Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–10.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.63656

Automorphisms of Projective Manifolds

Tsemo Aristide

ABSTRACT: Let $(M, P\nabla_M)$ be a compact projective manifold and $Aut(M, P\nabla_M)$ its group of automorphisms. The purpose of this paper is to study the topological properties of $(M, P\nabla_M)$ if $Aut(M, P\nabla_M))$ is not discrete by applying the results of [13] and the Benzekri's functor which associates to a projective manifold a radiant affine manifold. This enables us to show that the orbits of the connected component of $Aut(M, P\nabla_M)$ are immersed projective submanifolds. We also classify 3-dimensional compact projective manifolds such that $dim(Aut(M, P\nabla_M)) \geq 2$.

Key Words: Affine manifolds, projective manifolds.

Contents

1. Introduction

The purpose of this paper is to study the group of automorphisms of projective manifolds. Firstly we recall the definition of (X, G) manifolds, their group of automorphisms and morphisms between (X, G)-structures. We applied the results described in the general framework of (X, G)-manifolds to the category of affine manifolds and projective manifolds. Benzekri has constructed a functor which associates to a projective manifold $(M, P\nabla_M)$ a radiant affine manifold $(B(M), \nabla_{B(M)})$ whose underlying topological space is $M \times S^1$. It enables us to show that there exists a surjective morphism between the connected component $Aut(B(M), \nabla_{B(M)})_0$ of the group of affine automorphisms of $(B(M), \nabla_{B(M)})$ and the connected component $Aut(M, P\nabla_M)_0$ of the group of projective automorphisms of $(M, P\nabla_M)$.

Let (M, ∇_M) be a compact affine manifold, in [13], I have studied the relations between $Aut(M, \nabla_M)$ and the topology of M. This enables us to show that the orbits of $Aut(M, P\nabla_M)_0$ are projective immersed submanifolds. In the last section, we study the automorphisms group of 2 and 3 dimensional projective manifolds. We remark that a 2-dimensional projective manifold whose group of automorphisms is not discrete is homeomorphic to the sphere, the 2-dimensional projective space or the two dimensional torus. Finally we show that a 3-dimensional projective manifold $(M, P\nabla_M)$ whose developing map is injective and such that $dim(Aut(M, P\nabla_M) \geq 2$ is homeomorphic to a spherical manifold, $S^2 \times S^1$, or a finite cover of M is the total space of a torus bundle.

Remark that (X, G) manifolds play an important role in low dimensional topology: seven of the eight geometry of Thurston are examples of projective geometry (see Cooper and Goldman [8] p. 1220). In [12] p.17, Sullivan and Thurston note that the existence of a (X, G)-structure on every 3-manifold implies the Poincare conjecture.

2010 *Mathematics Subject Classification:* 53C15, 57S25.
Submitted May 21, 2022. Published August, 20, 2022

2. (X,G)-manifolds

A (X,G) model is a finite dimensional differentiable manifold X, endowed with an effective and transitive action of a Lie group G which satisfies the unique extension property. This is equivalent to saying that: two elements g, g' of G are equal if and only if their respective restrictions to a non empty open subset of X are equal.

A (X,G) manifold (M,X,G) is a differentiable manifold M, endowed with an open covering $(U_i)_{i\in I}$ such that for every $i\in I$, there exists a differentiable map $f_i : U_i \to X$ which is a diffeomorphism onto its image and $f_i \circ f_j^{-1}$ coincides with the restriction of an element g_{ij} of G to $f_j(U_i \cap U_j)$. The map f_i is called an (X,G) chart.

A (X,G) structure defined on M can be lifted to the universal cover $\tilde{M}$ of M. This structure is defined by a local diffeomorphism $D_M : \tilde{M} \to X$. This implies that a (X,G) chart of this structure is an open subset U of $\tilde{M}$ such that the restriction of D_M to U is a diffeomorphism onto its image.

Let (X,G) and (X',G') be two models, $\phi : X \to X'$ a differentiable map and $\Phi : G \to G'$ a morphism of groups such that for every $g \in G$, the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \phi\downarrow & & \downarrow\phi \\ X' & \xrightarrow{\Phi(g)} & X' \end{array}$$

Let (M,X,G) (resp. (M',X',G')) be a (X,G) manifold (resp. a (X',G') manifold). A (Φ,ϕ)-morphism $f : (M,X,G) \to (M',X',G')$ is a differentiable map: $f : M \to M'$ such that for every chart (U_i, f_i) of M such that $f(U_i)$ is contained in the chart (V_j, f'_j) of M', there exists an element $g \in G$ such that the restrictions of $f'_j \circ f \circ f_i^{-1}$ and $\Phi(g)\circ\phi$ to $f_i(U_i)$ coincide.

We will denote by $Aut(X,M,G)$ the group of (Id_G, Id_X)-automorphisms of (M,X,G) and by $Aut(M,X,G)_0$ its connected component. It is a Lie group endowed with the compact open topology. For every element $g \in Aut(\tilde{M},X,G)$, the developing map defines a representation $H_M : Aut(\tilde{M},X,G) \to G$ such that the following diagram is commutative:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{g} & \tilde{M} \\ D_M\downarrow & & \downarrow D_M \\ X & \xrightarrow{H_M(g)} & X \end{array}$$

Remark that the group of Deck transformations that we identify to the fundamental group $\pi_1(M)$, of M, is a subgroup of $Aut(\tilde{M},X,G)$. The restriction h_M of H_M to the fundamental group $\pi_1(M)$, of M is called the holonomy representation of the (X,G) manifold (M,X,G).

