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On some realizable metabelian 5-groups

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ABSTRACT: Let G be a 5-group of maximal class and $\gamma_2(G) = [G, G]$ its derived group. Assume that the abelianization $G/\gamma_2(G)$ is of type $(5, 5)$ and the transfers $V_{H_1\to\gamma_2(G)}$ and $V_{H_2\to\gamma_2(G)}$ are trivial, where H_1 and H_2 are two maximal normal subgroups of G. Then G is completely determined with the isomorphism class groups of maximal class. Moreover the group G is realizable with some fields k, which is the normal closure of a pure quintic field.

Key Words: Groups of maximal class, metabelian 5-groups, transfer, 5-class groups.

Contents

1. Introduction

The coclass of a p-group G of order p^n and nilpotency class c is defined as $cc(G) = n-c$, and a p-group G is called of maximal class, if it has $cc(G) = 1$. These groups have been studied by various authors, by determining there classification, the position in coclass graph [6] [3], and the realization of these groups. Blackburn's paper [2], is considered as reference of the basic materials about these groups of maximal class. Eick and Leendhan-Green in [6] gave a classification of 2-groups. Blackburn's classification in [2], of the 3-groups of coclass 1, implies that these groups exhibit behaviour similar to that proved for 2-groups. The 5-groups of maximal class have been investigated in detail in [3], [4], [5], [9], [14].
Let G be a metabelian p-group of order p^n, $n \geq 3$, with abelianization $G/\gamma_2(G)$ is of type (p, p), where $\gamma_2(G) = [G, G]$ is the commutator group of G. The subgroup G^p of G, generated by the p^{th} powers is contained in $\gamma_2(G)$, which therefore coincides with the Frattini subgroups $\phi(G) = G^p\gamma_2(G) = \gamma_2(G)$. According to the basis theorem of Burnside [[1], Theorem 1.12], the group G can thus be generated by two elements x and y, $G =< x, y >$. If we declare the lower central series of G recursively by

$$\begin{cases} \gamma_1(G) = G \\ \gamma_j(G) = [\gamma_{j-1}(G), G] \text{ for } j \geq 2, \end{cases}$$

Then we have Kaloujnine's commutator relation $[\gamma_j(G), \gamma_l(G)] \subseteq \gamma_{j+l}(G)$, for $j, l \geq 1$ [[2], Corollary 2], and for an index of nilpotence $c \geq 2$ the series

$$G = \gamma_1(G) \supset \gamma_2(G) \supset ..... \supset \gamma_{c-1}(G) \supset \gamma_c(G) = 1$$

becomes stationary.
The two-step centralizer

$$\chi_2(G) = \{g \in G \,|\, [g, u] \in \gamma_4(G) \text{for all u} \in \gamma_2(G)\}$$

