Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–7.
ISSN-0037-8712 in press
doi:10.5269/bspm.62192

Imbeddedness and Direct Sum of Uniserial Modules

Ayazul Hasan, Rafiquddin and Mohd Noman Ali

abstract: In this paper, we study a generalization of h-pure submodules as well as some other closely related concepts. Here, we examine the extent of this generalization in several ways. We then use this to give a characterization of the imbedded-complete modules. It is found that imbeddedness can considerably more abundant than h-purity on direct sum of uniserial modules.

Key Words: QTAG-modules, imbedded-complete modules, ℓ-imbedded submodules.

Contents

1. Introduction and backgrounds

Let R be any ring with unity. A uniserial module M is a module over a ring R, whose submodules are totally ordered by inclusion. This means simply that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A module M is called a serial module if it is a direct sum of uniserial modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length.

Modules are the natural generalizations of abelian groups. Many authors interested in module theory have worked on generalizing the theory of abelian groups. In fact, the theory of modules is highly motivated by abelian groups. The results which hold good for abelian groups need not be true for modules. By putting some restrictions on rings/modules these results hold good for modules too. In 1976 Singh [14] introduced a class of modules called TAG-modules, defined by satisfying two properties relating to uniserial modules while the rings were associative with unity.

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

It was shown that the theory of these modules very closely paralleled the theory of torsion abelian groups; for this reason they were referred to as TAG-modules. In 1987 Singh showed that the second property, with minimal additional hypotheses, can be deduced from the first and studied the modules satisfying only the first property and called them $QTAG$-modules. The study of $QTAG$-modules and their structure began with work of Singh in [15]. This work, executed by many authors, clearly parallels the earlier work on torsion abelian groups. This is a very fascinating structure that has been the subject of research of many authors. Different notions and structures of $QTAG$-modules have been studied, and a theory was developed, introducing several notions, interesting properties, and different characterizations of submodules. Many interesting results have been obtained, but still there remains a lot to explore.

2010 *Mathematics Subject Classification:* 16K20, 13C12, 13C13.
Submitted January 21, 2022. Published July 01, 2022

All rings below are assumed to be associative and with nonzero identity element; all modules are assumed to be unital $QTAG$-modules. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. For $k \geq 0$, $H_k(M) = \{x \in M \mid H_M(x) \geq k\}$ denotes the submodule of M generated by the elements of height at least k and for some submodule N of M, $H^k(M) = \{x \in M \mid d(xR/(xR \cap N)) \leq k\}$ is the submodule of M generated by the elements of exponents at most k. The module M is said to be bounded, if there exists an integer k such that $H_M(x) \leq k$ for every uniform element $x \in M$.

Let us denote by M^1, the submodule of M, containing elements of infinite height. The module M is called separable if $M^1 = 0$. The module M is h-divisible if $M = M^1 = \cap_{k=0}^{\infty} H_k(M)$. A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A submodule N of M is h-neat in M if $N \cap H_1(M) = H_1(N)$. The minimal h-neat submodule K of M containing N is called h-neat hull of N. For these concepts and related results, we refer the readers to [8].

The sum of all simple submodules of M is called the socle of M, denoted by $Soc(M)$ and a submodule S of $Soc(M)$ is called a subsocle of M. For any $k \geq 0$, $Soc^k(M)$ is defined inductively as follows: $Soc^0(M) = 0$ and $Soc^{k+1}(M)/Soc^k(M) = Soc(M/Soc^k(M))$. A submodule N of M is K-high [11] in M, if it is maximal with the property of being disjoint from K. It is well-known that all K-high submodules of M are bounded if and only if there exists $k \in Z^+$ such that $(K + Soc(H_k(M)))/K$ is finitely generated and K contains the socle of the h-divisible submodule of M.

Imitating [12], the submodules $H_k(M), k \geq 0$ form a neighborhood system of zero, thus a topology known as h-topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as $\overline{N} = \cap_{k=0}^{\infty}(N + H_k(M))$. Therefore, the submodule $N \subseteq M$ is closed with respect to the h-topology if $\overline{N} = N$.

Mehran et al. [13] proved that the results which hold for TAG-modules are also valid for $QTAG$-modules. Many results, stated in the present paper, are clearly motivated from the papers [6,7]. Most of our notations and terminology will be standard being in agreement with [2] and [3].