The pullback $p_M(f)$ of an element f of $Aut(M,X,G)$, by the universal covering map, $p_M : \tilde{M} \to M$ is an element of $Aut(\tilde{M},X,G)$ which belongs to the normalizer $N(\pi_1(M))$ of $\pi_1(M)$ in $Aut(\tilde{M},X,G)$. Conversely, every element g of $N(\pi_1(M))$ induces an element $A_M(g)$ of $Aut(M,X,G)$ such that the following diagram is commutative:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{g} & \tilde{M} \\ p_M\downarrow & & \downarrow p_M \\ M & \xrightarrow{A_M(g)} & M. \end{array}$$

The kernel of the morphism $A_M : N(\pi_1(M)) \to Aut(M,X,G)$ is $\pi_1(M)$ and A_M is a local diffeomorphim. We will denote by $N(\pi_1(M))_0$ the connected component of $N(\pi_1(M))$, it is also the connected component of the commutator of $\pi_1(M)$ in $Aut(\tilde{M},X,G)$. Since A_M is locally invertible, it induces an isomorphism between the Lie algebra $n(\pi_1(M))$ of $N(\pi_1(M))$ and the Lie algebra $aut(M,X,G)$ of $Aut(M,X,G)$. If (M,X,G) is a compact (X,G) manifold, $aut(M,X,G)$ is isomorphic to the subspace of elements of $\mathcal{G}$, the Lie algebra of G, which are invariant by $h_M(\pi_1(M))$.

3. Affine and projective structures

Let $\mathbb{R}^n$ be the n-dimensional real vector space. We denote by $Gl(n,\mathbb{R})$ the group of linear automorphisms of $\mathbb{R}^n$ and by $Aff(n,\mathbb{R})$ its group of affine transformations. If we fix an origin 0 of $\mathbb{R}^n$, for every element $f\in Aff(n,\mathbb{R})$, we can write $f=(L(f),a_f)$ where $L(f)$ is an element of $Gl(\mathbb{R}^n)$ and $a_f=f(0)$. The couple $(\mathbb{R}^n, Aff(n,\mathbb{R}))$ is a model. A $(\mathbb{R}^n, Aff(n,\mathbb{R}))$ manifold is also called an affine manifold. Equivalently, an $(\mathbb{R}^n, Aff(n,\mathbb{R}))$ manifold is a n-dimensional differentiable manifold M endowed with a connection ∇_M whose curvature and torsion tensors vanish identically.

Remark that the linear part $L(h_M)$ of the holonomy representation h_M of an affine manifold (M,∇_M) is the holonomy of the connection ∇_M. We say that the n-dimensional affine manifold (M,∇_M) is radiant if its holonomy h_M fixes an element of $\mathbb{R}^n$, this is equivalent to saying that h_M and $L(h_M)$ are conjugated by a translation.

The n-dimensional real projective space $\mathbb{R}P^n$ is the quotient of $\mathbb{R}^{n+1}-\{0\}$ by the equivalence relation defined by $x\simeq y$ if and only there exists $\lambda\in\mathbb{R}$ such that $x=\lambda y$. If x is an element of $\mathbb{R}^{n+1}-\{0\}$, we will denote by $[x]_{\mathbb{R}P^n}$ its equivalent class. The group $Gl(n+1,\mathbb{R})$ acts transitively on $\mathbb{R}P^n$ by the action defined by $g.[x]_{\mathbb{R}P^n}=[g.x]_{\mathbb{R}P^n}$ the kernel of this action is the group H_{n+1} of homothetic maps. We denote by $PGl(n+1,\mathbb{R})$ the quotient $Gl(n+1,\mathbb{R})$ by H_n. The couple $(\mathbb{R}P^n, PGl(n+1,\mathbb{R}))$ is a model. A $(\mathbb{R}P^n, PGl(n+1,\mathbb{R}))$ is also called a projective manifold. Equivalently, a projective manifold can be defined by a differentiable manifold M endowed with a projectively flat connection $P\nabla_M$. We will denote it by $(M,P\nabla_M)$.

The n-dimensional sphere S^n is the quotient of $\mathbb{R}^{n+1}-\{0\}$ by the equivalence relation defined by $x\simeq y$ if and only if there exists $\lambda>0$ such that $x=\lambda y$. Let x be an element of $\mathbb{R}^{n+1}-\{0\}$, we will denote by $[x]_{S^n}$ its equivalence class for this relation. Remark that if $\langle,\rangle$ is an Euclidean metric defined on $\mathbb{R}^{n+1}$, there exists a bijection between the unit sphere $S^n_{\langle,\rangle}=\{x: x\in\mathbb{R}^{n+1},\langle x,x\rangle=1\}$ and S^n defined by the restriction of the equivalence relation to $S^n_{\langle,\rangle}$.

There exists a map $D_{S^n}:S^n\to\mathbb{R}P^n$ such that for every element x of $\mathbb{R}^{n+1}-\{0\}$, $[x]_{\mathbb{R}P^n}=D_{S^n}([x]_{S^n})$. The map D_{S^n} is a covering, thus is the developing map of a projectively flat connection $P\nabla_{S^n}$ defined on S^n.

A p-dimensional projective submanifold $(F,P\nabla_F)$ of the projective manifold $(M,P\nabla_M)$ is a p-dimensional submanifold F of M endowed with a structure of a projective manifold, such that the canonical embedding $i_F:(F,P\nabla_F)\to(M,P\nabla_M)$ is a morphism of projective manifolds.

Let $\hat{F}$ be the universal cover of F, we can lift i_F to a projective map $\hat{i}_F:\hat{F}\to\hat{M}$. The image of $D_M\circ\hat{i}_F$ is contained in a p-dimensional projective subspace U_F of $\mathbb{R}P^n$. The map $D_M\circ\hat{i}_F:\hat{F}\to U_F$ is a developing map of F. There exists a canonical morphism $\pi_F:\pi_1(F)\to\pi_1(M)$ induced by i_F. Let γ be an element of $\pi_1(F)$, the holonomy $h_F(\gamma)$ is the restriction of $h_M(\pi_F(\gamma))$ to U_F. If there is no confusion, we are going to denote $h_M(\pi_F(\gamma))$ by $h_M(\gamma)$.