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of the two-step factor group $\gamma_2(G)/\gamma_4(G)$, that is the largest subgroup of G such that $[\chi_2(G),\gamma_2(G)]\subset\gamma_4(G)$. It is characteristic, contains the commutator subgroup $\gamma_2(G)$. Moreover $\chi_2(G)$ coincides with G if and only if $n=3$. For $n\geq 4$, $\chi_2(G)$ is one of the $p+1$ normal subgroups of G [[2], Lemma 2.5].
Let the isomorphism invariant $k=k(G)$ of G, be defined by $[\chi_2(G),\gamma_2(G)]=\gamma_{n-k}(G)$, where $k=0$ for $n=3$ and $0\leq k\leq n-4$ if $n\geq 4$, also for $n\geq p+1$ we have $k=min\{n-4,p-2\}$ [[11], p.331].
$k(G)$ provides a measure for the deviation from the maximal degree of commutativity $[\chi_2(G),\gamma_2(G)]=1$ and is called *defect of commutativity* of G.
With a further invariant e, it will be expressed, which factor $\gamma_j(G)/\gamma_{j+1}(G)$ of the lower central series is cyclic for the first time [13], and we have $e+1=min\{3\leq j\leq m\,|\,1\leq|\gamma_j(G)/\gamma_{j+1}|\leq p\}$.
In this definition of e, we exclude the factor $\gamma_2(G)/\gamma_3(G)$, which is always cyclic. The value $e=2$ is characteristic for a group G of maximal class.
By $G_a^{(n)}(z,w)$ we denote the representative of an isomorphism class of the metabelian p-groups G, which satisfies the relations of theorem 2.1, with a fixed system of exponents a,w and z.
In this paper we shall prove that some metabelian 5-groups are completely determined with the isomorphism class groups of maximal class, furthermore they can be realized.
For that we consider $K=\mathbb{Q}(\sqrt[5]{p},\zeta_5)$, the normal closure of the pure quintic field $\Gamma=\mathbb{Q}(\sqrt[5]{p})$, and also a cyclic Kummer extension of degree 5 of the 5^{th} cyclotomic field $K_0=\mathbb{Q}(\zeta_5)$, where p is a prime number, such that $p\equiv -1(\operatorname{mod} 25)$. According to [7], if the 5-class group of K, denoted $C_{K,5}$, is of type $(5,5)$, we have that the rank of the subgroup of ambiguous ideal classes, under the action of $Gal(K/K_0)=\langle\sigma\rangle$, denoted $C_{K,5}^{(\sigma)}$, is rank $C_{K,5}^{(\sigma)}=1$. Whence by class field theory the relative genus field of the extension K/K_0, denoted $K^*=(K/K_0)^*$, is one of the six cyclic quintic extension of K.
By $F_5^{(1)}$ we denote the Hilbert 5-class field of a number field F. Let $G=\mathrm{Gal}\left((K^*)_5^{(1)}/K_0\right)$, we show that G is a metabelian 5-group of maximal class, and has two maximal normal subgroups H_1 and H_2, such that the transfers $V_{H_1\to\gamma_2(G)}$ and $V_{H_2\to\gamma_2(G)}$ are trivial. Moreover G is completely determined with the isomorphism class groups of maximal class.
The theoretical results are underpinned by numerical examples obtained with the computational number theory system PARI/GP [16].

2. On the 5-class group of maximal class

Let G be a metabelian 5-group of order 5^n, such that $G/\gamma_2(G)$ is of type $(5,5)$, then G admits six maximal normal subgroups $H_1,...,H_6$, which contain the commutator group $\gamma_2(G)$ as a normal subgroup of index 5. We have that $\chi_2(G)$ is one of the groups H_i and we fix $\chi_2(G)=H_1$. We have the following theorem

Theorem 2.1. *Let G be a metabelian 5-group of order 5^n where $n\geq 5$, with the abelianization $G/\gamma_2(G)$ is of type $(5,5)$ and $k=k(G)$ its invariant defined before. Assume that G is of maximal class, then G can be generated by two elements, $G=<x,y>$, be selected such that $x\in G\setminus\chi_2(G)$ and $y\in\chi_2(G)\setminus\gamma_2(G)$. Let $s_2=[y,x]\in\gamma_2(G)$ and $s_j=[s_{j-1},x]\in\gamma_j(G)$ for $j\geq 3$. Then we have:*

(1) $s_j^5 s_{j+1}^{10} s_{j+2}^{10} s_{j+3}^5 s_{j+4}=1$ *for* $j\geq 2$.
(2) $x^5=s_{n-1}^w$ *with* $w\in\{0,1,2,3,4\}$.
(3) $y^5 s_2^{10} s_3^{10} s_4^5 s_5=s_{n-1}^z$ *with* $z\in\{0,1,2,3,4\}$.
(4) $[y,s_2]=\prod_{i=1}^{k} s_{n-i}^{a_{n-i}}$ *with* $a=(a_{n-1},...a_{n-k})$ *exponents such that* $0\leq a_{n-i}\leq 4$.