2. ℓ-imbedding and h-purity

A submodule N of a $QTAG$-module M is called imbedded if there exists a function $\ell : Z^+ \to Z^+$ such that $N \cap H_{\ell(k)}(M) \subseteq H_k(N)$ for each $k \in Z^+$. Here, ℓ is called an imbedding function for N in M. Let ℓ be an imbedding function for N in M then N is called ℓ-imbedded submodule of M. Trivially, ℓ-imbedded submodules are exactly h-pure submodules. These ℓ-imbedded submodules were originally defined and carefully explored in [9]. But there are some other closely related concepts which have been of interest: recall that a $QTAG$-module M is called ℓ-quasi-complete if the closure $\overline{N}$ of every ℓ-imbedded submodule N of M, is an imbedded submodule of M; while an ℓ-imbedded submodule N of the $QTAG$-module M is said to be an ℓ-hull (or minimal ℓ-imbedding) of a submodule T in M if N is a minimal ℓ-imbedded submodule of M containing T. The basic properties of these concepts and their interrelationships were explored, and in particular, ℓ-quasi-complete modules were characterized according to the existence of a minimal ℓ-imbedded submodule. Our purpose in this article is to continue the exploration of these ℓ-imbedded submodules, and show that they differ significantly from h-pure submodules. It is worthwhile noticing that some of the results in this direction are already announced in [10].

To develop the study, we defining the following.

Definition 2.1. *A submodule T of a $QTAG$-module M is ℓ-dense in M if M/N is h-divisible for some ℓ-imbedded submodule N of M containing T.*

Now we prove the following proposition.

Proposition 2.2. *Let N be an ℓ-imbedded submodule of a $QTAG$-module M such that $Soc(H_k(M)) \subset N$ for some $k \in Z^+$. Then $H_{\ell(k+1)-1}(M) \subset N$.*

Proof: Clearly, $Soc(H_{\ell(k+1)-1}(M)) \subset Soc(H_k(M)) \subset N$, and thus $Soc^n(H_{\ell(k+1)-1}(M)) \subset N$, for some $n \in Z^+$. Let x be any uniform element in M such that $d(xR/x'R) = n+1$ and $d(x'R) = 1$. Let $y \in M$ such that $d(yR/xR) = \ell(k+1)-1$. Then $x' \in N$ such that $d(xR/x'R) = 1$. Therefore, $x' = y' = z'$, where $d(xR/x'R) = 1$, $d(yR/y'R) = \ell(k+1)$ and $d(zR/z'R) = k$ for some $z \in N$. Hence $x - z' \in Soc(H_k(M)) \subset N$ such that $d(zR/z'R) = k$. Therefore, $x \in N$ and we are done. □

Next, we concentrate on the following theorem.

Theorem 2.3. *Let T be a submodule of a $QTAG$-module M. Then T is contained in no proper ℓ-imbedded submodule of M if and only if T is ℓ-dense in M, and $Soc(H_k(M)) \subset T$, for some $k \in Z^+$.*

Proof: Suppose T is ℓ-dense, $Soc(H_k(M)) \subset T$, and N be an ℓ-imbedded submodule N of M containing T. Then $H_{\ell(k+1)-1}(M) \subset N$, so that M/N is bounded and h-divisible, and $N = M$. Clearly, if no proper ℓ-imbedded submodule contains T, T is ℓ-dense. Since all h-pure submodules are exactly ℓ-imbedded, $Soc(H_k(M)) \subset T$ for some $k \in Z^+$. The proof is over. □

It is interesting to note that all ℓ-imbedded submodules are h-neat and thus, the h-neat hulls of an ℓ-imbedded submodule are ℓ-imbedded. So, we characterize the ℓ-dense submodules of a $QTAG$-module.

Theorem 2.4. *Let T be a submodule of a $QTAG$-module M. Then T is ℓ-dense in M if and only if $Soc(H_k(M)) \subset T + H_{\ell(k+2)-1}(M)$, for some $k \in Z^+$.*

Proof: Suppose T satisfies the condition, and let N be an ℓ-imbedded submodule of M containing T such that $x + N \in Soc(M/N)$. Since N is h-neat in M, we may assume that $x \in Soc(H_k(M))$ for some k. Then $x + y' \in T$ such that $d(yR/y'R) = \ell(k+2)-1$ for some $y \in M$, and $x + N \in H_{\ell(k+2)-1}(M/N)$. Suppose $x + N \in H_{\ell(n)-1}(M/N)$, and let $x + N = z' + N$ such that $d(zR/z'R) = \ell(n)-1$. Then $z' \in N$ such that $d(zR/z'R) = \ell(n)$, so $z' = a'$ such that $d(zR/z'R) = \ell(n)$ and $d(aR/a'R) = n$ for some $a \in N$.