Proposition 3.1. *The group of automorphisms of the n-dimensional projective manifold S^n is isomorphic to $Sl(n+1,\mathbb{R})$, the group of invertible $(n+1)\times(n+1)$ matrices such that for every element $A\in Sl(n+1,\mathbb{R})$, $|det(A)|=1$.*

Proof. Let g be an element of $Sl(n+1,\mathbb{R})$. For every $[x]_{S^n}\in S^n$, we write $u_g(x)=[g(x)]_{S^n}$. Let $[g]$ be the image of g by the quotient map $Sl(n+1,\mathbb{R})\to PGl(n+1,\mathbb{R})$, we have $[g]\circ D_{S^n}=D_{S^n}\circ u_g$. This implies that u_g is an element of $Aut(S_n,P\nabla_{S^n})$. Suppose that $u_g=Id_{S^n}$, it implies that for every $[x]\in S^n$, $g(x)=\lambda(x)x, \lambda(x)>0$, we deduce that $g(x)=\lambda Id_{\mathbb{R}^n}, \lambda>0$, and $\lambda^{n+1}=1$ since $g\in Sl(n+1,\mathbb{R})$. This implies that $\lambda=1$. We deduce that $u:Sl(n+1,\mathbb{R})\to Aut(S^n,P\nabla_{S^n})$ defined by $u(g)=u_g$ is injective. Let f be an element of $Aut(S^n,P\nabla_{S^n})$, there exists an element $[g]\in PGl(n+1,\mathbb{R})$ such that $[g]\circ D_{S^n}=D_{S^n}\circ f$. Consider an element $g\in Sl(n+1,\mathbb{R})$ whose image by the quotient map is $[g]$, $f=u_g$. This implies that u is an isomorphism. □

The Benzecri correspondence.

Consider the embedding $i_n^G : Gl(n,\mathbb{R}) \to Gl(n+1,\mathbb{R})$ defined by $i_n^G(A) =$

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

and the open embedding $i_n : \mathbb{R}^n \to \mathbb{R}P^n$ defined by $i_n(x_1,...,x_n) = [x_1,...,x_n,1]$. For every elements $x \in \mathbb{R}^n$ and $g \in Gl(n,\mathbb{R})$, we have $i_n(g(x)) = i_n^G(g)(i_n(x))$. We deduce that for every affine manifold (M,∇_M) whose developing map is D_M, there exists a projective structure defined on M whose developing map is $i_n \circ D_M$.

Benzecri [4] p.241-242 has defined a functor between the category of projective manifolds of dimension n and the category of radiant affine manifolds of dimension $n+1$ which can be described as follows:

Firstly, we remark that since the universal cover $\tilde{M}$ of the projective manifold M is simply connected and $D_{S^n} : S^n \to P\mathbb{R}^n$ is a covering map, the theorem 4.1 of Bredon [5] p.143 implies that the development map $D_M : \tilde{M} \to P\mathbb{R}^n$, can be lifted to a local diffeomorphism $D'_M : \tilde{M} \to S^n$ which is a projective morphism. Let $N(\pi_1(M))$ be the normalizer of $\pi_1(M)$ in $Aut(\tilde{M}, P\nabla_{\tilde{M}})$, for every $g \in N(\pi_1(M))$, there exists $H'_M(g) \in Aut(S^n, P\nabla_{S^n})$ such that the following diagram is commutative:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{g} & \tilde{M} \\ D'_M \downarrow & & \downarrow D'_M \\ S^n & \xrightarrow{H'_M(g)} & S^n. \end{array}$$

We will denote by h'_M the restriction of H'_M to $\pi_1(M)$.

There exists a local diffeomorphism $D_{\tilde{M}\times S^1} : \tilde{M} \times \mathbb{R}_+^* \to \mathbb{R}^{n+1} - \{0\}$ defined by $D_{\tilde{M}\times S^1}(x,t) = tD'_M(x)$, which is the developing map of a radiant structure defined on $M \times S^1$ whose holonomy representation $h_{M\times S^1} : \pi_1(M\times S^1) \to Gl(n+1,\mathbb{R})$ is defined by $h_{M\times S^1}(\gamma,n) = 2^n h'_M(\gamma)$. This radiant affine manifold $M \times S^1$ is the construction of Benzecri, we will often denote this affine structure by $(B(M), \nabla_{B(M)})$ and by $p_{B(M)} : M \times S^1 \to M$ the projection on the first factor.

Let $f : (M, P\nabla_M) \to (N, P\nabla_N)$ be a morphism between n-dimensional projective manifolds; f can lifted to the projective the morphism $\tilde{f} : \tilde{M} \to \tilde{N}$. We deduce the existence of a morphism of affine manifolds $f' : \tilde{M} \times \mathbb{R}_+^* \to \tilde{N} \times \mathbb{R}_+^*$ defined by $f'(x,t) = (\tilde{f}, t)$. The morphism f' is equivariant with respect to the action of $\pi_1(B(M))$ on $\tilde{M} \times \mathbb{R}_+^*$ and $\pi_1(B(N))$ on $\tilde{N} \times \mathbb{R}_+^*$, and covers a morphism $b(f) : B(M) \to B(N)$.

Let (N, ∇_N) be a n-dimensional radiant affine manifold. We suppose that the holonomy of N fixes the origin of $\mathbb{R}^n$. The vector field defined on $\mathbb{R}^n$ by $X_R^{\mathbb{R}^n}(x) = x$ is invariant by the holonomy. Its pullback by the developing map is a vector field $X_R^{\tilde{N}}$ of $\tilde{N}$ invariant by $\pi_1(N)$. We deduce that $X_R^{\tilde{N}}$ is the pullback of a vector field X_R^N of N called the radiant vector field of N.