Proof. See [[12], Theorem 1] for $p=5$. □

The six maximal normal subgroups $H_1....H_6$ are arranged as follows:
$H_1=\langle y,\gamma_2(G)\rangle=\chi_2(G)$, $H_i=\langle xy^{i-2},\gamma_2(G)\rangle$ for $2\leq i\leq 6$. The order of the abelianization of each H_i, for $1\leq i\leq 6$, is given by the following theorem.

Theorem 2.2. *Let G, H_i and the invariant k as before. Then for $1 \leq i \leq 6$, the order of the commutator factor groups of H_i is given by:*

(1) *If $n = 2$ we have : $|H_i/\gamma_2(H_i)| = 5$ for $1 \leq i \leq 6$.*

(2) *If $n \geq 3$ we have : $|H_i/\gamma_2(H_i)| = 5^2$ for $2 \leq i \leq 6$, and $|H_1/\gamma_2(H_1)| = 5^{n-k-1}$*

Proof. See [[10], Theorem 3.1] for $p = 5$. □

Lemma 2.3. *Let G be a 5-group of order $|G| = 5^n$, $n \geq 4$. Assume that the commutator group $G/\gamma_2(G)$ is of type $(5,5)$. Then G is of maximal class if and only if G admits a maximal normal subgroup with factor commutator of order 5^2. Furthermore G admits at least five maximal normal subgroups with factor commutator of order 5^2.*

Proof. Assume that G is of maximal class, then by theorem 2.2, we conclude that G has five maximal normal subgroups with the order of commutator factor is 5^2 if $n \geq 4$, and has six when $n = 3$. Conversely, Assume that $cc(G) \geq 2$, the invariant e defined before is greater than 3, and since each maximal normal subgroup H of G verify $|H/\gamma_2(H)| \geq 5^e$ we get that $|H/\gamma_2(H)| > 5^2$ □

2.1. On the transfer concept

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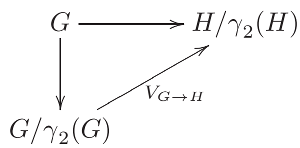
Let G be a group and let H be a subgroup of G. The transfer $V_{G\to H}$ from G to H can be decomposed as follows:



Definition 2.4. *Let G be a group, H be a normal subgroup of G, and let $g \in G$ such that, f is the order of gH in G/H, $r = \frac{[G:H]}{f}$ and $g_1, \ldots. g_r$ be a representative system of G/H, then the transfer from G to H, noted $V_{G\to H}$, is defined by:*

$$\begin{aligned} V_{G\to H} \,:\, G/\gamma_2(G) &\longrightarrow H/\gamma_2(H) \\ g\gamma_2(G) &\longrightarrow \prod_{i=1}^{r} g_i^{-1} g^f g_i \gamma_2(H) \end{aligned}$$

In the special case that G/H is cyclic group of order 5 and $G = \langle h, H\rangle$, then the transfer $V_{G\to H}$ is given as:

(1) If $g \in H$; then $V_{G\to H}(g\gamma_2(G)) = g^{1+h+h^2+h^3+h^4}\gamma_2(H)$

(2) $V_{G\to H}(h\gamma_2(G)) = h^5\gamma_2(H)$

3. Main results

In this section we investigate the purely group theoretic results to determine the invariants of metabelian 5-group of maximal class developed in theorem 2.1. Furthermore we show that a such metabelian 5-group is realized by the Galois group of some fields tower.

3.1. Invariants of metabelian 5-group of maximal class

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In this paragraph, we keep the same hypothesis on the group G and the generators $G = \langle x, y\rangle$, such that $x \in G \setminus \chi_2(G)$ and $y \in \chi_2(G) \setminus \gamma_2(G)$. The six maximal normal subgroups of G are as follows: $H_1 = \chi_2(G) = \langle y, \gamma_2(G)\rangle$ and $H_i = \langle xy^{i-2}, \gamma_2(G)\rangle$ for $2 \leq i \leq 6$.
In the case that the transfers from two subgroups H_i and H_j to $\gamma_2(G)$ are trivial, we can determine completely the 5-group G.