Now, $z' - a' \in Soc(H_{n-1}(M))$ where $d(zR/z'R) = \ell(n)-1$ and $d(aR/a'R) = n-1$, so that $z' - a' = b' + c$ where $d(zR/z'R) = \ell(n)-1$, $d(aR/a'R) = n-1$ and $d(bR/b'R) = \ell(n+1)-1$ for some $b \in M$, $c \in T$. Therefore $(z' - x) - a' = b' - x + c \in N$ where $d(zR/z'R) = \ell(n)-1$, $d(aR/a'R) = n-1$ and $d(bR/b'R) = \ell(n+1)-1$. Thus, $H_{M/N}(x+N) \geq \ell(n+1)-1$. But $x + N \in H_\omega(M/N)$ and $x + N$ is arbitrary, hence M/N must be h-divisible, as promised.

Conversely, suppose $Soc(H_k(M)) \nsubseteq T + H_{\ell(k+2)-1}(M)$, and let $Soc(H_k(M)) = U \oplus V$, where $U = Soc(H_k(M)) \cap H_{\ell(k+2)-1}(M)$. Consider a submodule L of M such that $L/H_{\ell(k+2)-1}(M)$ is W-high in M, where $W = (U \oplus H_{\ell(k+2)-1}(M))/H_{\ell(k+2)-1}(M)$. By [4, Theorem 2.1], L is U-high in M. Since $U \subset H_k(M)$, L is $(k+1)$-pure in M. Note that if $x \in L \cap H_{\ell(k+r)}(M)$, for $r > 1$ and L is ℓ-imbedded in M. Then $H_{\ell(k+2)-1}(M)) \subset L$ and $x = y'$ where $d(yR/y'R) = r-1$, for some $y \in L \cap H_{k+1}(M)$. Hence $y = w'$ such that $d(wR/w'R) = k+1$, $w \in L$ and $x = H_{r-1}(w'R) = H_{k+r}(wR)$ where $d(wR/w'R) = k+1$. Thus, by [4, Theorem 2.3], L/T is bounded and so, L/T is not h-divisible, to get the claim. □

Definition 2.5. *Let T be a submodule of a $QTAG$-module M. An ℓ-imbedded submodule N of M is said to be an h-neat-imbedded hull of T if $L = N$, for some ℓ-imbedded submodule L of M containing T.*

And so, we will verify the validity of the following theorem.

Theorem 2.6. *Let T be a submodule of a $QTAG$-module M and N be an ℓ-imbedded submodule of M containing T. Then N is an h-neat-imbedded hull of T if and only if $Soc(H_k(N)) \subset T$, for some k, and $Soc(N) \subset \overline{T}$, where $\overline{T}$ is the closure of T in the h-topology.*

Proof: As we have noted earlier, N is an ℓ-hull of T in M if and only if $Soc(N) \subset T + H_k(M)$, for all k, hence $Soc(N) \subset \overline{T}$, as claimed. This ends the proof. □

The study of subsocles is an important part of the theory of $QTAG$-modules. As defined in [4], a subsocle S of a $QTAG$-module M is called h-purifiable in M if and only if there is an h-pure submodule N of M such that $Soc(N) = S$. It is evident that a subsocle S supports the submodule N of M if and only if $Soc(N) = S$.

We continue the study with the following corollary.

Corollary 2.7. *Let S be a subsocle of the QTAG-module M such that S is contained in an h-neat-imbedded hull N. Then S is h-purifiable in M.*

Proof: If $a \in Soc(N)$, then there exists $b \in S$ such that $a+b = c'$ where $d(cR/c'R) = \ell(k)$ for some $c \in M$ and $k \in Z^+$. Thus $z' \in Soc(N) \cap H_{\ell(k)}(M) \subset Soc(H_k(N)) \subset S$. Therefore, $a \in S$, and $Soc(N) = S$. □

Following [9], a submodule N of a $QTAG$-module M is called regularly imbedded in M with index k, if $N \cap H_{k+r}(M) \subseteq H_r(N \cap H_k(M))$ for all non-negative integer r. Evidently, if N is regularly imbedded with index k, then $N \cap H_{k+r}(M) \subseteq H_r(N)$ gives that the regularly imbedded submodules are ℓ-imbedded for some $\ell : Z^+ \to Z^+$. Moreover, the regularly imbedded submodules of index zero are exactly the h-pure submodules.

