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On the Solutions and Behavior of Rational Systems of Difference Equations

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ABSTRACT: In this paper, we obtain the expressions of the solutions of the following nonlinear systems of difference equations

$$x_{n+1} = \frac{x_{n-9}}{1 + x_{n-9}y_{n-4}}, \quad y_{n+1} = \frac{y_{n-9}}{\pm 1 \pm x_{n-4}y_{n-9}}, \; n = 0, 1, ...,$$

where the initial conditions $x_{-9},\ x_{-8},\ x_{-7},\ x_{-6},\ x_{-5}, x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0, y_{-9},\ y_{-8},\ y_{-7}, y_{-6},\ y_{-5}, y_{-4},\ y_{-4},\ y_{-3}, y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers.

Key Words: Difference equations, recursive sequences, periodic solution, system of difference equations.

Contents

1. Introduction

Many problems in Probability give rise to difference equations. Difference equations relate to differential equations as discrete mathematics relates to continuous mathematics. Anyone who has made a study of differential equations will know that even supposedly elementary examples can be hard to solve. By contrast, elementary difference equations are relatively easy to deal with. Aside from Probability, Computer Scientists take an interest in difference equations for a number of reasons. For example, difference equations frequently arise when determining the cost of an algorithm in big-O notation. Since difference equations are readily handled by program, a standard approach to solving a nasty differential equation is to convert it to an approximately equivalent difference equation see [1]- [30].

Many articles discuss difference equations systems, for example, Elsayed et al. [17] dealt with the solutions of the system of the difference equations

$$x_{n+1} = \frac{1}{x_{n-p}y_{n-p}}, \qquad y_{n+1} = \frac{x_{n-p}y_{n-p}}{x_{n-q}y_{n-q}}.$$

Kurbanli et al. [22] discussed the behavior of positive solutions of the following system

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1}y_n}, \qquad y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1}x_n}.$$

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El-Dessoky [7] considered the solutions of the following system

$$x_{n+1}=\frac{y_{n-1}y_{n-2}}{x_n(\pm 1\pm y_{n-1}y_{n-2})},\qquad y_{n+1}=\frac{x_{n-1}x_{n-2}}{y_n(\pm 1\pm x_{n-1}x_{n-2})}.$$

In [10] El-Dessoky and Elsayed investigated the form of the solution of the following systems of difference equations.

$$x_{n+1}=\frac{x_{n-2}}{\pm 1\pm x_{n-2}z_{n-1}y_n},\quad y_{n+1}=\frac{y_{n-2}}{\pm 1\pm y_{n-2}x_{n-1}z_n},\quad z_{n+1}=\frac{z_{n-2}}{\pm 1\pm z_{n-2}y_{n-1}x_n}$$

Elabbasy et al. [6] devoted to investigate the local asymptotic stability, boundedness and periodic solutions of particular cases of the following general system of difference equations:

$$x_{n+1}=\frac{a_1y_{n-1}+a_2x_{n-3}+a_3}{a_4y_{n-1}x_{n-3}+a_5},\qquad y_{n+1}=\frac{b_1x_{n-1}+b_2y_{n-3}+b_3}{b_4x_{n-1}y_{n-3}+b_5}.$$

Elsayed et al. [15] obtained the expressions of the solutions of the following nonlinear systems of difference equations

$$x_{n+1}=\frac{y_nx_{n-2}}{y_n+y_{n-3}},\qquad y_{n+1}=\frac{x_ny_{n-2}}{\pm x_n\pm x_{n-3}}.$$

Khan et al. [20], studied the qualitative behavior of two systems of second-order rational difference equations.

Zhang et al. [30] discussed the boundedness, persistence, and global asymptotic stability of positive solution for a system of third-order rational difference equations

$$x_{n+1}=A+\frac{x_n}{y_{n-1}y_{n-2}},\qquad y_{n+1}=A+\frac{y_n}{x_{n-1}x_{n-2}}.$$

Elsayed and Ahmed [16] investigated the solutions and the periodicity of the following rational systems of difference equations of three-dimensional

$$x_{n+1}=\frac{y_nx_{n-2}}{x_{n-2}\pm z_{n-1}},\qquad y_{n+1}=\frac{z_ny_{n-2}}{y_{n-2}\pm x_{n-1}},\qquad z_{n+1}=\frac{x_nz_{n-2}}{z_{n-2}\pm y_{n-1}}.$$

Asiri et al. [1] studied the form of the solutions and the periodicity of the following third order systems of rational difference equations

$$x_{n+1}=\frac{y_{n-2}}{1-y_{n-2}x_{n-1}y_n},\qquad y_{n+1}=\frac{x_{n-2}}{1-x_{n-2}y_{n-1}x_n}.$$

The aim of this article is to obtain the expressions of the solutions of the following systems of difference equations

$$x_{n+1}=\frac{x_{n-9}}{1+x_{n-9}y_{n-4}},\qquad y_{n+1}=\frac{y_{n-9}}{\pm 1\pm x_{n-4}y_{n-9}},\qquad n=0,1,2,..., \tag{1.1}$$

where the initial conditions $x_{-9},\ x_{-8},\ x_{-7},\ x_{-6},\ x_{-5}, x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0, y_{-9},\ y_{-8},\ y_{-7}, y_{-6},\ y_{-5},\ y_{-4},\ y_{-4},\ y_{-3},\ y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers. Moreover, we obtain some numerical simulation to the equation are given to illustrate our results.

2. On The System $x_{n+1}=\frac{x_{n-9}}{1+x_{n-9}y_{n-4}}$, $y_{n+1}=\frac{y_{n-9}}{1+x_{n-4}y_{n-9}}$

In this section, we study the solution of the following system of difference equations

$$x_{n+1}=\frac{x_{n-9}}{1+x_{n-9}y_{n-4}},\qquad y_{n+1}=\frac{y_{n-9}}{1+x_{n-4}y_{n-9}}, \tag{2.1}$$

where the initial conditions $x_{-9},\ x_{-8},\ x_{-7},\ x_{-6},\ x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0, y_{-9},\ y_{-8},\ y_{-7}, y_{-6},\ y_{-5},\ y_{-4}, y_{-4},\ y_{-3}, y_{-2},\ y_{-1},\ y_0$ are arbitrary non zero real numbers.

2.1. The Form of the Solutions of System (2)

The following theorem describes the form of the solutions of system (2).

Theorem 1. Suppose that $\{x_n, y_n\}$ are solutions of the system (2). Then for $n = 0, 1, 2, ...$, we have the following formulas

$$\begin{aligned}
x_{10n-9} &= s\prod_{i=0}^{n-1}\frac{1+2ist}{1+(2i+1)st}, \quad x_{10n-8} = k\prod_{i=0}^{n-1}\frac{1+2ikw}{1+(2i+1)kw},\\
x_{10n-7} &= h\prod_{i=0}^{n-1}\frac{1+2ihv}{1+(2i+1)hv}, \quad x_{10n-6} = g\prod_{i=0}^{n-1}\frac{1+2igu}{1+(2i+1)gu},
\end{aligned}$$

$$\begin{aligned}
x_{10n-5} &= f\prod_{i=0}^{n-1}\frac{1+2ifr}{1+(2i+1)fr}, \quad x_{10n-4} = e\prod_{i=0}^{n-1}\frac{1+(2i+1)eq}{1+(2i+2)eq},\\
x_{10n-3} &= d\prod_{i=0}^{n-1}\frac{1+(2i+1)dp}{1+(2i+2)dp}, \quad x_{10n-2} = c\prod_{i=0}^{n-1}\frac{1+(2i+1)co}{1+(2i+2)co},\\
x_{10n-1} &= b\prod_{i=0}^{n-1}\frac{1+(2i+1)Lb}{1+(2i+2)Lb}, \quad x_{10n} = a\prod_{i=0}^{n-1}\frac{1+(2i+1)az}{1+(2i+2)az},
\end{aligned}$$

