ln(\sqrt{2}+1) + \frac{2559\sqrt{2}}{35} - \frac{864}{7}\right), \tag{2.1} $$

$$ {}_4F_3\left[\begin{matrix} \frac{3}{4}, 1, \frac{5}{4}, \frac{11}{4} \\ 3, \frac{7}{2}, \frac{15}{4} \end{matrix} ; 1\right] = \frac{55}{9}\left(\frac{9\pi}{2} - 9\ln(1+\sqrt{2}) - \frac{2559\sqrt{2}}{35} + \frac{3408}{35}\right), \tag{2.2} $$

$$ {}_6F_5\left[\begin{matrix} \frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}, 2, \frac{5}{2} \\ \frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{11}{6} \end{matrix} ; 1\right] = \frac{10}{27}\left(\frac{7\sqrt{3}\pi}{27} + 2\ln 2\right), \tag{2.3} $$

$$ {}_7F_6\left[\begin{matrix}\frac{2}{3},\frac{5}{6},1,\frac{4}{3},\frac{7}{6},\frac{5}{2},3\\ 2,\frac{3}{2},\frac{5}{3},\frac{11}{6},\frac{7}{3},\frac{13}{6}\end{matrix};1\right]=\frac{70}{9}\left(\frac{46\sqrt{3}\pi}{81}-2-\frac{4}{3}\ln 2\right), \tag{2.4}$$

$$ {}_7F_6\left[\begin{matrix}\frac{1}{12},\frac{1}{6},\frac{1}{4},\frac{1}{3},\frac{5}{12},\frac{2}{3},1\\ \frac{5}{6},\frac{13}{12},\frac{5}{4},\frac{4}{3},\frac{4}{3},\frac{17}{12}\end{matrix};1\right]=\frac{1}{2}\left(\frac{40}{3}\ln 2-\frac{15}{2}\ln 3+\frac{5\pi}{3}\left(\frac{5}{2}-\frac{4}{\sqrt{3}}\right)\right), \tag{2.5}$$

$$ {}_7F_6\left[\begin{matrix}\frac{7}{12},\frac{2}{3},\frac{3}{4},\frac{5}{6},\frac{11}{12},1,\frac{7}{6}\\ \frac{4}{3},\frac{19}{12},\frac{7}{4},\frac{11}{6},\frac{11}{6},\frac{23}{12}\end{matrix};1\right]=\frac{1155}{4}\left(\frac{40}{3}\ln 2-\frac{15}{2}\ln 3-\frac{5\pi}{3}\left(\frac{5}{2}-\frac{4}{\sqrt{3}}\right)\right), \tag{2.6}$$

$$ {}_5F_4\left[\begin{matrix}\frac{1}{3},\frac{2}{3},1,1,\frac{3}{2}\\ \frac{1}{2},\frac{4}{3},\frac{5}{3},2\end{matrix};1\right]=\frac{1}{2}\left(3\ln 3-\frac{\sqrt{3}\pi}{9}\right), \tag{2.7}$$

$$ {}_4F_3\left[\begin{matrix}\frac{5}{6},\frac{7}{6},2,\frac{3}{2}\\ \frac{11}{6},\frac{13}{6},\frac{5}{2}\end{matrix};1\right]=\frac{105}{64}\left(3\ln 3+\frac{\sqrt{3}\pi}{3}-4\right). \tag{2.8}$$

Proof. In order to establish the results (2.1)-(2.8), we shall use the following general results recorded in [6, p. 441].

$$ {}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_{q+1}\\ b_1,b_2,\ldots,b_q\end{matrix};1\right]+{}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_{q+1}\\ b_1,b_2,\ldots,b_q\end{matrix};-1\right]$$
$$ =2\,{}_{2q+2}F_{2q+1}\left[\begin{matrix}\frac{a_1}{2},\frac{a_1}{2}+\frac{1}{2},\ldots,\frac{a_{q+1}}{2},\frac{a_{q+1}}{2}+\frac{1}{2}\\ \frac{1}{2},\frac{b_1}{2},\frac{b_1}{2}+\frac{1}{2},\ldots,\frac{b_q}{2},\frac{b_q}{2}+\frac{1}{2}\end{matrix};1\right] \tag{2.9}$$

and

$$ {}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_{q+1}\\ b_1,b_2,\ldots,b_q\end{matrix};1\right]-{}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_{q+1}\\ b_1,b_2,\ldots,b_q\end{matrix};-1\right]$$
$$ =\frac{2a_1a_2\ldots a_{q+1}}{b_1b_2\ldots b_q}\,{}_{2q+2}F_{2q+1}\left[\begin{matrix}\frac{a_1}{2}+\frac{1}{2},\frac{a_1}{2}+1,\ldots,\frac{a_{q+1}}{2}+\frac{1}{2},\frac{a_{q+1}}{2}+1\\ \frac{3}{2},\frac{b_1}{2}+\frac{1}{2},\frac{b_1}{2}+1,\ldots,\frac{b_{q+1}}{2}+\frac{1}{2},\frac{b_q}{2}+1\end{matrix};1\right] \tag{2.10}$$

The results (2.9) and (2.10) can be established by resolving the generalized hypergeometric function

$$ {}_{q+1}F_q\left[\begin{matrix}a_1,a_2,\ldots,a_{q+1}\\ b_1,b_2,\ldots,b_q\end{matrix};1\right]$$

into even and odd components and making use of the following identities:

$$ (a)_{2n}=2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a}{2}+\frac{1}{2}\right)_n$$

and

$$ (a)_{2n+1}=a2^{2n}\left(\frac{a}{2}+\frac{1}{2}\right)_n\left(\frac{a}{2}+1\right)_n$$

Therefore, for the derivation of the results (2.1) and (2.2), we substitute the results (1.10) and (1.11) by letting $q = 2$ and substituting $a_1 = 1/2, a_2 = 1$, $a_3 = 9/2$, $b_1 = 5$ and $b_2 = 11/2$ in (2.9) and (2.10) respectively, and after some simplification, we obtain the results (2.1) and (2.2).

Similarly, other results (2.3) to (2.8) can be established by choosing the appropriate parameters and making use of the results (1.12) to (1.17) respectively. We omit these details. □

3. Concluding remark

In this short note, we have provided eight closed-form evaluations for the generalized hypergeometric function with argument 1 with the help of the results obtained earlier by Sofo. The results established have been verified numerically using MAPLE.

Acknowledgment

The authors thank anonymous referees for valuable suggestions and comments.

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