Now we are ready to deal with the following theorem.

Theorem 2.8. *Let M be a separable QTAG-module, and S be a subsocle of M. The following are equivalent:*

(*i*) *S supports a regularly imbedded submodule of M;*

(*ii*) *ℓ-imbeddedness is S-high in M;*

(*iii*) *S is h-purifiable in M.*

Proof: $(i) \Rightarrow (ii)$. Let R be a regularly imbedded submodule of M, then $Soc(R) = S$. Thus, by [10, Theorem 5.5], there is an h-pure submodule N of M such that $H_k(N) \subset R \subset N$, for some k. Consequently, $Soc(H_k(N)) \subset S$.

$(ii) \Rightarrow (iii)$. If N is h-pure in M and $Soc(H_k(N)) \subset S$, then all S-high submodules of N are bounded. Thus, by [4, Theorem 2.2], S supports an h-pure submodule of N, which is then h-pure in M.

The implication $(iii) \Rightarrow (i)$ is obvious. □

Recall that a $QTAG$-module M is called h-pure-complete, if for every subsocle of M supports an h-pure submodule of M. It is easy to see that M is h-pure-complete if and only if $H_k(M)$ is h-pure-complete for every $k \in Z^+$.

Analogous to h-pure-complete module, we will now study a little different module class.

Definition 2.9. *A separable QTAG-module M is called imbedded-complete, if for every subsocle of M supports an imbedded submodule of M.*

So, we are ready to formulate the following.

Theorem 2.10. *Let M be an imbedded-complete QTAG-modules. Then M properly contains the h-pure-complete modules.*

Proof: Let $U_n = \langle u_n \rangle$ be a uniserial module of exponent $2n$. Set $P = \bigoplus_1 \langle u_n \rangle$, $Q = \bigoplus_2 \langle u_n \rangle$, where 1 and 2 denote summation over the odd and even integers. Define $X = \prod_1 \langle u_n \rangle$, $Y = \prod_2 \langle u_n \rangle$, and write $K = P \oplus Y$, for some submodule K of M. Assume that N is an ℓ-imbedded submodule of M such that $M = \langle N, Q \rangle$, where $\ell(n) = n + 1$.

Let S be a subsocle of M. Then $Soc(M) = Soc(P) \oplus Soc(Y)$, and consequently, $S = (Soc(Y) \cap S) \oplus K$. Now N is closed, and hence h-pure-complete, so $Soc(Y) \cap S$ supports an h-pure pure submodule L of N, which is then imbedded in M. Also, $(K + L)/L \subset Soc(M/L)$, and $(K + L)/L \simeq K$ is countably generated. From the h-pure-completeness of M/L, there is an h-pure submodule T/L of M/L such that $S = Soc(T/L)$. Thus, $T \subset M$ and $S = Soc(T)$. If $a \in Soc(T)$, then $a + L \in Soc(T/L)$, so $a + L = b + L$, for some $b \in K$, and $a - b \in Soc(L) = Soc(Y) \cap S$. Therefore, $a = b + c$, some $c \in Soc(Y) \cap S$ and $a \in S$, as required. □

The following example demonstrates that the sum of minimal submodules of $H_k(M)$ is not imbedded-complete.

Example: There is an imbedded-complete module M and $k \in Z^+$ such that $Soc(H_k(M))$ is not imbedded-complete.

Let M be any imbedded-complete module such that $d(H_k(M)) = 1$ and $M/H_k(M)$ is a direct sum of uniserial modules. Let N be the closed, and hence h-pure-complete in M. It is readily checked that M/N supports an h-pure submodule L of N such that $M/N = Soc(L)$. In fact, $H_k(M/N) = 0$, so that $H_k(M)$ remains imbedded submodule of M. Now, for any uniform element $x \in Soc(H_k(M^1))$, we get $x \in H_k(M^1)$. It follows that $Soc(M^1) = Soc(H_k(M^1))$, and hence M is imbedded-complete.