$$\begin{aligned}
y_{10n-9} &= q\prod_{i=0}^{n-1}\frac{1+2ieq}{1+(2i+1)eq}, \quad y_{10n-8} = p\prod_{i=0}^{n-1}\frac{1+2idp}{1+(2i+1)dp},\\
y_{10n-7} &= o\prod_{i=0}^{n-1}\frac{1+2ico}{1+(2i+1)co}, \quad y_{10n-6} = L\prod_{i=0}^{n-1}\frac{1+2iLb}{1+(2i+1)Lb},\\
y_{10n-5} &= z\prod_{i=0}^{n-1}\frac{1+2iaz}{1+(2i+1)az}, \quad y_{10n-4} = t\prod_{i=0}^{n-1}\frac{1+(2i+1)st}{1+(2i+2)st},\\
y_{10n-3} &= w\prod_{i=0}^{n-1}\frac{1+(2i+1)kw}{1+(2i+2)kw}, \quad y_{10n-2} = v\prod_{i=0}^{n-1}\frac{1+(2i+1)hv}{1+(2i+2)hv},\\
y_{10n-1} &= u\prod_{i=0}^{n-1}\frac{1+(2i+1)gu}{1+(2i+2)gu}, \quad y_{10n} = r\prod_{i=0}^{n-1}\frac{1+(2i+1)fr}{1+(2i+2)fr},
\end{aligned}$$

where the initial conditions $x_{-9} = s$, $x_{-8} = k$, $x_{-7} = h$, $x_{-6} = g$, $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$, $y_{-9} = q$, $y_{-8} = p$, $y_{-7} = o$, $y_{-6} = L$, $y_{-5} = z$, $y_{-4} = t$, $y_{-3} = w$, $y_{-2} = v$, $y_{-1} = u$, $y_0 = r$.

Proof. For $n = 0$ the result holds. Suppose that the result holds for $n - 1$.

$$\begin{aligned}
x_{10n-19} &= s\prod_{i=0}^{n-2}\frac{1+2ist}{1+(2i+1)st}, \quad x_{10n-18} = k\prod_{i=0}^{n-2}\frac{1+2ikw}{1+(2i+1)kw},\\
x_{10n-17} &= h\prod_{i=0}^{n-2}\frac{1+2ihv}{1+(2i+1)hv}, \quad x_{10n-16} = g\prod_{i=0}^{n-2}\frac{1+2igu}{1+(2i+1)gu},
\end{aligned}$$

$$\begin{aligned}
x_{10n-15} &= f\prod_{i=0}^{n-2}\frac{1+2ifr}{1+(2i+1)fr}, \quad x_{10n-14} = e\prod_{i=0}^{n-2}\frac{1+(2i+1)eq}{1+(2i+2)eq},\\
x_{10n-13} &= d\prod_{i=0}^{n-2}\frac{1+(2i+1)dp}{1+(2i+2)dp}, \quad x_{10n-12} = c\prod_{i=0}^{n-2}\frac{1+(2i+1)co}{1+(2i+2)co},
\end{aligned}$$

$$x_{10n-11} = b\prod_{i=0}^{n-2}\frac{1+(2i+1)Lb}{1+(2i+2)Lb}, \quad x_{10n-10} = a\prod_{i=0}^{n-2}\frac{1+(2i+1)az}{1+(2i+2)az},$$

$$\begin{aligned} y_{10n-19} &= q\prod_{i=0}^{n-2}\frac{1+2ieq}{1+(2i+1)eq}, \quad y_{10n-18} = p\prod_{i=0}^{n-2}\frac{1+2idp}{1+(2i+1)dp}, \\ y_{10n-17} &= o\prod_{i=0}^{n-2}\frac{1+2ico}{1+(2i+1)co}, \quad y_{10n-16} = L\prod_{i=0}^{n-2}\frac{1+2iLb}{1+(2i+1)Lb}, \end{aligned}$$

$$\begin{aligned} y_{10n-15} &= z\prod_{i=0}^{n-2}\frac{1+2iaz}{1+(2i+1)az}, \quad y_{10n-14} = t\prod_{i=0}^{n-2}\frac{1+(2i+1)st}{1+(2i+2)st}, \\ y_{10n-13} &= w\prod_{i=0}^{n-2}\frac{1+(2i+1)kw}{1+(2i+2)kw}, \quad y_{10n-12} = v\prod_{i=0}^{n-2}\frac{1+(2i+1)hv}{1+(2i+2)hv}, \end{aligned}$$

$$y_{10n-11} = u\prod_{i=0}^{n-2}\frac{1+(2i+1)gu}{1+(2i+2)gu}, \quad y_{10n-10} = r\prod_{i=0}^{n-2}\frac{1+(2i+1)fr}{1+(2i+2)fr},$$

from system (2) we can prove as follow

$$\begin{aligned} x_{10n-9} &= \frac{x_{10n-19}}{1+x_{10n-19}y_{10n-14}} = \frac{s\prod_{i=0}^{n-2}(\frac{1+2ist}{1+(2i+1)st})}{1+st\prod_{i=0}^{n-2}(\frac{1+2ist}{1+(2i+1)st})(\frac{1+(2i+1)st}{1+(2i+2)st})} \\ &= \frac{s\prod_{i=0}^{n-2}(\frac{1+2ist}{1+(2i+1)st})}{1+st\prod_{i=0}^{n-2}(\frac{1+2ist}{1+(2i+2)st})} = \frac{s\prod_{i=0}^{n-2}(\frac{1+2ist}{1+(2i+1)st})}{1+(\frac{st}{1+(2n-2)st})}\left(\frac{1+(2n-2)st}{1+(2n-2)st}\right) \\ &= \frac{s\left(1+(2n-2)st\right)\prod_{i=0}^{n-2}(\frac{1+2ist}{1+(2i+1)st})}{1+(2n-2)st+st} \\ &= \frac{s\left(1+(2n-2)st\right)}{1+(2n-1)st}\prod_{i=0}^{n-2}\left(\frac{1+2ist}{1+(2i+1)st}\right) \\ &= s\prod_{i=0}^{n-1}\left(\frac{1+2ist}{1+(2i+1)st}\right). \end{aligned}$$

Also, we get

$$\begin{aligned}
y_{10n-9} &= \frac{y_{10n-19}}{1+y_{10n-19}x_{10n-14}} = \frac{q\prod\limits_{i=0}^{n-2}\frac{1+2ieq}{1+(2i+1)eq}}{1+q\prod\limits_{i=0}^{n-2}\frac{1+2ieq}{1+(2i+1)eq}e\prod\limits_{i=0}^{n-2}\frac{1+(2i+1)eq}{1+(2i+2)eq}} \\
&= \frac{q\prod\limits_{i=0}^{n-2}\frac{1+2ieq}{1+(2i+1)eq}}{1+eq\prod\limits_{i=0}^{n-2}\frac{1+2ieq}{1+(2i+2)eq}} = \frac{q\prod\limits_{i=0}^{n-2}\frac{1+2ieq}{1+(2i+1)eq}}{1+\frac{eq}{1+(2n-2)eq}} = \frac{q\prod\limits_{i=0}^{n-2}\frac{1+2ieq}{1+(2i+1)eq}}{\frac{1+(2n-1)eq}{1+(2n-2)eq}} \\
&= q\prod_{i=0}^{n-2}\frac{1+2ieq}{1+(2i+1)eq}\left(\frac{1+(2n-2)eq}{1+(2n-1)eq}\right) = q\prod_{i=0}^{n-1}\frac{1+2ieq}{1+(2i+1)eq}.
\end{aligned}$$

The other relations can be proved by similar way. This completes the proof.