Proposition 3.2. *Let $(M, P\nabla_M)$ be a compact projective manifold. There exists a surjective morphism of groups between the connected component of $Aut(B(M), \nabla_{B(M)})$ and the connected component of $Aut(M, P\nabla_M)$.*

Proof. Let f be an element of $Aut(B(M), \nabla_{B(M)})_0$, the connected component of $Aut(B(M), \nabla_{B(M)})$. Consider an element $\tilde{f}$ of $Aut(\tilde{M} \times \mathbb{R}_+^*)_0$ over f. For every $\tilde{x} \in \tilde{M}$ and $t \in \mathbb{R}_+^*$, we can write $\tilde{f}(\tilde{x},t) = (\tilde{g}(\tilde{x},t), h(\tilde{x},t))$. The flow of $X_R^{\mathbb{R}^{n+1}}$ is in the center of $Gl(n+1,\mathbb{R})$, we deduce that $\tilde{f}$ commutes with the flow $X_R^{\tilde{B}(M)}$, $g(\tilde{x},t)$ does not depend of t and $h(\tilde{x},t) = th(\tilde{x},1)$.

Let γ be an element of $\pi_1(M)$, since $(\gamma,2).(\tilde{x},t) = (\gamma(\tilde{x}), 2t)$ is an element of $\pi_1(B(M))$ and $\tilde{f}$ is an element of $N(\pi_1(B(M)))_0$, we deduce that $(\gamma,2)$ commutes with $\tilde{f}$ and $\tilde{g}$ commute with γ. This implies that there exists an element g of $Aut(M, P\nabla_M)$ whose lifts is $\tilde{g}$. Remark that since $\tilde{f}$ is an affine transformation, $h(x,1)$ is a constant. The correspondence $P : Aut(B(M), \nabla_{B(M)})_0 \to Aut(M, P\nabla_M)_0$ defined by $P(f) = g$ is well defined and is surjective morphism of groups since for every element $f \in Aut(M, P\nabla_M)_0$, $P(b(f)) = f$. $\square$

Let $(M, P\nabla_M)$ be a projective manifold M, the orbits of the radiant flow $\phi_t^{B(M)}$ of $X_R^{B(M)}$ are compact. The images of the elements of $\phi_t^{B(M)}$ by P are the identity on $(M, P\nabla_M)$. This implies that $dim(Aut(M, P\nabla_M)) + 1 \leq dim(Aut(B(M), \nabla_{B(M)})))$. We deduce that if $(M, P\nabla_M)$ is a projective manifold, such that $Aut(M, P\nabla_M)$ is not discrete, the dimension of $Aut(B(M), \nabla_{B(M)})$ is superior or equal to 2.

4. Automorphisms of projective manifolds and automorphisms of radiant affine manifolds

Let (N, ∇_N) be an affine manifold. In [13], I have shown that $aut(N, \nabla_N)$, the Lie algebra of $Aut(N, \nabla_N)$ is endowed with an associative product defined by $X.Y = \nabla_{MX}Y$. We deduce that $\nabla_{B(M)}$ defines on $aut(B(M), \nabla_{B(M)})$ an associative structure which can be pulled back to $n(\pi_1(B(M))$. It results that the Lie algebra $H_{B(M)}(n(B(M), \nabla_{B(M)}))$ of the image of $N(\pi_1(B(M)))$ by $H_{B(M)}$ is stable by the canonical product of matrices which is the image of the associative product of $n(\pi_1(B(M)))$ by $H_{B(M)}$. Remark that $H_{B(M)}(n(B(M), \nabla_{B(M)}))$ is isomorphic to $n(B(M), \nabla_{B(M)})$. The theorem 23 of chap. III of [1] implies that we can write: $H_{B(M)}(n(B(M), \nabla_{B(M)})) = S_M \oplus N_M$ where S_M is a semi-simple associative algebra and N_M a nilpotent associative algebra.

In [14], by using this associative product, I have shown that the orbits of the canonical action of $Aff(N, \nabla_N)_0$ on N are immersed affine submanifolds of (N, ∇_N) and are the leaves of a (singular) foliation. This leads to the following result:

Proposition 4.1. *Let $(M, P\nabla_M)$ be a projective manifold. The orbits of the action of $Aut(M, P\nabla_M)_0$ on M are immersed projective submanifolds and are the leaves of a singular foliation.*

Proof. The orbits of $Aut(B(M), \nabla_{B(M)})_0$ are immersed affine submanifolds of $B(M)$. The proposition 3.2 shows that there exists a surjective map $P : Aut(B(M), \nabla_{B(M)})_0 \to Aut(M, P\nabla_M)_0$ such that, for every $g \in Aut(B(M), \nabla_{B(M)})_0$ and $x \in B(M)$, $p_{B(M)}(g(x)) = P(g)(p_B(x))$. This implies that the orbits of $Aut(M, P\nabla_M)_0$ are the images of the orbits of $Aut(B(M), \nabla_{B(M)})_0$ by the quotient map $B(M) \to M$. $\square$

Theorem 4.2. *Let $(M, P\nabla_M)$ be a compact oriented projective manifold of dimension superior or equal to 2. Suppose that $H_M(N(\pi_1(M)))$ acts transitively on $\mathbb{R}P^n$, then $(M, P\nabla_M)$ is isomorphic to a finite quotient of $\mathbb{K}P^m$ by a subgroup of $\mathbb{K}$ where $\mathbb{K}$ is the field of real numbers, complex numbers, quaternions or octonions. The action of $\pi_1(M)$ on $\mathbb{K}P^n$ is induced by its action on $\mathbb{K}^{m+1}$ by homothetic maps.*

Proof. The fact that $H_M(N(\pi_1(M)))$ acts transitively on $\mathbb{R}P^n$ implies that $H'_M(N(\pi_1(M)))$ acts transitively on S^n. The theorem of Montgomery Zipplin [11] p.226 implies that a connected compact subgroup K' of $H'_M(N(\pi_1(M)))$ acts transitively on S^n. The theorem I p. 456 of Montgomery and Samelson [10] implies that a connected compact simple subgroup C' of K' acts transitively on S^n. The Lie algebra of the connected component C of ${H'}_M^{-1}(C')$ is isomorphic to the Lie algebra of C' since the kernel of H'_M is discrete. This implies that C is compact. Remark that the orbits of the action of C on $\tilde{M}$ are open. We deduce that C acts transitively on $\tilde{M}$ and $\tilde{M}$ is compact. This implies that $D'_M : \tilde{M} \to S^n$ is a covering since it is a local diffeomorphism defined between compact manifolds. This implies that D'_M is a diffeomorphim since S^n and $\tilde{M}$ are simply connected.