On the other hand, $d(H_k(Soc(H_k(M))) = 1$. Therefore, $Soc(H_k(M))$ is not imbedded in M, and consequently, it is not imbedded-complete. We are finished.

We come now to a significant characterization of the imbedded-complete module in terms of minimal imbedded submodules.

Theorem 2.11. *Let S be a subsocle of the $QTAG$-module M. Then M is imbedded-complete if and only if S is contained in an h-neat-imbedded hull.*

Proof: The sufficiency follows directly from Corollary 2.7.

As for the necessity, since M is imbedded-complete, and $S \subset Soc(M)$, then $S = Soc(N)$, for some ℓ-imbedded submodule N of M. Let L be an h-neat hull of N. Then $S = Soc(L)$, and L is an ℓ-imbedded in M. Letting K be the h-neat hull in M such that $S \subset K \subset L$, we get $K = L$. Finally, the definition formulated allows us to conclude that N is indeed an h-neat-imbedded hull of S, as desired. □

3. Direct sum of uniserial modules

The class of $QTAG$-modules over a ring R need not be closed under direct sums of uniserial modules. It is well-known by [14] that a $QTAG$-module M is a direct sum of uniserial modules if and only if M is the union of an ascending chain of bounded submodules. This indicates that M is a direct sum of uniserial modules if and only if $Soc(M) = \bigoplus_{k\in\omega} S_k$ and $H_M(x) = k$ for every $x \in S_k$.

Likewise, a theorem of [1] states the problem of detecting finite direct sums of uniserial modules. Recently [5], some new achievements in this theme for other important sorts of $QTAG$-modules are established, which possess the following property: M/S is a direct sum of uniserial modules such that $S = Soc(N)$ for some h-pure submodules N of M, then M is a direct sum of uniserial modules and N is a direct summand of M.

We now strengthen the idea of direct sums of uniserial modules, and see that the assumption of imbeddedness can replace that of h-purity in certain circumstances.

Theorem 3.1. *Let T be a submodule of the $QTAG$-module M such that M/T is a direct sum of uniserial modules. If N is an ℓ-imbedded submodule of M containing T such that $Soc(H_k(N)) \subset T$, for some $k \in Z^+$. Then M/N is a direct sum of uniserial modules.*

Proof: Consider the canonical homomorphism

$$f : (Soc(H_{\ell(k)}(M)) + T)/T \to (Soc(H_{\ell(k)}(M)) + N)/N$$

Let $x \in Soc(H_{\ell(k)}(M))$ be any uniform element such that $H_{M/N}(x + N) \geq \ell(n + 2)$ for some $n \geq \ell(k)$. Then $x + N = y' + N$ where $d(yR/y'R) = \ell(n+2)$ and $y \in M$. Therefore, $y' \in N$ such that $d(yR/y'R) = \ell(n+2)+1$. Since N is ℓ-imbedded in M, $y' = z'$ where $d(yR/y'R) = \ell(n+2)+1$ and $d(zR/z'R) = n+2$, for $z \in N$. Thus $y' - z' \in Soc(H_{n+1}(M))$ where $d(yR/y'R) = \ell(n + 2)$ and $d(zR/z'R) = n + 1$. Let now u be an uniform element of $y' + N$ such that $d(yR/y'R) = \ell(n + 2)$. Then $y' + N \in Soc(H_{n+1}(M)) \subset Soc(H_{\ell(k)}(M))$. Thus we obtain: $x - u \in N \cap Soc(H_{\ell(k)}(M)) \subset Soc(H_k(M)) \subset T$. It follows that $x + T = u + T$, and therefore, $H_{M/T}(x + T) \geq n + 1$.

Now since $(Soc(H_{\ell(k)}(M)) + T)/T$ is the union of an ascending chain of bounded submodules, so is $(Soc(H_{\ell(k)}(M)) + N)/N$.

But

$$Soc(H_{\ell(\ell(k)+1)}(M/N)) \subset (Soc(H_{\ell(k)}(M)) + N)/N,$$

therefore, $H_{\ell(\ell(k)+1)}(M/N)$ is a direct sum of uniserial modules, and hence M/N is a direct sum of uniserial modules, as stated. The proof of the theorem is completed. □

As a consequence, we have the following.