Lemma 1. The equilibrium points of system (2) are $(0,\alpha)$ and $(\gamma,0)$ where $\alpha,\gamma\in[0,\infty)$.

Proof. For the equilibrium points of system (2), we can write

$$\bar{x} = \frac{\bar{x}}{1+\bar{x}\bar{y}}, \qquad \bar{y} = \frac{\bar{y}}{1+\bar{x}\bar{y}}.$$

Then

$$\bar{x}(1+\bar{x}\bar{y}) = \bar{x}, \qquad \bar{y}(1+\bar{x}\bar{y}) = \bar{y},$$

we have

$$\bar{x}(1+\bar{x}\bar{y}-1) = 0, \qquad \bar{y}(1+\bar{x}\bar{y}-1) = 0.$$

Therefore every $(0,\alpha)$ and $(\gamma,0)$ are solutions. Thus the equilibrium points of system (2) are $(0,\alpha)$ and $(\gamma,0)$.

Lemma 2. Every positive solution of the system (2) is bounded and convergent.

Proof. Let $\{x_n,y_n\}$ be a positive solution of system (2). It follows from system (2) that

$$x_{n+1} = \frac{x_{n-9}}{1+x_{n-9}y_{n-4}} < x_{n-9},$$

and

$$y_{n+1} = \frac{y_{n-9}}{1+y_{n-9}x_{n-4}} < y_{n-9}.$$

Then the subsequences $\{x_{10n-9}\}_{n=0}^{\infty}$, $\{x_{10n-8}\}_{n=0}^{\infty}$, $\{x_{10n-7}\}_{n=0}^{\infty}$, $\{x_{10n-6}\}_{n=0}^{\infty}$, $\{x_{10n-5}\}_{n=0}^{\infty}$, $\{x_{10n-4}\}_{n=0}^{\infty}$, $\{x_{10n-3}\}_{n=0}^{\infty}$, $\{x_{10n-2}\}_{n=0}^{\infty}$, $\{x_{10n-1}\}_{n=0}^{\infty}$ and $\{x_{10n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$M = \max\{x_{-9},x_{-8},x_{-7},x_{-6},x_{-5},x_{-4},x_{-3},x_{-2},x_{-1},x_0\}.$$

Similarly the subsequences $\{y_{10n-9}\}_{n=0}^{\infty}$, $\{y_{10n-8}\}_{n=0}^{\infty}$, $\{y_{10n-7}\}_{n=0}^{\infty}$, $\{y_{10n-6}\}_{n=0}^{\infty}$, $\{y_{10n-5}\}_{n=0}^{\infty}$, $\{y_{10n-4}\}_{n=0}^{\infty}$, $\{y_{10n-3}\}_{n=0}^{\infty}$, $\{y_{10n-2}\}_{n=0}^{\infty}$, $\{y_{10n-1}\}_{n=0}^{\infty}$ and $\{y_{10n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$N = \max\{y_{-9},y_{-8},y_{-7},y_{-6},y_{-5},y_{-4},y_{-3},y_{-2},y_{-1},y_0\}.$$

Lemma 3. If $a,b,c,d,e,f,g,h,k,s,o,p,q,r,s,L,t,u,v,w$ *and* z arbitrary real numbers and let $\{x_n,y_n\}$ be a solution of system (2) then the following statements are true:

(i) *If* $a=0,\ z\neq0,$ $(or\ z=0,\ a\neq0)$, *then* $x_{10n}=0$ *and* $y_{10n-5}=z$ (*or* $x_{10n}=a$ *and* $y_{10n-5}=0$).

(ii) *If* $b=0,\ L\neq0,$ $(or\ L=0,\ b\neq0)$, *then* $x_{10n-1}=0$ *and* $y_{10n-6}=L$ (*or* $x_{10n-1}=b$ *and* $y_{10n-6}=0$).

(iii) *If* $c=0,\ o\neq0,$ (*or* $o=0,\ c\neq0)$, *then* $x_{10n-2}=0$ *and* $y_{10n-7}=o$ (*or* $x_{10n-2}=c$ *and* $y_{10n-7}=0$).

$(iv) If\ d=0,\ p\neq 0,\ (\ or\ p=0,\ d\neq 0),\ then\ x_{10n-3}=0\ and\ y_{10n-8}=p\ (or\ x_{10n-3}=d\ and\ y_{10n-8}=0).$

$(v) If\ e=0,\ q\neq 0,\ (\ or\ q=0,\ e\neq 0),\ then\ x_{10n-4}=0\ and\ y_{10n-9}=q\ (or\ x_{10n-4}=e\ and\ y_{10n-9}=0).$

$(vi) If\ f=0,\ r\neq 0,\ (\ or\ r=0,\ f\neq 0),\ then\ x_{10n-5}=0\ and\ y_{10n}=r\ (or\ x_{10n-5}=f\ and\ y_{10n}=0).$

$(vii) If\ g=0,\ u\neq 0,\ (\ or\ u=0,\ g\neq 0),\ then\ x_{10n-6}=0\ and\ y_{10n-1}=u\ (or\ x_{10n-6}=g\ and\ y_{10n-1}=0).$

$(viii) If\ h=0,\ v\neq 0,\ (\ or\ v=0,\ h\neq 0),\ then\ x_{10n-7}=0\ and\ y_{10n-2}=v\ (or\ x_{10n-7}=h\ and\ y_{10n-2}=0).$

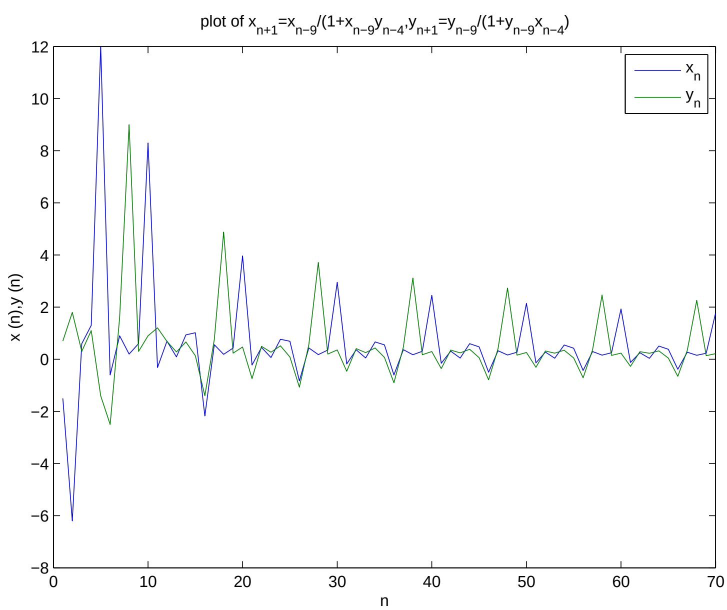
$(ix) If\ k=0,\ w\neq 0,\ (\ or\ w=0,\ k\neq 0),\ then\ x_{8n-1}=0\ and\ y_{10n-3}=w\ (or\ x_{8n-1}=k\ and\ y_{10n-3}=0).$
$(x) If\ s=0,\ t\neq 0,\ (\ or\ t=0,\ s\neq 0),\ then\ x_{10n-9}=0\ and\ y_{10n-4}=t\ (or\ x_{10n-9}=s\ and\ y_{10n-4}=0).$

Proof. The proof follows from the form of the solutions of system (1).