We can write $\mathbb{R}^n = \oplus_{i \in I} U_i$ where U_i is an irreducible component of the action of $h'_M(\pi_1(M))$ since $\pi_1(M)$ is finite.

Let $i, j \in I$, consider two non zero elements $x_i \in U_i, x_j \in U_j$, since $H'_M(N(\pi_1(M))_0)$ acts transitively on S^n there exists $B \in H'_M(N(\pi_1(M))_0)$ such that $B(x_i) = cx_j$, $B(U_i) \cap U_j$ is invariant by $H'_M(\pi_1(M))$, we deduce that $B(U_i) = U_j$ since U_j is irreducible and B is an isomorphism.

The group of automorphisms of the irreducible representation U_i is $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. We deduce that $\mathbb{R}^{n+1}$ is a $\mathbb{K}$ vector space and the action of $H'_M(\pi_1(M))_0)$ on $\mathbb{K}^n$ is induced by its action on $\mathbb{K}$ by right multiplication of elements of $\mathbb{K}$. $\square$

Remark.

Suppose that the dimension of M is even, and $H_M(N(\pi_1(M))_0))$ acts transitively on $\mathbb{R}P^n$. The proof of the previous theorem can be simplified as follows: Every element of $H'_M(\pi_1(M))$ has a fixed point since every element of $Gl(2n+1,\mathbb{R})$ has a real eigenvalue. We deduce that $H'_M(\pi_1(M))$ is the identity and there exists a map $f : M \to \mathbb{R}P^n$ such that $D_M = f \circ p_M$. This implies that f is a covering map and M is homeomorphic to S^n or $\mathbb{R}P^n$.

Let $(M, P\nabla_M)$ be a projective manifold, suppose that $Aut(M, P\nabla_M)_0$ is not solvable. This implies that $Aut(B(M), \nabla_{B(M)})_0$ and the connected component of the normalizer $N(\pi_1(B(M)))_0$ of $\pi_1(B(M))$ in $Aut(\widetilde{B(M)}, \nabla_{\widetilde{B(M)}})$ are not solvable. We deduce that the image of $N(\pi_1(B(M))$ by $H_{B(M)}$ contains a subgroup H_{S^1} isomorphic to S^1. We denote by $X'_{B(M)}$, a vector field which generates the Lie algebra of H_{S^1}, its pullback by the developing map $D_{B(M)}$ of $B(M)$ is a vector field $\tilde{X}_{B(M)}$ invariant by the fundamental group of $B(M)$. We deduce that there exists a vector field $X_{B(M)}$ of $B(M)$ whose pullback by the universal covering map is $\tilde{X}_{B(M)}$. Suppose that the developing map is injective, the flow of $X_{B(M)}$ defines an action of S^1 on $B(M)$ which is transverse and commutes with the radial flow. This implies there exists a vector X_M on M which is the image of $X_{B(M)}$ by the map induced by $p_{B(M)} : B(M) \to M$. The vector field X_M induces an action of S^1 on M: We have:

Proposition 4.3. *Let $(M, P\nabla_M)$ be a compact projective manifold whose developing map is injective, suppose that $Aut(M, P\nabla_M)_0$ is not solvable, then M is endowed with a non trivial action of S^1.*

5. Automorphisms of projective manifolds of dimension 2 and 3

In dimension 2, we have the following result:

Proposition 5.1. *Let $(M, P\nabla_M)$ be a 2-dimensional compact connected oriented projective manifold, suppose that $Aut(M, P\nabla_P)$ is not discrete, then M is homeomorphic to the 2-dimensional torus or to the sphere.*

Proof. Suppose $N_M \neq 0$, there exists a non zero element $A_M \in N_M$ such that $A_M^2 = 0$, we deduce that $dim(ker(A_M)) = 2, dim(Im(A_M)) = 1$. Remark that $Im(A_M)$ is fixed by the holonomy.

Suppose that $N_M = 0$, we deduce that $dim(S_M) \geq 2$, there exists a non zero element distinct of the identity $e_M \in S_M$ such that $e_M^2 = e_M$. To see this remark that S_M contains either an associative algebra isomorphic to the associative algebra of 2×2 real matrices or two idempotents which are linearly independent. The linear map e_M is diagonalizable and its eigenvalues are equal to 0 and 1. Since the flow of e_M is distinct of the radial flow, we deduce that 0 is an eigenvalue of e_M. This implies that either the dimension of the eigenspace associated to 0 is 1, or the the dimension of the eigenspace associated to 1 is 1. We deduce that the holonomy preserves a vector subspace of dimension 1.

We conclude that if $Aut(M, P\nabla_M))$ is not discrete, its holonomy fixed a point of $P\mathbb{R}^2$. The lemma 2.5 p. 808 in Goldman [9] implies that the Euler number of M is positive. $\square$

Dimension 3.

In this section, we study the group of automorphisms of a connected 3-dimensional compact projective manifold $(M, P\nabla_M)$ whose group of automorphisms is not discrete.

$Aut(M, P\nabla_M)_0$ **is not solvable.**

Suppose that $Aut(M, P\nabla_M)_0$ is not solvable, then $Aut(B(M), \nabla_{B(M)})_0$ and $N(\pi_1(B(M)))_0$ are not solvable. We deduce that the connected subgroup of $Gl(n+1,\mathbb{R})$, $H_{B(M)}(N(\pi_1(M))_0)$ contains a subgroup H" isomorphic to S^1. We denote by $X"_{B(M)}$ a vector field which generates the Lie algebra of H". The pullback $X'_{B(M)}$ of $X"_{B(M)}$ by $D_{B(M)}$ is the pullback of a vector field $X_{B(M)}$ of $B(M)$ by $p_{B(M)}$.