Proposition 3.1. *Let G be a metabelian 5-group of maximal class of order 5^n, $n \geq 4$. If the transfers $V_{\chi_2(G)\to\gamma_2(G)}$ and $V_{H_2\to\gamma_2(G)}$ are trivial, then $n \leq 6$ and $\gamma_2(G)$ is of exponent 5. Furthermore:*
- If $n = 6$ then $G \sim G_a^{(6)}(1,0)$ where $a = 0$ or 1.
- If $n = 5$ then $G \sim G_a^{(5)}(0,0)$ where $a = 0$ or 1.
- If $n = 4$ then $G \sim G_0^{(4)}(0,0)$.

Proof. Assume that $n \geq 7$, then $\gamma_5(G) = \langle s_5, \gamma_6(G)\rangle$, because G is of maximal class and $|\gamma_5(G)/\gamma_6(G)| = 5$. By [[2], lemma 3.3] we have $y^5 s_5 \in \gamma_6(G)$, thus $\gamma_5(G) = \langle s_5^4, \gamma_6(G)\rangle = \langle y^5 s_5 s_5^4, \gamma_6(G)\rangle = \langle y^5, \gamma_6(G)\rangle$, and since $V_{\chi_2(G)\to\gamma_2(G)}(y) = y^5 = 1$, because the transfers are trivial by hypothesis, we get that $\gamma_5(G) = \gamma_6(G)$, which is impossible, whence $n \leq 6$ and According to [[2], lemma 3.2], $\gamma_2(G)$ is of exponent 5.
If $n = 6$, we have $V_{\chi_2(G)\to\gamma_2(G)}$ and $V_{H_2\to\gamma_2(G)}$ are trivial, so by theorem 2.1 we obtain $x^5 = s_5^w = 1$ which imply $w = 0$, because $0 \leq w \leq 4$. Since $\gamma_2(G)$ is of exponent 5, we have $s_2^5 = 1$ and by theorem 2.1 the relation $s_4^5 s_5^{10} s_6^{10} s_7^5 s_8 = 1$ gives $s_4^5 = 1$, also $s_3^5 s_4^{10} s_5^{10} s_6^5 s_7 = 1$ gives $s_3^5 = 1$. We replace in $y^5 s_2^{10} s_3^{10} s_4^5 s_5 = s_5^z$ and we get $s_5 = s_5^z$, whence $z = 1$. We have $[\chi_2(G), \gamma_2(G)] \subset \gamma_{6-k}(G) \subset \gamma_4(G)$ then $6 - k \geq 4$, and $0 \leq k \leq 2$, thus $[y, s_2] = s_4^{\alpha\beta}$, $a = (\alpha, \beta)$. If $k = 0$, then $a = 0$ and $G \sim G_0^{(6)}(1,0)$, if $k = 1$ then $a = 1$ and $G \sim G_1^{(6)}(1,0)$ and if $k = 2$ then $G \sim G_a^{(6)}(1,0)$.
If $n = 5$, we have $[\chi_2(G), \gamma_2(G)] \subset \gamma_{5-k}(G) \subset \gamma_4(G)$ then $5 - k \geq 4$, and $0 \leq k \leq 1$. We have $s_4^5 = 1$, $s_2^5 = s_3^5 = 1$ and $[y, s_2] = s_4^a$. the relation $y^5 s_2^{10} s_3^{10} s_4^5 s_5 = s_4^z$ imply $s_4^z = 1$ so $z = 0$. As $n = 6$ we obtain $w = 0$. If $k = 0$ then $G \sim G_0^{(5)}(0,0)$ and if $k = 1$ $G \sim G_a^{(5)}(0,0)$.
If $n = 4$, Since $[\chi_2(G), \gamma_2(G)] \subset \gamma_{5-k}(G) \subset \gamma_4(G)$ we have $4 - k \geq 4$, and $k = 0$, thus $[y, s_2] = 1$, i.e $a = 0$. By the same way in this case we have $w = z = 0$, therefor $G \sim G_0^{(4)}(0,0)$. □

Proposition 3.2. *Let G be a metabelian 5-group of maximal class of order 5^n. If the transfers $V_{H_2\to\gamma_2(G)}$ and $V_{H_i\to\gamma_2(G)}$, $3 \leq i \leq 6$, are trivial, then we have:*
- If $n = 5$ or 6 then $G \sim G_a^{(n)}(0,0)$.
- If $n \geq 7$ then $G \sim G_0^{(n)}(0,0)$.