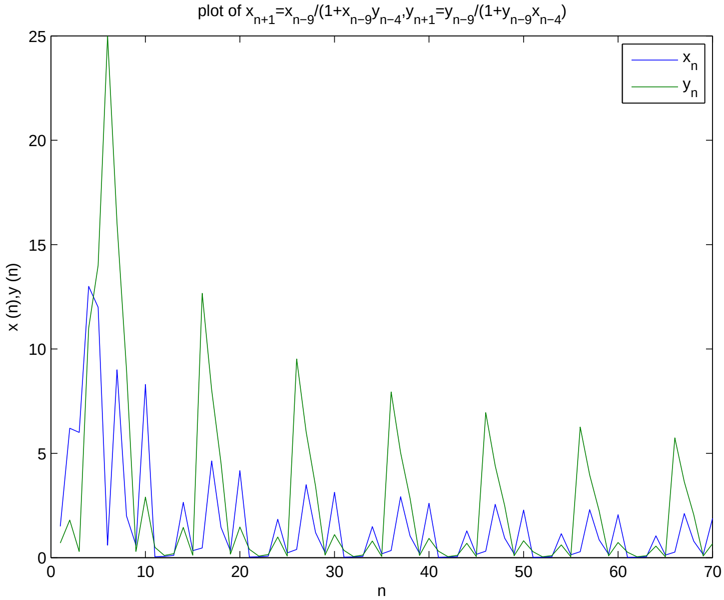
2.2. Numerical Examples

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section.

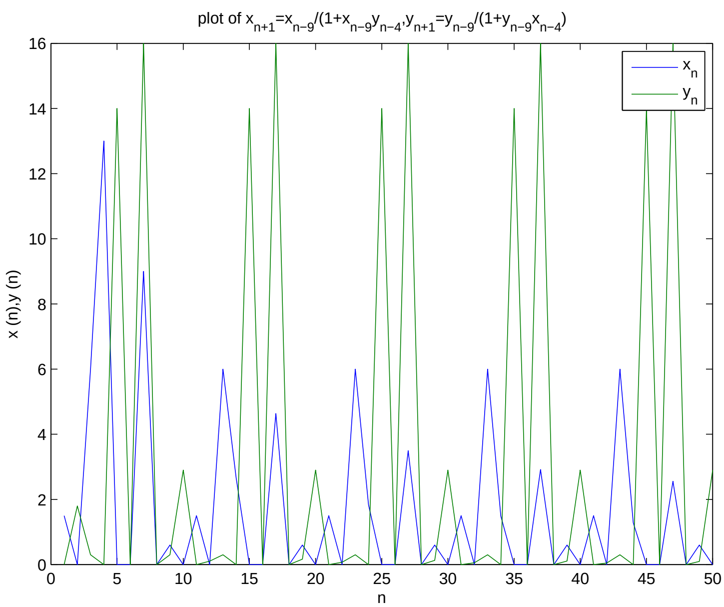
Example 1. For the initial conditions $x_{-9}=-1.5$, $x_{-8}=-6.2$, $x_{-7}=0.6$, $x_{-6}=1.3$, $x_{-5}=12$, $x_{-4}=-0.6$, $x_{-3}=0.9$, $x_{-2}=0.2$, $x_{-1}=0.6$, $x_0=8.3$, $y_{-9}=0.7$, $y_{-8}=1.8$, $y_{-7}=0.3$, $y_{-6}=1.1$, $y_{-5}=-1.4$, $y_{-4}=-2.5$, $y_{-3}=1.6$, $y_{-2}=9$, $y_{-1}=0.3$, $y_0=0.9$ when we take the system (2). (See Fig. 1).



Example 2. We consider interesting numerical example for the difference equations system (2) with the initial conditions $x_{-9}=1.5$, $x_{-8}=6.2$, $x_{-7}=6$, $x_{-6}=13$, $x_{-5}=12$, $x_{-4}=0.6$, $x_{-3}=9$, $x_{-2}=2$, $x_{-1}=0.6$, $x_0=8.3$, $y_{-9}=0.7$, $y_{-8}=1.8$, $y_{-7}=0.3$, $y_{-6}=11$, $y_{-5}=14$, $y_{-4}=25$, $y_{-3}=16$, $y_{-2}=9$, $y_{-1}=0.3$, $y_0=2.9$. (See Fig. 2).



Example 3. Consider the difference system (2) with the initial conditions $x_{-9} = 1.5$, $x_{-8} = 0$, $x_{-7} = 6$, $x_{-6} = 13$, $x_{-5} = 0$, $x_{-4} = 0$, $x_{-3} = 9$, $x_{-2} = 0$, $x_{-1} = 0.6$, $x_0 = 0$, $y_{-9} = 0$, $y_{-8} = 1.8$, $y_{-7} = 0.3$, $y_{-6} = 0$, $y_{-5} = 14$, $y_{-4} = 0$, $y_{-3} = 16$, $y_{-2} = 0$, $y_{-1} = 0.3$, $y_0 = 2.9$. (See Fig. 3).



3. On The System $x_{n+1} = \frac{x_{n-9}}{1+x_{n-9}y_{n-4}}$, $y_{n+1} = \frac{y_{n-9}}{1-x_{n-4}y_{n-9}}$

In this section, we obtain the form of the solution of the following system of difference equations

$$x_{n+1} = \frac{x_{n-9}}{1+x_{n-9}y_{n-4}}, \quad y_{n+1} = \frac{y_{n-9}}{1-x_{n-4}y_{n-9}}, \tag{3.1}$$

where the initial conditions x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, y_{-9}, y_{-8}, y_{-7},

$y_{-6}, y_{-5}, y_{-4}, y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0$ are arbitrary non zero real numbers with $x_{-9}y_{-4}, x_{-8}y_{-3}, x_{-7}y_{-2}, x_{-6}y_{-1}, x_{-5}y_0 \neq -1$ and $x_{-4}y_{-9}, x_{-3}y_{-8}, x_{-2}y_{-7}, x_{-1}y_{-6}, x_0y_{-5} \neq 1$.

3.1. The Form of the Solutions of System (3)

Theorem 2. Let $\{x_n, y_n\}$ are solutions of system (3). Then for $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned}
x_{10n-9} &= \frac{s}{(1+st)^n},\ x_{10n-8} = \frac{k}{(1+kw)^n},\ x_{10n-7} = \frac{h}{(1+hv)^n},\ x_{10n-6} = \frac{g}{(1+gu)^n},\\
x_{10n-5} &= \frac{f}{(1+fr)^n},\ x_{10n-4} = (-1)^n e(eq-1)^n,\quad x_{10n-3} = (-1)^n d(dp-1)^n,\\
x_{10n-2} &= (-1)^n c(co-1)^n,\quad x_{10n-1} = (-1)^n b(Lb-1)^n,\quad x_{10n} = (-1)^n a(az-1)^n,
\end{aligned}$$

$$\begin{aligned}
y_{10n-9} &= \frac{(-1)^n q}{(eq-1)^n}, y_{10n-8} = \frac{(-1)^n p}{(dp-1)^n}, y_{10n-7} = \frac{(-1)^n o}{(co-1)^n}, y_{10n-6} = \frac{(-1)^n L}{(Lb-1)^n},\\
y_{10n-5} &= \frac{(-1)^n z}{(az-1)^n},\quad y_{10n-4} = t(st+1)^n,\quad y_{10n-3} = w(kw+1)^n,\\
y_{10n-2} &= v(hv+1)^n,\quad y_{10n-1} = u(gu+1)^n,\quad y_{10n} = r(fr+1)^n.
\end{aligned}$$