Suppose that the set of fixed points of H" is not empty, we can write $\mathbb{R}^4 = U \oplus V$ where U is a 2-dimensional vector subspace corresponding to the non trivial irreducible submodule of H" and V the set of fixed points. Remark that $h_{B(M)}(\pi_1(B(M)))$ preserves U and V since it commutes with H". This implies that there exists a foliation $\mathcal{F}_U$ (resp. $\mathcal{F}_V$) on $B(M)$ whose pullback by the universal covering map is the pullback by $D_{B(M)}$ of the foliation of $\mathbb{R}^4$ whose leaves are 2-dimensional affine spaces parallel to U (resp. parallel to V).

Proposition 5.2. *Suppose that $V \cap D_{B(M)}(\widetilde{B(M)})$ is empty. Then a finite cover of M is a total space of a fibre bundle over S^1 whose fibre is T^2.*

Proof. The vector field defined by $Y"(u,v) = u; u \in U, v \in V$ is invariant by the holonomy of $B(M)$. To show this, remark that the restriction of $H"$ to U defines on it a complex structure and since $h_{B(M)}(\pi_1(B(M)))$ commutes with $H"$, its restriction to U are morphisms of that complex structure. The pullback of $Y"$ by $D_{B(M)}$ is the pullback of a vector field $Y_{B(M)}$ of $B(M)$ by the universal covering map. The image Y_M of $Y_{B(M)}$ by $p_{B(M)}$ and X_M commute and generate a locally free action of $\mathbb{R}^2$ on M since $V \cap D_{B(M)}(\widetilde{B(M)})$ is empty. Chatelet, Rosenberg and Weil [6] implies that M is the total space of a fibre bundle over S^1 whose fibre is T^2. □

If $\mathcal{F}_V$ has compact leaf, we have the following result:

Proposition 5.3. *Let $(M, P\nabla_M)$ be a 3-dimensional compact projective manifold whose developing map is injective. Suppose that $Aut(M, P\nabla_M)_0$ is not solvable and $V \cap D_{B(M)}(\widetilde{B(M)})$ is not empty. Then the holonomy of $(M, P\nabla_M)$ is solvable.*

Proof. Let $\hat{F}_0$ be a connected component of $V \cap D_{B(M)}(\widetilde{B(M)})$ its image by the universal covering map is a compact leaf F_0 compact leaf of $\mathcal{F}_V$ which is a 2-dimension compact affine manifold, we deduce that its fundamental group is solvable. Let r be the restriction of $h(\pi_1(B(M))$ to V, since $h_{B(M)}(\pi_1(B(M))$ preserves V, we have an exact sequence:

$$1 \to Ker(r) \to h_{B(M)}(\pi_1(B(M)) \to Im(r) \to 1$$

The groups $Ker(r)$ is solvable since it restriction to U commutes with a non trivial linear action of S^1. The group $Im(r)$ is also solvable since it is contained in $h_{F_0}(\pi_1(F_0))$, we deduce that $h_{B(M)}(\pi_1(B(M))$ is solvable. □

$Aut(M, P\nabla_M)_0$ **is solvable.**

In this section we study 3-dimensional projective manifolds whose group of automorphisms is solvable. We can decompose the associative algebra $n(\pi_1(B(M))$ by writing: $n(\pi_1(B(M)) = S_M \oplus N_M$, where S_M is a semi-simple associative algebra and N_M a nilpotent associative algebra. We deduce that S_M is the direct product of associative algebras isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ and is commutative. It results that the fact that $Aut(M, P\nabla_M)_0$ is not commutative implies that N_M is not commutative.

$Aut(M, P\nabla_M)_0$ **is solvable and is not commutative.**

Theorem 5.4. *Suppose that N_M is not commutative, then $h_{B(M)}(\pi_1(B(M)))$, the image of the holonomy of $B(M)$ is solvable.*

Proof. First step:

Suppose that the square of every element of N_M is zero.

Let $A, B \in N_M$ such that $AB \neq BA$. Suppose that $dim(ker(A)) = 3$. It implies that $dim(Im(A)) = 1$. Since $(A+B)^2 = 0$, we deduce that $AB + BA = 0$ and $B(Ker(A)) \subset Ker(A), B(Im(A)) \subset Im(A))$, we deduce that the restriction of B to $Im(A)$ is zero since B is nilpotent and $dim(Im(A)) = 1$. This implies that $BA = 0$, we deduce that $AB = 0$ and $AB = BA$. Contradiction.

Suppose that that $dim(Ker(A)) = dim(Im(A)) = dim(Ker(B)) = dim(Im(B)) = 2$. We deduce that $Im(A) = Ker(A), Im(B) = Ker(B)$ since $A^2 = B^2 = 0$. If $Ker(A) \cap Ker(B) = 0$, $\mathbb{R}^4 = Ker(A) \oplus Ker(B)$ and $AB = BA = 0$ since $AB + BA = 0$. If $Ker(A) = Ker(B)$, $AB = BA = 0$ since $Im(A) = Im(B) = Ker(A) = Ker(B)$. Contradiction.

We deduce that $dim(Ker(A) \cap Ker(B)) = 1$. We can write $\mathbb{R}^4 = Vect(e_1, e_2, e_3, e_4)$ where $Vect(e_1) = Ker(A) \cap Ker(B)$, $Vect(e_1, e_2) = Ker(A)$ and $Vect(e_1, e_3) = Ker(B)$. Every element in $h_{B(M)}(\pi_1(B(M))$ preserves $Ker(A) \cap Ker(B)$, $Ker(A)$ and $Ker(A) + Ker(B)$ since it commutes with A and B. We deduce that $\pi_1(B(M))$ is solvable since it preserves a flag.

Step 2.

Suppose that there exists an element $A \in N_M$ such that $A^2 \neq 0$.