Proof. If $n = 5$ or 6, by [[2], theorem 1.6] we have $[\chi_2(G), \gamma_2(G)] = 1$ and $[\chi_2(G), \gamma_2(G)] \subset \gamma_4(G)$ elementary, and $(\gamma_2(\chi_2(G)))^5 = 1$ and $\prod_{i=2}^{3}[\gamma_i(G), \gamma_4(G)] = 1$, we conclude that $(xy)^5 = x^5 y^5 s_2^{10} s_3^{10} s_4^5 s_5$ and we have $y^5 s_2^{10} s_3^{10} s_4^5 s_5 = s_{n-1}^z$ then $(xy)^5 = x^5 s_{n-1}^z$ and since $V_{H_2\to\gamma_2(G)}$ and $V_{H_3\to\gamma_2(G)}$ are trivial then $(xy)^5 = x^5 = s_{n-1}^z = s_{n-1}^w = 1$, thus $z = w = 0$. Since $[\chi_2(G), \gamma_2(G)] = \gamma_{n-k} \subset \gamma_4(G)$ we have $n - k \geq 4$, whence $0 \leq k \leq 2$ because $n = 5$ or 6 then $G \sim G_a^{(n)}(0,0)$.
If $n \geq 7$, according to corollary page 69 of [2] we have, $(\gamma_j(\chi_2(G)))^5 = \gamma_{j+4}(G)$ for $j \geq 2$, and since $y^5 s_2^{10} s_3^{10} s_4^5 s_5 = s_{n-1}^z$ we obtain:

$$y^5 = s_{n-1}^z s_5^{-1} s_4^{-1} s_3^{-10} s_2^{-10} \equiv s_{n-1}^z s_5^{-1} \operatorname{mod} \gamma_6(\mathrm{G})$$

because $s_2^5 \in \gamma_6(G)$, $s_3^5 \in \gamma_6(G)$ and $s_4^5 \in \gamma_6(G)$, and since $n \geq 7$ we have $s_{n-1} \in \gamma_6(G)$, therefor $V = V_{H_3\to\gamma_2(G)}(y) \equiv s_5^{-1} \operatorname{mod} \gamma_6(\mathrm{G})$. Thus $\mathrm{Im}(V) \subset \gamma_5(G)$, In fact $\mathrm{Im}(V) = \gamma_5(G)$, and also we have $y \notin \ker(V)$ and $\forall f \geq 2$ $y^k s_f^l \notin \ker(V)$. The kernel of V is formed by elements of $\gamma_2(G)$ of exponent 5, its exactly $\gamma_{n-4}(G)$, and since G is of maximal class then the rank of $\gamma_2(G)$ is 2 and $\gamma_2(G)$ admits exactly 25 elements of exponent 5, these elements form $\gamma_{n-4}(G)$. We conclude that $|\chi_2(G)/\gamma_2(\chi_2(G))| = |\gamma_{n-4}(G)| \times |\gamma_5(G)| = 5^4 \times 5^{n-5} = 5^{n-1} = |\chi_2(G)|$, whence $\chi_2(G)$ is abelian because $\gamma_2(\chi_2(G)) = 1$, consequently $[y, s_2] = 1$, thus $a = 0$. As the cases $n = 5$ or 6 we obtain $(xy)^5 = x^5 s_{n-1}^z$, therefor $z = w = 0$, hence $G \sim G_0^{(n)}(0,0)$.
In the case when $V_{H_2\to\gamma_2(G)}$ and $V_{H_i\to\gamma_2(G)}$, $4 \leq i \leq 6$ are trivial, according to [[2], theorem 1.6] we have $(xy^\mu)^5 = x^5(y^5 s_2^{10} s_3^{10} s_4^5 s_5)^\mu = s_{n-1}^w s_{n-1}^{\mu z}$ with $\mu = 2, 3, 4$, then we can admit the same reasoning to prove the result. □