Proof. For $n = 0$ the result holds. Suppose that the result holds for $n-1$

$$\begin{aligned}
x_{10n-19} &= \frac{s}{(1+st)^{n-1}},\quad x_{10n-18} = \frac{k}{(1+kw)^{n-1}},\quad x_{10n-17} = \frac{h}{(1+hv)^{n-1}},\\
x_{10n-16} &= \frac{g}{(1+gu)^{n-1}}, x_{10n-15} = \frac{f}{(1+fr)^{n-1}}, x_{10n-14} = (-1)^{n-1} e(eq-1)^{n-1},\\
x_{10n-13} &= (-1)^{n-1} d(dp-1)^{n-1},\quad x_{10n-12} = (-1)^{n-1} c(co-1)^{n-1},\\
x_{10n-11} &= (-1)^{n-1} b(Lb-1)^{n-1},\quad x_{10n-10} = (-1)^{n-1} a(az-1)^{n-1},
\end{aligned}$$

$$\begin{aligned}
y_{10n-19} &= \frac{(-1)^{n-1} q}{(eq-1)^{n-1}},\quad y_{10n-18} = \frac{(-1)^{n-1} p}{(dp-1)^{n-1}},\quad y_{10n-17} = \frac{(-1)^{n-1} o}{(co-1)^{n-1}},\\
y_{10n-16} &= \frac{(-1)^{n-1} L}{(Lb-1)^{n-1}},\quad y_{10n-15} = \frac{(-1)^{n-1} z}{(az-1)^{n-1}},\quad y_{10n-14} = t(st+1)^{n-1},\\
y_{10n-13} &= w(kw+1)^{n-1},\quad y_{10n-12} = v(hv+1)^{n-1},\\
y_{10n-11} &= u(gu+1)^{n-1},\quad y_{10n-10} = r(fr+1)^{n-1},
\end{aligned}$$

from system (3) we can see that

$$\begin{aligned}
x_{10n-9} &= \frac{x_{10n-19}}{1 + x_{10n-19}y_{10n-14}} = \frac{\frac{s}{(1+st)^{n-1}}}{1 + (\frac{s}{(1+st)^{n-1}})(t(st+1)^{n-1})}\\
&= \frac{\frac{s}{(1+st)^{n-1}}}{1+st} = \frac{s}{(1+st)^n}.
\end{aligned}$$

Also, we get

$$\begin{aligned}
x_{10n-8} &= \frac{x_{10n-18}}{1+x_{10n-18}y_{10n-13}} = \frac{\frac{k}{(1+kw)^{n-1}}}{1+(\frac{k}{(1+kw)^{n-1}}w(kw+1)^{n-1})} = \frac{k}{(1+kw)^{n}}, \\
x_{10n-3} &= \frac{x_{10n-13}}{1+x_{10n-13}y_{10n-8}} = \frac{(-1)^{n-1}d(dp-1)^{n-1}}{1+\frac{(-1)^{n}p}{(dp-1)^{n}}(-1)^{n-1}d(dp-1)^{n-1}} \\
&= \frac{(-1)^{n}d(dp-1)^{n}}{-dp+1+dp} = (-1)^{n}d(dp-1)^{n}, \\
y_{10n-6} &= \frac{y_{10n-16}}{1-y_{10n-16}x_{10n-11}} = \frac{\frac{(-1)^{n-1}L}{(Lb-1)^{n-1}}}{1-\frac{(-1)^{n-1}L}{(Lb-1)^{n-1}}(-1)^{n-1}b(Lb-1)^{n-1}} \\
&= \frac{\frac{(-1)^{n-1}L}{(Lb-1)^{n-1}}}{1-(-1)^{2n-1}Lb} = \frac{(-1)^{n}L}{(Lb-1)^{n}}.
\end{aligned}$$

By similar way we can prove the other relations. This completes the proof.

Lemma 4. Assume that $\{x_n, y_n\}$ be a solution of system (3) with $x_{-9}y_{-4}$, $x_{-8}y_{-3}$, $x_{-7}y_{-2}$, $x_{-6}y_{-1}$, $x_{-5}y_0 = -2$ and $x_{-4}y_{-9}$, $x_{-3}y_{-8}$, $x_{-2}y_{-7}$, $x_{-1}y_{-6}$, $x_0y_{-5} = 2$, then $\{x_n\}$ and $\{y_n\}$ is periodic with period twenty.

Proof. From the form of the solutions of system (3) we see when $x_{-9}y_{-4}$, $x_{-8}y_{-3}$, $x_{-7}y_{-2}$, $x_{-6}y_{-1}$, $x_{-5}y_0 = -2$ and $x_{-4}y_{-9}$, $x_{-3}y_{-8}$, $x_{-2}y_{-7}$, $x_{-1}y_{-6}$, $x_0y_{-5} = 2$ that

$$\begin{aligned}
x_{10n-9} &= \frac{s}{(-1)^n},\ x_{10n-8} = \frac{k}{(-1)^n},\ x_{10n-7} = \frac{h}{(-1)^n},\ x_{10n-6} = \frac{g}{(-1)^n},\ x_{10n-5} = \frac{f}{(-1)^n}, \\
x_{10n-4} &= (-1)^n e,\ x_{10n-3} = (-1)^n d,\ x_{10n-2} = (-1)^n c,\ x_{10n-1} = (-1)^n b,\ x_{10n} = (-1)^n a,
\end{aligned}$$

$$\begin{aligned}
y_{10n-9} &= (-1)^n q,\ y_{10n-8} = (-1)^n p,\ y_{10n-7} = (-1)^n o,\ y_{10n-6} = (-1)^n L,\ y_{10n-5} = (-1)^n z, \\
y_{10n-4} &= t(-1)^n,\ y_{10n-3} = w(-1)^n,\ y_{10n-2} = v(-1)^n,\ y_{10n-1} = u(-1)^n,\ y_{10n} = r(-1)^n.
\end{aligned}$$

and so it is periodic with period twenty. This completes the proof.

Lemma 5. Let $\{x_n, y_n\}$ be a positive solution of system (3), then $\{x_n\}$ is bounded and converges to zero.

Proof. It follows from system (3) that

$$x_{n+1} = \frac{x_{n-9}}{1+x_{n-9}y_{n-4}} < x_{n-9}.$$

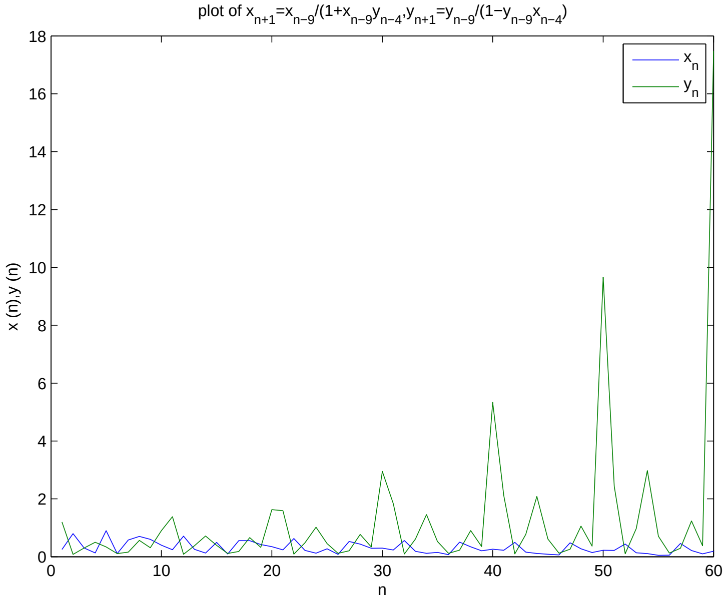
Then the subsequences $\{x_{10n-9}\}_{n=0}^{\infty}$, $\{x_{10n-8}\}_{n=0}^{\infty}$, $\{x_{10n-7}\}_{n=0}^{\infty}$, $\{x_{10n-6}\}_{n=0}^{\infty}$, $\{x_{10n-5}\}_{n=0}^{\infty}$, $\{x_{10n-4}\}_{n=0}^{\infty}$, $\{x_{10n-3}\}_{n=0}^{\infty}$, $\{x_{10n-2}\}_{n=0}^{\infty}$, $\{x_{10n-1}\}_{n=0}^{\infty}$ and $\{x_{10n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$M = \max\{x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}.$$