If $dim(KerA) = 1$, we have $(A^2)^2 = 0$ implies that $Im(A^2) \subset Ker(A^2)$. Remark that $x \in Ker(A^2)$ if and only if $A(x) \in Ker(A)$ and $x \in A^{-1}(Ker(A))$; $dim(A^{-1}(Ker(A)) = 2$ since $dim(Ker(A)) = 1$. We deduce that $dim(Ker(A^2)) = dim(Im(A^2)) = 2$, and $Ker(A) \subset Ker(A^2) = Im(A^2) \subset Im(A)$ and $\pi_1(B(M))$ preserves a flag since it commutes with A and $dim(Im(A)) = 3$.

Suppose that $dim(ker(A)) = 2$, we deduce that $dim(Im(A)) = 2$. Suppose that $Ker(A) \cap Im(A) = 0$, we deduce that $\mathbb{R}^4 = Ker(A) \oplus Im(A)$. This is impossible since A is nilpotent. We also remark that $Ker(A)$ is distinct of $Im(A)$ since $A^2 \neq 0$. This implies that $dim(Ker(A) \cap Im(A)) = 1$. Every element of $\pi_1(B(M))$ preserves, $Ker(A) \cap Im(A), Ker(A)$ and $Ker(A) + Im(A)$ and thus preserves a flag. We deduce that $h_{B(M)}(\pi(B(M)))$ is solvable. □

$Aut(M, P\nabla_M)_0$ **is commutative.**

Suppose that the developing map is injective and $Aut(M, P\nabla_M)_0$ is commutative and its dimension is superior or equal to 2. Let X_M and Y_M two projective vector fields linearly independent. We denote by $X_{B(M)}$ and $Y_{B(M)}$ two affine vector fields of $B(M)$ whose respective images by $p_{B(M)}$ are X_M and Y_M. There exist affine vector fields $X'_{B(M)}$ and $Y'_{B(M)}$ of $\mathbb{R}^4$ whose respective images by the covering map are $X_{B(M)}$ and $Y_{B(M)}$. Remark that if the group generated by X_M and Y_M acts freely on M, then Chatelet, Rosenberg and Weil [6] implies that M is the total space of a torus bundle.

In the rest of this section we assume that the set of zero of X_M is not empty. This implies that we can assume that the set of zero U of $X'_{B(M)}$ is not empty by eventually replacing $X'_{B(M)}$ with $X'_{B(M)} + cX_R$ where $c \in \mathbb{R}$ and X_R is the radiant flow. We denote by $B(N)$ the image of $U \cap D_{B(M)}(\tilde{M})$ by the covering map. Remark that $B(N)$ is not empty.

Proposition 5.5. *Suppose that $dim(U) = 3$ then $\pi_1(M)$ is solvable.*

Proof. Suppose that the restriction of $Y'_{B(M)}$ to U is not zero. This implies that the restriction of $Y_{B(M)}$ to $B(N)$ is not zero. This implies that the group of projective automorphisms of N, the quotient of $B(N)$ by the radiant flow is not discrete. The proposition 5.1 implies that N has a finite cover homeomorphic to S^2 or T^2. This implies that $\pi_1(B(N))$ is solvable. The restriction of $\pi_1(B(M))$ to U induces an exact sequence whose image is contained in the image of the holonomy representation of $B(N)$ and whose kernel is solvable. We deduce that $\pi_1(B(M))$ and $\pi_1(M)$ are solvable. If the restriction of $Y'_{B(M)}$ to U vanishes, let V be the image of $X'_{B(M)}$, if V is not contained in U, then $\mathbb{R}^4 = U \oplus V$, and since $Y'_{B(M)}$ commutes with $X'_{B(M)}$, it preserves V. This implies that $X'_{B(M)}$ and $Y'_{B(M)}$ are linearly dependent contradiction.

Suppose that V is a subset of U, let W be the image of $Y'_{B(M)}$, if W is not contained in U, then we can apply the previous argument to obtain a contradiction by replacing $X'_{B(M)}$ by $Y'_{B(M)}$. Suppose that W is contained in U, $V \cap W = \{0\}$ since $X'_{B(M)}$ and $Y'_{B(M)}$ are linearly independent. We deduce that, the holonomy of $B(M)$ preserves, $V, V \oplus W$ and U. This implies that the holonomy of $B(M)$ and $\pi_1(B(M))$ are solvable. □

Proposition 5.6. *Suppose that $dim(U) = 2$, then $\pi_1(B(M))$ is solvable.*

Proof. Suppose that $U \oplus Im(X'_{B(M)}) = \mathbb{R}^4$.

Step 1.

If the restriction of $X'_{B(M)}$ or $Y'_{B(M)}$to $Im(X'_{B(M)})$ are not a multiple of the identity, we deduce that $r(\pi_1(B(M))$, the image of the restriction of $\pi_1(B(M))$ to $Im(X'_{B(M)})$ is solvable since it commutes with $X'_{B(M)}$ and $Y'_{B(M)}$. The restriction of $\pi_1(B(M))$ to $Ker(X_{B(M)})$ is contained in the holonomy group the 2-dimensional closed affine manifold $B(N)$, we deduce that it is solvable. This implies $\pi_1(B(M))$ is solvable.

Step 2.

Suppose that the restriction of $X'_{B(M)}$ and $Y'_{B(M)}$ to $Im(X'_{B(M)})$ are multiple of the identity. If the restriction of $Y'_{B(M)}$ to U is equal to aI_U, we deduce that $Y'_{B(M)}$ is contained in the vector space generated by $Id_{\mathbb{R}^4}$ and $X'_{B(M)}$. This implies that the dimension of the vector space generated by X_M and Y_M is 1. Contradiction. We deduce that the restriction of $Y'_{B(M)}$ to U is not a multiple of the Id_U. There exists a real a such that the restriction of $Z = Y'_{B(M)} + aId_{\mathbb{R}^4}$ to $Im(X'_{B(M)})$ is zero. The restriction of Z to U is distinct of a multiple of the identity, we conclude that $\pi_1(B(M))$ is solvable by replacing $Y'_{B(M)}$ by Z in the first step of the proof.