Proposition 3.3. *Let G be a metabelian 5-group of maximal class of order 5^n. If the transfers $V_{H_i\to\gamma_2(G)}$ and $V_{H_j\to\gamma_2(G)}$, where $i, j \in \{3, 4, 5, 6\}$ and $i \neq j$, are trivial, then we have: $G \sim G_0^{(n)}(0,0)$.*

Proof. Assume that $H_i = \langle xy^{\mu_1}, \gamma_2(G)\rangle$ and $H_j = \langle xy^{\mu_2}, \gamma_2(G)\rangle$ where $\mu_1, \mu_2 \in \{1,2,3,4\}$ and $\mu_1 \neq \mu_2$. According to [[2], theorem 1.6] we have already prove that $(xy^{\mu_1})^5 = s_{n-1}^{w+\mu_1 z}$ and $(xy^{\mu_2})^5 = s_{n-1}^{w+\mu_2 z}$. Since $V_{H_i \to \gamma_2(G)}$ and $V_{H_j \to \gamma_2(G)}$ are trivial, we obtain $s_{n-1}^{w+\mu_1 z} = s_{n-1}^{w+\mu_2 z} = 1$ then $w+\mu_1 z \equiv w+\mu_2 z \equiv 0 \pmod 5$ and since 5 does not divide $\mu_1 - \mu_2$ we get $z = 0$ and at the same time $w = 0$. To prove $a = 0$ we admit the same reasoning as proposition 3.2. □

3.2. Application

.

Through this section we denote by:

- p a prime number such that $p \equiv -1 (\bmod\, 25)$.
- $K_0 = \mathbb{Q}(\zeta_5)$ the 5^{th} cyclotomic field, ($\zeta_5 = e^{\frac{2\pi i}{5}}$).
- $K = K_0(\sqrt[5]{p})$ a cyclic Kummer extension of K_0 of degree 5.
- $C_{F,5}$ the 5-ideal class group of a number field F.
- $K^* = (K/K_0)^*$ the relative genus field of K/K_0.
- $F_5^{(1)}$ the absolute Hilbert 5-class field of a number field F.
- $G = \mathrm{Gal}\left((K^*)_5^{(1)}/K_0\right)$.

We begin by the following theorem.

Theorem 3.4. *Let $K = \mathbb{Q}(\sqrt[5]{p}, \zeta_5)$ be the normal closure of a pure quintic field $\mathbb{Q}(\sqrt[5]{p})$, where p a prime congruent to -1 modulo 25. Let K_0 be the the 5^{th} cyclotomic field. Assume that the 5-class group $C_{K,5}$ of K, is of type $(5,5)$, then $Gal(K^*/K_0)$ is of type $(5,5)$, and two sub-extensions of K^*/K_0 admit a trivial 5-class number.*