Corollary 3.2. *Let T be a submodule of the $QTAG$-module M such that M/T is a direct sum of uniserial modules. If N is a minimal ℓ-imbedded submodule of M containing T. Then M/N is a direct sum of uniserial modules.*

Proof: It relies on the same idea as in Theorem 2.3 and 3.1. □

The above assertion can be extended to the following.

Corollary 3.3. *Let T be a submodule of the $QTAG$-module M such that $T \subset Soc(M)$ and M/T is a direct sum of uniserial modules. If T supports an ℓ-imbedded submodule N of M. Then M/N is a direct sum of uniserial modules.*

In particular, if L is an h-neat hull of N, then M/L is a direct sum of uniserial modules.

Proof: The first part follows since $T = Soc(N)$.

As for the second part, since L is an h-neat hull of N, and N is ℓ-imbedded in M, then L is also ℓ-imbedded in M, and $T = Soc(L)$. Thus, M/L is a direct sum of uniserial modules, as needed. □

4. Open problems

In the last we would like to state some open problems which are yet to be explored.

Problem 1. Is a direct summand of an imbedded-complete module again an imbedded-complete module?

Problem 2. Suppose M is a $QTAG$-module with M/M^1 a direct sum of of uniserial modules. Does it follow that $(M/N)^1$ a direct sum of of uniserial modules, whenever N is ℓ-imbedded in M?

Problem 3. If M is a direct sum of uniserial modules. What are the conditions under which any ℓ-imbedded submodule between M and $H_k(M)$ is uniserial?

Acknowledgments

The authors are grateful to the referees for the valuable comments, and to the Editor, for the expert editorial advice.

References

1. Facchini A. and Salce L., *Uniserial modules: sums and isomorphisms of subquotients,* Comm. Algebra, **18**(2), 499-517, (1990).
2. Fuchs L., *Infinite Abelian Groups,* Volume I, Pure Appl. Math. 36, Academic Press, New York, 1970.
3. Fuchs L., *Infinite Abelian Groups,* Volume II, Pure Appl. Math. 36, Academic Press, New York, 1973.
4. Hasan A., *On quasi-large submodules of QTAG-modules,* Asian-Europen J. Math., **9**(2), 1650033(1-8), (2016).
5. Hasan A., *Subsocles and direct sum of uniserial modules,* J. Algebra Comb. Discrete Appl., **8**(3), 213-218, (2021).
6. Keane W.J., *Imbedded N-high subgroups of abelian groups,* Comment. Math. Univ. St. Pauli, **29**(2), 145-155, (1980).
7. Keane W.J., *On minimal imbedded subgroups of abelian p-groups,* Comment. Math. Univ. St. Pauli, **31**(2), 155-158, (1982).
8. Khan M.Z., *On h-purity in QTAG-modules,* Comm. Algebra, **16**(12), 2649-2660, (1988).
9. Khan M.Z., Ahmad M. and Ansari A.H., *On subsocles of S_2-modules-II,* Kyungpook Math. J., **31**(2), 207-217, (1991).
10. Khan M.Z. and Rafiquddin, *Imbedded submodules,* Aligarh Bull. Math., **16**, 1-8, (1995).
11. Mehdi A. and Mehdi F., *N-high submodules and h-topology,* South East Asian J. Math & Math Sci, **1**(1), 83–85, (2002).

12. Mehdi A., Naji S.A.R.K. and Hasan A., *Small homomorphisms and large submodules of QTAG-modules,* Sci. Ser. A. Math Sci., **23**, 19-24, (2012).

13. Mehran H.A. and Singh S., *On σ-pure submodules of QTAG-modules,* Arch. Math., **46**(6), 501-510, (1986).

14. Singh S., *Some decomposition theorems in abelian groups and their generalizations,* Ring Theory: Proceedings of Ohio University Conference, Marcel Dekker, New York **25**, 183-189, (1976).

15. Singh S., *Abelian groups like modules,* Act. Math. Hung, **50**, 85-95, (1987).

Ayazul Hasan,
College of Applied Industrial Technology,
Jazan University, Jazan,
Kingdom of Saudi Arabia.
E-mail address: `ayazulh@jazanu.edu.sa`

and

Rafiquddin,
Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University, Aligarh-202002,
India.
E-mail address: `rafiqamt786@rediffmail.com`

and

Mohd Noman Ali,
College of Applied Industrial Technology,
Jazan University, Jazan,
Kingdom of Saudi Arabia.
E-mail address: `mmdali@jazanu.edu.sa`