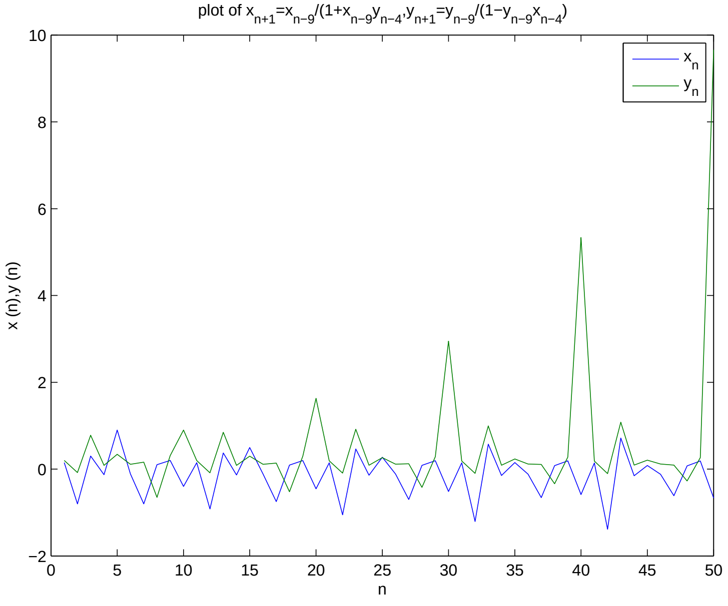
3.2. Numerical Examples

Here we plot some examples to illustrate the results of the previous sections.

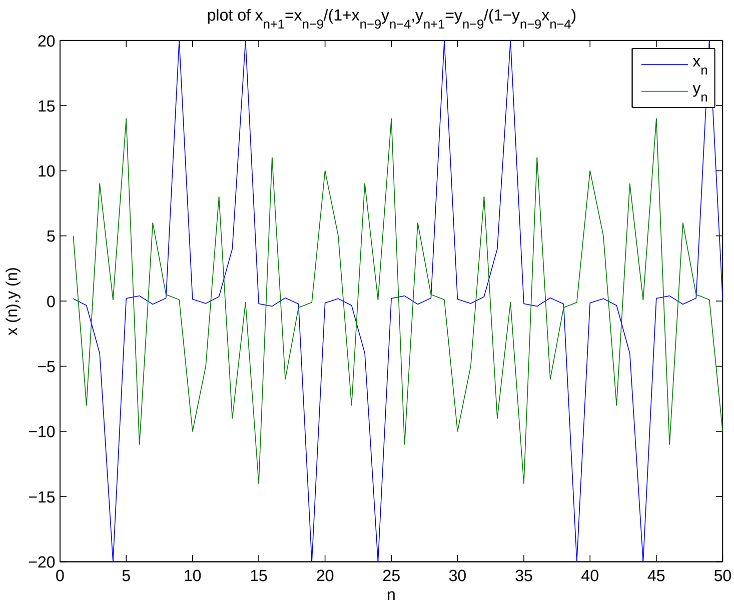
Example 4. See Figure 4 when we suppose the initial conditions for system (3) as follows: $x_{-9} = 0.25$, $x_{-8} = 0.8$, $x_{-7} = 0.3$, $x_{-6} = 0.13$, $x_{-5} = 0.9$, $x_{-4} = 0.11$, $x_{-3} = 0.58$, $x_{-2} = 0.7$, $x_{-1} = 0.6$, $x_0 = 0.4$, $y_{-9} = 1.2$, $y_{-8} = .08$, $y_{-7} = 0.3$, $y_{-6} = 0.5$, $y_{-5} = 0.34$, $y_{-4} = 0.11$, $y_{-3} = 0.16$, $y_{-2} = 0.565$, $y_{-1} = 0.311$, $y_0 = 0.9$.



Example 5. We take a numerical example for the system (3) with the initial conditions $x_{-9} = 0.15$, $x_{-8} = -0.8$, $x_{-7} = 0.3$, $x_{-6} = -0.13$, $x_{-5} = 0.9$, $x_{-4} = -0.11$, $x_{-3} = -0.8$, $x_{-2} = 0.1$, $x_{-1} = 0.2$, $x_0 = -0.4$, $y_{-9} = 0.2$, $y_{-8} = -0.08$, $y_{-7} = 0.78$, $y_{-6} = 0.085$, $y_{-5} = 0.34$, $y_{-4} = 0.11$, $y_{-3} = 0.16$, $y_{-2} = -0.65$, $y_{-1} = 0.311$, $y_0 = 0.9$. (See Fig. 5).



Example 6. Figure 6 shows the periodic solutions with period twenty for system (3) when $x_{-9} = 2/11$, $x_{-8} = -1/3$, $x_{-7} = -4$, $x_{-6} = -20$, $x_{-5} = 0.2$, $x_{-4} = 0.4$, $x_{-3} = -0.25$, $x_{-2} = 2/9$, $x_{-1} = 20$, $x_0 = 1/7$, $y_{-9} = 5$, $y_{-8} = -8$, $y_{-7} = 9$, $y_{-6} = 0.1$, $y_{-5} = 14$, $y_{-4} = -11$, $y_{-3} = 6$, $y_{-2} = 0.5$, $y_{-1} = 0.1$, $y_0 = -10$.



4. On The Systems $x_{n+1} = \frac{x_{n-9}}{1+x_{n-9}y_{n-4}}$, $y_{n+1} = \frac{y_{n-9}}{-1\pm x_{n-4}y_{n-9}}$

Here, we investigate the expressions of the solutions of the following systems of difference equations

$$x_{n+1} = \frac{x_{n-9}}{1 + x_{n-9}y_{n-4}}, \quad y_{n+1} = \frac{y_{n-9}}{-1 + x_{n-4}y_{n-9}}, \tag{4.1}$$

$$x_{n+1} = \frac{x_{n-9}}{1 + x_{n-9}y_{n-4}}, \quad y_{n+1} = \frac{y_{n-9}}{-1 - x_{n-4}y_{n-9}}, \tag{4.2}$$

where the initial conditions x_{-9}, x_{-8}, x_{-7}, x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, y_{-9}, y_{-8}, y_{-7}, y_{-6}, y_{-5}, y_{-4}, y_{-4}, y_{-3}, y_{-2}, y_{-1}, y_0 are arbitrary non zero real numbers.