Proof. By $C_{K,5}^{(\sigma)}$ we denote the subgroup of ambiguous ideal classes under the action of $Gal(K/K_0) = \langle\sigma\rangle$. According to [[7], theorem 1.1], in this case of the prime p we have rank $C_{K,5}^{(\sigma)} = 1$, and by class field theory, since $[K^* : K] = |C_{K,5}^{(\sigma)}|$, we have that K^*/K is a cyclic quintic extension, whence $Gal(K^*/K_0)$ is of type $(5,5)$.
Since $p \equiv -1 (\bmod\, 25)$, then p splits in K_0 as $p = \pi_1\pi_2$, where π_1, π_2 are primes of K_0. By [[8], theorem 5.15] we have explicitly the relative genus field K^* as $K^* = K(\sqrt[5]{\pi_1^{a_1}\pi_2^{a_2}}) = K_0(\sqrt[5]{\pi_1\pi_2}, \sqrt[5]{\pi_1^{a_1}\pi_2^{a_2}})$ with $a_1, a_2 \in \{1,2,3,4\}$ such that $a_1 \neq a_2$. Its clear that the extension K^*/K_0 admits six sub-extensions, where K is one of them, and the others are $K_0(\sqrt[5]{\pi_1^{a_1}\pi_2^{a_2}})$, $K_0(\sqrt[5]{\pi_1^{a_1+1}\pi_2^{a_2+1}})$, $K_0(\sqrt[5]{\pi_1^{a_1+2}\pi_2^{a_2+2}})$, $K_0(\sqrt[5]{\pi_1^{a_1+3}\pi_2^{a_2+3}})$ and $K_0(\sqrt[5]{\pi_1^{a_1+4}\pi_2^{a_2+4}})$. Since $a_1, a_2 \in \{1,2,3,4\}$, we can see that the extensions $L_1 = K_0(\sqrt[5]{\pi_1})$ and $L_2 = K_0(\sqrt[5]{\pi_2})$ are sub-extensions of K^*/K_0.
In [[8], section 5.1], we have an investigation of the rank of ambiguous classes of $K_0(\sqrt[5]{x})/K_0$, denoted t. We have $t = d + q^* - 3$, where d is the number of prime divisors of x in K_0, and q^* an index defined as [[8], section 5.1]. For the extensions L_i/K_0, $(i = 1,2)$, we have $d = 1$ and by [[8], theorem 5.15] we have $q^* = 2$, hence $t = 0$.
By $h_5(L_i)$, $(i = 1,2)$, we denote the class number of L_i, then we have $h_5(L_1) = h_5(L_2) = 1$. Otherwise $h_5(L_i) \neq 1$, then there exists an unramified cyclic extension of L_i, denoted F. This extension is abelian over K_0, because $[F : K_0] = 5^2$, then F is contained in $(L_i/K_0)^*$ the relative genus field of L_i/K_0. Since $[(L_i/K_0)^* : L_i] = 5^t = 1$, we get that $(L_i/K_0)^* = L_i$, which contradicts the existence of F. Hence the 5-class number of L_i, $(i = 1,2)$, is trivial. □

In what follows, we denote by L_1 and L_2 the two sub-extensions of K^*/K_0, which verify theorem 3.4, and by $\tilde{L}$ the three remaining sub-extensions different to K. Let $G = Gal((K^*)_5^{(1)}/K_0)$, we have $\gamma_2(G) = Gal((K^*)_5^{(1)}/K^*)$, then $G/\gamma_2(G) = Gal(K^*/K_0)$ is of type $(5,5)$, therefore G is metabelian 5-group with factor commutator of type $(5,5)$, thus G admits exactly six maximal normal subgroups as follows:

$$H = Gal((K^*)_5^{(1)}/K),\ H_{L_i} = Gal((K^*)_5^{(1)}/L_i),\ (i=1,2),\ \tilde{H} = Gal((K^*)_5^{(1)}/\tilde{L}).$$

With $\chi_2(G)$ is one of them.

Now we can state our principal result.

Theorem 3.5. *Let $G = Gal((K^*)_5^{(1)}/K_0)$ be a 5-group of order 5^n, $n \geq 4$, then G is a metabelian of maximal class. Furthermore we have:*

- If $\chi_2(G) = H_{L_i} (i=1,2)$ then: $G \sim G_a^{(n)}(z,0)$ with $n \in \{4,5,6\}$ and $a, z \in \{0,1\}$.

- If $\chi_2(G) = \tilde{H}$ then : $G \sim G_1^{(n)}(0,0)$ with $n = 5$ or 6.

$G \sim G_0^{(n)}(0,0)$ with $n \geq 7$ such that $n = s+1$ where $h_5(\tilde{L}) = 5^s$.