Suppose that $dim(U \cap Im(X'_{B(M)}) = 1$. The vector subspaces, $U \cap Im(X'_{B(M)}), U, U \oplus Im(X'_{B(M)})$ are stable by the holonomy. We deduce that $\pi_1(B(M))$ preserves a flag and is solvable.

Suppose that $U = Im(X'_{B(M)})$.

We can write $\mathbb{R}^4 = Vect(e_1, e_2, e_3, e_4)$ where $Vect(e_1, e_2) = U$ and $X'_{B(M)}(e_3) = e_1, X'_{B(M)}(e_4) = e_2$. Let γ be an element of $\pi_1(B(M))$, if we write the fact that the matrix $M(\gamma)$ of γ commutes with the matrix of $X'_{B(M)}$ in the basis (e_1, e_2, e_3, e_4), we obtain that:

$M(\gamma) =$

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ 0 & 0 & a_1 & b_1 \\ 0 & 0 & a_2 & b_2 \end{pmatrix}$$

Since the restriction of $\pi_1(B(M))$ to U is contained in the holonomy of $\pi_1(B(N))$ which is solvable, we deduce that $\pi_1(B(M))$ is solvable. □

Proposition 5.7. *Suppose that $dim(U) = 1$, then $\pi_1(B(M))$ is nilpotent.*

Proof. We can write $n(\pi_1(B(M)) = S_M \oplus N_M$ where S_M is semi-simple and N_M nilpotent. Suppose that N_M is not zero, and consider $n \in N_M$, if $n^2 \neq 0$, $(n^2)^2 = 0$, $dim(Ker(n^2)) \geq 2$, if $n^2 = 0$, $dim(Ker(n)) \geq 2$. We can apply the proposition 5.4 and the proposition 5.5.

Suppose that $N_M = 0$, $n(B(M))$ contains a non zero idempotent u_1 which is not a multiple of the identity, since it does not generates the radiant flow. If the eigenvalues of u_1 are 0 or 1, this implies that $dim(Ker(u_1)) \geq 2)$ or $dim(Ker(u_1) - Id_{\mathbb{R}^4}) \geq 2$, we can apply the proposition 5.4 and the proposition 5.5 to deduce that $\pi_1(B(M))$ is nilpotent. □

Theorem 5.8. *Let $(M, P\nabla_M)$ be a 3-dimensional projective manifold whose developing map is injective, suppose that $dim(Aut(M, P\nabla_M)) \geq 2$, then M is homeomorphic to a spherical manifold, $S^2 \times S^1$ or a finite cover of M is a torus bundle.*

Proof. Suppose that $Aut(M, P\nabla_M)_0$ is not solvable, the proposition 5.2 and the proposition 5.3 imply that either M is the total space of a bundle over S^1 whose fibre is homeomorphic to T^2 or $\pi_1(M)$ is solvable. If $Aut(M, P\nabla_M)_0$ is solvable, the theorem 5.1, the propositions 5.4, 5.5 and 5.6 show that $\pi_1(M)$ is solvable. In [2], it is shown that a 3-dimensional closed manifold whose fundamental group is solvable is homeomorphic to a spherical manifold, $S^1 \times S^2$, a finite cover of M is a torus bundle, or $\mathbb{R}P^3 \# \mathbb{R}P^3$. Benoist [3] and Goldman and Cooper [8] have shown that there does not exist a projective structure on $\mathbb{R}P^3 \# \mathbb{R}P^3$. □

References

1. Albert A. *Structure of algebras.* Vol. 24. American Math. Soc. 1939.
2. Aschenbrenner, M. Friedl, S. Wilton, H. *3 manifold groups.* https://arxiv.org/pdf/1205.0202.pdf
3. Benoist, Y. *Nilvariétés projectives.* Commentarii Mathematici Helvetici, 1994, vol. 69, no 1, p. 447-473.
4. Benzecri J.P. *Sur les variétés localement affines et localement projectives. Bulletin de la S.M.F.* 88 (1960) 229-332.
5. Bredon, G. Bredon, Glen E. *Topology and geometry.* Vol. 139. Graduate texts in Math.
6. Chatelet, Gilles, Rosenberg, Harold, et Weil, Daniel. *A classification of the topological types of* $\mathbf{R}^2$*-actions on closed orientable 3-manifolds.* Publications Mathématiques de l'IHÉS, 1974, vol. 43, p. 261-272
7. Conlon, Lawrence. *Transversally parallelizable foliations of codimension two.* Transactions of the American Mathematical Society, 1974, vol. 194, p. 79-102.

8. Cooper, D. Goldman, W. *A 3–Manifold with no Real Projective Structure*. In Annales de la Faculté des sciences de Toulouse: Mathématiques, vol. 24, no. 5, pp. 1219-1238. 2015

9. Goldman, W. *Convex real projective structures on compact surfaces. J. Differential Geometry* 31 (1990) 791-845.

10. Montgomery D. Samelson H. *Transformation groups of spheres. Annals of Mathematics* 44 (1943) 454-470.

11. Montgomery, D. Zippin, L. *Topological transformation groups*. Interscience Tracts in pure and applied mathematics 1955.

12. Sullivan, D; Thurston,, W. *Manifolds with canonical coordinate charts: some examples*. Enseign. Math 29 (1983): 15-25.

13. Tsemo, A. *Dynamique des variétés affines*. Journal of the London Mathematical Society 63 (2001) 469-486.

14. Tsemo, A. *Décomposition des variétés affines*. Bulletin des sciences mathématiques 125 (2001) 71-83.

15. A Tsemo, A. *Linear foliations on affine manifolds*. Topology and its Applications. (300) 107756.

Tsemo Aristide,
Department of Mathematics,
PKFokam Institute Of Excellence,
Cameroon.
E-mail address: tsemo58@yahoo.ca