4.1. The Form of the Solutions of Systems (4.1) and (4.2)

Theorem 3. Assume that $\{x_n, y_n\}$ are solutions of the system (4.1) where the initial conditions satisfy that $x_{-9}y_{-4}$, $x_{-8}y_{-3}$, $x_{-7}y_{-2}$, $x_{-6}y_{-1}$, $x_{-5}y_0 \neq \pm 1$ and $x_{-4}y_{-9}$, $x_{-3}y_{-8}$, $x_{-2}y_{-7}$, $x_{-1}y_{-6}$, $x_0y_{-5} \neq 1, \neq \frac{1}{2}$. Then we have the following expressions for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{20n-9} &= \frac{(-1)^n s}{(s^2t^2-1)^n}, \quad x_{20n-8} = \frac{(-1)^n k}{(k^2w^2-1)^n}, \quad x_{20n-7} = \frac{(-1)^n h}{(h^2v^2-1)^n}, \\
x_{20n-6} &= \frac{(-1)^n g}{(g^2u^2-1)^n}, \quad x_{20n-5} = \frac{(-1)^n f}{(f^2r^2-1)^n}, \quad x_{20n-4} = \frac{(-1)^n e(eq-1)^{2n}}{(2eq-1)^n}, \\
x_{20n-3} &= \frac{(-1)^n d(dp-1)^{2n}}{(2dp-1)^n}, \; x_{20n-2} = \frac{(-1)^n c(co-1)^{2n}}{(2co-1)^n}, \; x_{20n-1} = \frac{(-1)^n b(Lb-1)^{2n}}{(2Lb-1)^n}, \\
x_{20n} &= \frac{(-1)^n a(az-1)^{2n}}{(2az-1)^n}, x_{20n+1} = \frac{(-1)^n s}{(st+1)(s^2t^2-1)^n}, x_{20n+2} = \frac{(-1)^n k}{(1+kw)(k^2w^2-1)^n}, \\
x_{20n+3} &= \frac{(-1)^n h}{(1+hv)(h^2v^2-1)^n}, x_{20n+4} = \frac{(-1)^n g}{(1+gu)(g^2u^2-1)^n}, x_{20n+5} = \frac{(-1)^n f}{(1+fr)(f^2r^2-1)^n}, \\
x_{20n+6} &= \frac{(-1)^n e(eq-1)^{2n+1}}{(2eq-1)^{n+1}}, x_{20n+7} = \frac{(-1)^n d(dp-1)^{2n+1}}{(2dp-1)^{n+1}}, x_{20n+8} = \frac{(-1)^n c(co-1)^{2n+1}}{(2co-1)^{n+1}}, \\
x_{20n+9} &= \frac{(-1)^n b(Lb-1)^{2n+1}}{(2Lb-1)^{n+1}}, \quad x_{20n+10} = \frac{(-1)^n a(az-1)^{2n+1}}{(2az-1)^{n+1}},
\end{aligned}$$

$$\begin{aligned}
y_{20n-9} &= \frac{(-1)^n q(2eq-1)^n}{(eq-1)^{2n}}, \quad y_{20n-8} = \frac{(-1)^n p(2dp-1)^n}{(dp-1)^{2n}}, \quad y_{20n-7} = \frac{(-1)^n o(2co-1)^n}{(co-1)^{2n}}, \\
y_{20n-6} &= \frac{(-1)^n L(2Lb-1)^n}{(Lb-1)^{2n}}, \quad y_{20n-5} = \frac{(-1)^n z(2az-1)^n}{(az-1)^{2n}}, \quad y_{20n-4} = (-1)^n t(s^2t^2-1)^n, \\
y_{20n-3} &= (-1)^n w(k^2w^2-1)^n, \quad y_{20n-2} = (-1)^n v(h^2v^2-1)^n, \quad y_{20n-1} = (-1)^n u(g^2u^2-1)^n, \\
y_{20n} &= (-1)^n r(f^2r^2-1)^n, \quad y_{20n+1} = \frac{(-1)^n q(2eq-1)^n}{(eq-1)^{2n+1}}, \quad y_{20n+2} = \frac{(-1)^n p(2dp-1)^n}{(dp-1)^{2n+1}}, \\
y_{20n+3} &= \frac{(-1)^n o(2co-1)^n}{(co-1)^{2n+1}}, \quad y_{20n+4} = \frac{(-1)^n L(2Lb-1)^n}{(Lb-1)^{2n+1}}, \quad y_{20n+5} = \frac{(-1)^n z(2az-1)^n}{(az-1)^{2n+1}}, \\
y_{20n+6} &= (-1)^{n+1} t(st+1)(s^2t^2-1)^n, \quad y_{20n+7} = (-1)^{n+1} w(kw+1)(k^2w^2-1)^n, \\
y_{20n+8} &= (-1)^{n+1} v(hv+1)(h^2v^2-1)^n, \quad y_{20n+9} = (-1)^{n+1} u(gu+1)(g^2u^2-1)^n, \\
y_{20n+10} &= (-1)^{n+1} r(fr+1)(f^2r^2-1)^n.
\end{aligned}$$

Proof. As the proof of Theorem 1 and will be left to the reader.

Theorem 4. Consider $\{x_n, y_n\}$ are solutions of the system (4.2) with $x_{-9}y_{-4}$, $x_{-8}y_{-3}$, $x_{-7}y_{-2}$, $x_{-6}y_{-1}$, $x_{-5}y_0 \neq -1, \neq -\frac{1}{2}$ and $x_{-4}y_{-9}$, $x_{-3}y_{-8}$, $x_{-2}y_{-7}$, $x_{-1}y_{-6}$, $x_0y_{-5} \neq \pm 1$. Then for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{20n-9} &= \frac{s(2st+1)^n}{(st+1)^{2n}}, \quad x_{20n-8} = \frac{k(2kw+1)^n}{(kw+1)^{2n}}, \quad x_{20n-7} = \frac{h(2hv+1)^n}{(hv+1)^{2n}}, \\
x_{20n-6} &= \frac{g(2gu+1)^n}{(gu+1)^{2n}}, \quad x_{20n-5} = \frac{f(2fr+1)^n}{(fr+1)^{2n}}, \quad x_{20n-4} = (-1)^n e(e^2q^2-1)^n, \\
x_{20n-3} &= (-1)^n d(d^2p^2-1)^n, x_{20n-2} = (-1)^n c(c^2o^2-1)^n, x_{20n-1} = (-1)^n b(L^2b^2-1)^n, \\
x_{20n} &= (-1)^n a(a^2z^2-1)^n, \quad x_{20n+1} = \frac{s(2st+1)^n}{(st+1)^{2n+1}}, \quad x_{20n+2} = \frac{k(2kw+1)^n}{(kw+1)^{2n+1}}, \\
x_{20n+3} &= \frac{h(2hv+1)^n}{(hv+1)^{2n+1}}, \quad x_{20n+4} = \frac{g(2gu+1)^n}{(gu+1)^{2n+1}}, \quad x_{20n+5} = \frac{f(2fr+1)^n}{(fr+1)^{2n+1}}, \\
x_{20n+6} &= (-1)^n e(eq+1)(e^2q^2-1)^n, \quad x_{20n+7} = (-1)^n d(dp+1)(d^2p^2-1)^n, \\
x_{20n+8} &= (-1)^n c(co+1)(c^2o^2-1)^n, \quad x_{20n+9} = (-1)^n b(Lb+1)(L^2b^2-1)^n, \\
x_{20n+10} &= (-1)^n a(az+1)(a^2z^2-1)^n,
\end{aligned}$$