Proof. Let $G = Gal((K^*)_5^{(1)}/K_0)$ and $H = Gal((K^*)_5^{(1)}/K)$ its maximal normal subgroup, then $\gamma_2(H) = Gal((K^*)_5^{(1)}/K_5^{(1)})$, therefor $H/\gamma_2(H) = Gal(K_5^{(1)}/K) \simeq C_{K,5}$, and as $C_{K,5}$ is of type $(5,5)$ by hypothesis we get that $|H/\gamma_2(H)| = 5^2$. Lemma 2.3 imply that G is a metabelian 5-group of maximal class, generated by two elements $G = \langle x, y\rangle$, such that, $x \in G \setminus \chi_2(G)$ and $y \in \chi_2(G) \setminus \gamma_2(G)$. Since $\chi_2(G) = \langle y, \gamma_2(G)\rangle$, we have $\chi_2(G) \neq H$. Otherwise we get that $|H/\gamma_2(H)| = 5^2$, which contradict theorem 2.1.
According to theorem 3.4, we have $h_5(L_1) = h_5(L_2) = 1$, then the transfers $V_{H_{L_i} \to \gamma_2(G)}$ are trivial.
If $\chi_2(G) = H_{L_i}$ the results are nothing else than proposition 3.1.
If $\chi_2(G) = \tilde{H}$ and $n = 4$ then $\gamma_4(G) = 1$ and $[\chi_2(G), \gamma_2(G)] = \gamma_2(\tilde{H})$, also $[\chi_2(G), \gamma_2(G)] = \gamma_4(G) = 1$ then $\chi_2(\tilde{H}) = 1$, whence $\tilde{H}$ is abelian. Consequently $\tilde{H}/\gamma_2(\tilde{H}) = C_{\tilde{L},5}$, so $h_5(\tilde{L}) = |\tilde{H}| = 5^3$ because its a maximal subgroup of G. Since $\tilde{L}$ and k have always the same conductor, we deduce that $h_5(K)$ and $h_5(\tilde{L})$ verify the relations $5^5 h_{\tilde{L}} = u h_\Gamma^4$ and $5^5 h_K = u h_\Gamma^4$, given by C. Parry in [15], where u is a unit index and a divisor of 5^6. Using the 5-valuation on these relations we get that $h_5(\tilde{L}) = 5^s$ where s is even, which contradict the fact that $h_5(\tilde{L}) = 5^3$, hence $n \geq 5$.
The results of the theorem are exactly application of propositions 3.2, 3.3. According to proposition 3.2, if $n \geq 7$ we have $|\chi_2(G)| = 5^{n-1}$ and since $h_5(\tilde{L}) = |\tilde{H}/\gamma_2(\tilde{H})| = |\tilde{H}| = 5^{n-1} = 5^s$ we deduce that $n = s+1$. □

4. Numerical examples

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For these numerical examples of the prime p, we have that $C_{K,5}$ is of type $(5,5)$ and rank $C_{K,5}^{(\sigma)} = 1$, which mean that K^* is cyclic quintic extension of K, then by theorem 3.5 we have a completely determination of G. We note that the absolute degree of $(K^*)_5^{(1)}$ surpass 100, then the task to determine the order of G is definitely far beyond the reach of computational algebra systems like MAGMA and PARI/GP.

Table 1: $K = \mathbb{Q}(\sqrt[5]{p}, \zeta_5)$ with $C_{K,5}$ is of type $(5,5)$ and rank $C_{K,5}^{(\sigma)} = 1$

p	$p \pmod{25}$	$h_{K,5}$	$C_{K,5}$	rank $(C_{K,5}^{(\sigma)})$
149	-1	25	$(5,5)$	1
199	-1	25	$(5,5)$	1
349	-1	25	$(5,5)$	1
449	-1	25	$(5,5)$	1
559	-1	25	$(5,5)$	1
1249	-1	25	$(5,5)$	1
1499	-1	25	$(5,5)$	1
1949	-1	25	$(5,5)$	1
1999	-1	25	$(5,5)$	1
2099	-1	25	$(5,5)$	1

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