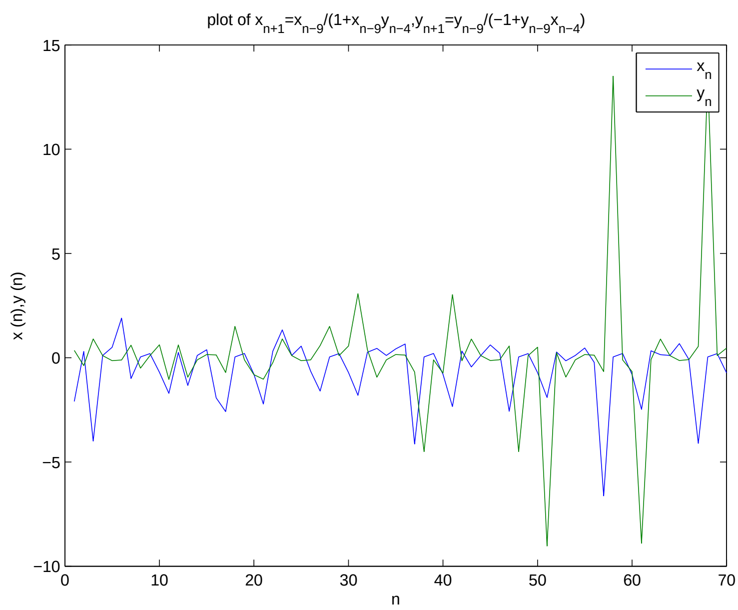
$$\begin{aligned}
y_{20n-9} &= \frac{(-1)^n q}{(e^2q^2-1)^n}, \quad y_{20n-8} = \frac{(-1)^n p}{(d^2p^2-1)^n}, \quad y_{20n-7} = \frac{(-1)^n o}{(c^2o^2-1)^n}, \\
y_{20n-6} &= \frac{(-1)^n L}{(L^2b^2-1)^n}, \quad y_{20n-5} = \frac{(-1)^n z}{(a^2z^2-1)^n}, \quad y_{20n-4} = \frac{t(st+1)^{2n}}{(2st+1)^n}, \\
y_{20n-3} &= \frac{w(kw+1)^{2n}}{(2kw+1)^n}, \quad y_{20n-2} = \frac{v(hv+1)^{2n}}{(2hv+1)^n}, \quad y_{20n-1} = \frac{u(gu+1)^{2n}}{(2gu+1)^n}, \\
y_{20n} &= \frac{r(fr+1)^{2n}}{(2fr+1)^n}, \quad y_{20n+1} = \frac{(-1)^n q(2eq-1)^n}{(eq-1)^{2n+1}}, \quad y_{20n+2} = \frac{(-1)^n p(2dp-1)^n}{(dp-1)^{2n+1}}, \\
y_{20n+3} &= \frac{(-1)^n o(2co-1)^n}{(co-1)^{2n+1}}, \quad y_{20n+4} = \frac{(-1)^n L(2Lb-1)^n}{(Lb-1)^{2n+1}}, \quad y_{20n+5} = \frac{(-1)^n z(2az-1)^n}{(az-1)^{2n+1}}, \\
y_{20n+6} &= \frac{-t(st+1)^{2n+1}}{(2st+1)^{n+1}}, \quad y_{20n+7} = \frac{-w(kw+1)^{2n+1}}{(2kw+1)^{n+1}}, \quad y_{20n+8} = \frac{-v(hv+1)^{2n+1}}{(2hv+1)^{n+1}}, \\
y_{20n+9} &= \frac{-u(gu+1)^{2n+1}}{(2gu+1)^{n+1}}, \quad y_{20n+10} = \frac{-r(fr+1)^{2n+1}}{(2fr+1)^{n+1}}.
\end{aligned}$$

Proof. As in the proof of Theorem 2 and will be omitted.

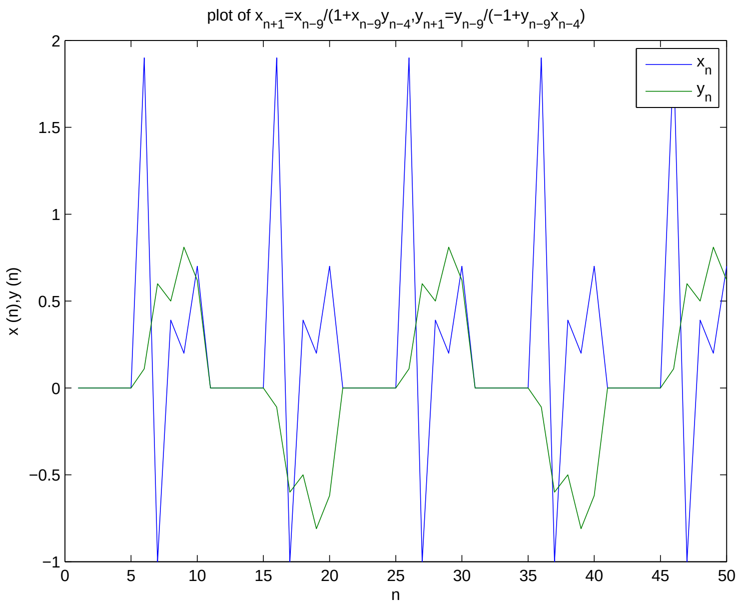
4.2. Numerical Examples

In this subsection to support our theoretical discussions, we consider some interesting numerical examples.

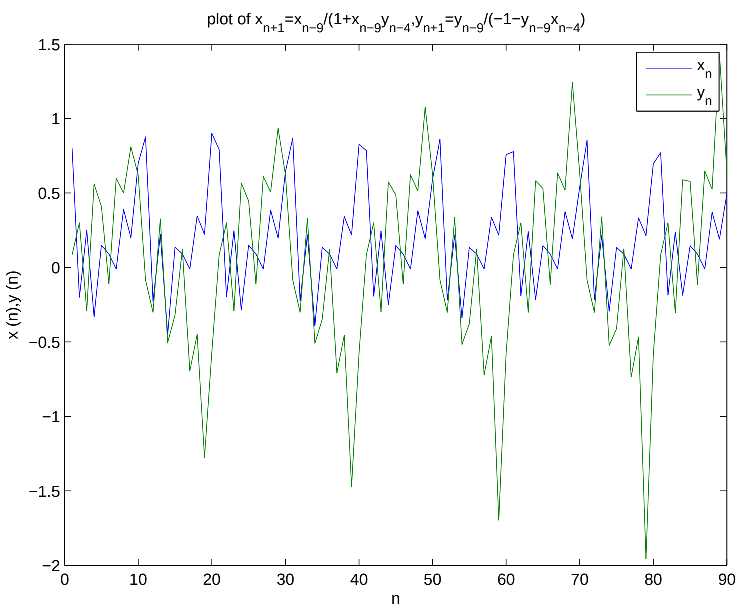
Example 7. When we put the initial conditions $x_{-9} = -2.1$, $x_{-8} = 0.3$, $x_{-7} = -4$, $x_{-6} = 0.11$, $x_{-5} = 0.5$, $x_{-4} = 1.9$, $x_{-3} = -1$, $x_{-2} = 0.039$, $x_{-1} = 0.2$, $x_0 = -0.7$, $y_{-9} = 0.35$, $y_{-8} = -0.38$, $y_{-7} = 0.9$, $y_{-6} = 0.1$, $y_{-5} = -0.14$, $y_{-4} = -0.11$, $y_{-3} = 0.6$, $y_{-2} = -0.5$, $y_{-1} = 0.1$, $y_0 = 0.62$ for system (4). (See Fig. 7).



Example 8. See Figure 8 it shows the periodicity of the solutions for system (4) where the initial conditions takes the numbers $x_{-9} = x_{-8} = x_{-7} = x_{-6} = x_{-5} = 0$, $x_{-4} = 1.9$, $x_{-3} = -1$, $x_{-2} = 0.39$, $x_{-1} = 0.2$, $x_0 = 0.7$, $y_{-9} = y_{-8} = y_{-7} = y_{-6} = y_{-5} = 0$, $y_{-4} = 0.11$, $y_{-3} = 0.6$, $y_{-2} = 0.5$, $y_{-1} = 0.81$, $y_0 = 0.62$.



Example 9. Let we take for the difference system (5) the initial conditions $x_{-9} = 0.8$, $x_{-8} = -0.2$, $x_{-7} = 0.25$, $x_{-6} = -0.33$, $x_{-5} = 0.15$, $x_{-4} = 0.9$, $x_{-3} = -0.01$, $x_{-2} = 0.39$, $x_{-1} = 0.2$, $x_0 = 0.7$, $y_{-9} = 0.085$, $y_{-8} = 0.3$, $y_{-7} = -0.29$, $y_{-6} = 0.56$, $y_{-5} = 0.41$, $y_{-4} = -0.11$, $y_{-3} = 0.6$, $y_{-2} = 0.5$, $y_{-1} = 0.81$, $y_0 = 0.62$. (See Fig. 9).



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