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Local and Global Well-Posedness for Fractional Porous Medium Equation in Critical Fourier-Besov Spaces

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ABSTRACT: In this paper, we study the Cauchy problem for the fractional porous medium equation in $\mathbb{R}^n$ for $n \geq 2$. By using the contraction mapping method, Littlewood-Paley theory and Fourier analysis, we get, when $1 < \beta \leq 2$, the local solution $v \in \mathcal{X}_T := \mathcal{L}_T^{\infty}\left(F\dot{B}_{p,r}^{2-2m-\beta+\frac{n}{p'}}(\mathbb{R}^n)\right) \cap \mathcal{L}_T^{\rho_1}\left(F\dot{B}_{p,r}^{s_1}(\mathbb{R}^n)\right) \cap \mathcal{L}_T^{\rho_2}\left(F\dot{B}_{p,r}^{s_2}(\mathbb{R}^n)\right)$ with $1 \leq p < \infty$, $1 \leq r \leq \infty$, and the solution becomes global when the initial data is small in critical Fourier-Besov spaces $F\dot{B}_{p,r}^{2-2m-\beta+\frac{n}{p'}}(\mathbb{R}^n)$. In addition, We establish a blowup criterion for the solutions. Furthermore, the global existence of solutions with small initial data in $F\dot{B}_{\infty,1}^{2-2m-\beta+n}(\mathbb{R}^n)$ is also established. In the limit case $\beta = 1$, we prove global well-posedness for small initial data in critical Fourier-Besov spaces $F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}(\mathbb{R}^n)$ with $1 < p < \infty$ and $F\dot{B}_{\infty,1}^{1-2m+n}(\mathbb{R}^n)$, respectively.

Key Words: Well-posedness, Fractional porous medium equation, Littlewood-Paley theory, Fourier-Besov spaces.

Contents

1. Introduction

In this article, we investigate the existence of mild solutions for the initial value problem of the following fractional porous medium equation (FPME):

$$\begin{cases} \partial_t v + \mu\Lambda^{\beta} v + \nabla\cdot(v\nabla p) = 0 & \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ p = \kappa(-\Delta)^{-m} v & \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x) & \text{for} \quad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 2$, $v = v(x,t)$ denotes the density or concentration, v_0 is the initial data, $\mu > 0$ is the dissipative coefficient, $\kappa = \pm 1$, and here for simplify the notation, we take $\mu = \kappa = 1$. The operator Λ^{β} is the Fourier multiplier with symbol $|\xi|^{\beta}$, and p represents the gas pressure which releted to v by an abstract operator; $p = Pv$.

When $\kappa = -1$ and $0 < m < 1$, the system (1.1) was first formulated by Caffarelli and Vázquez [5]. Indeed, the system (1.1) is created by adding to the continuity equation

$$\partial_t v + \nabla\cdot(vV) = 0, \tag{1.2}$$

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where $V = \nabla p$ is the velocity, the fractional dissipative term $\mu \Lambda^{\beta} v$. In that work, they demonstrated that a weak solution exists when v_0 is a bounded function with exponential decay at infinity. Please see literature [20] for more information on the solution of Equation (1.2).

When $\kappa = 1,\ \beta = 2$ and $m = 1$, Equation (1.1) corresponds to the following classical Keller-Segel equation:

$$\begin{cases} \partial_t v + \mu \Delta v + \nabla \cdot (v \nabla p) = 0 & \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ -\Delta p = v & \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ v(x,0) = v_0(x) & \text{for} \quad x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

which describes a model of chemotaxis. The system (1.3) was introduced by Keller and Segel [11]. The well-posedness of the system (1.3) has been studied by several researchers in various spaces, such as Corrias et al. in the Lebesgue space $L^1(\mathbb{R}^n) \cup L^{\frac{n}{2}}(\mathbb{R}^n)$ [7], Kozono and Sugiyama in the Sobolev space $L^1(\mathbb{R}^n) \cup W^{2,2}(\mathbb{R}^n)$ [13], Ogawa and Shimizu in the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ [17] and in the Besov space $\dot{B}^0_{1,2}(\mathbb{R}^2)$ [18], Iwabuchi in the Fourier-Herz $\dot{\mathcal{B}}^{-2}_2(\mathbb{R}^n)$ [10], for more results, please refer to Lemarié-Rieusset [15] and the references therein.

For the case $\kappa = 1,\ 1 < \beta < 2$ and $m = 1$, Equation (1.1) was initially analyzed by Escudero [9]. It was utilized to characterize the spatiotemporal patterns exhibited by a population density consisting of individuals that perform Lévy flights. Furthermore, in that paper, it has been established that Equation (1.1) in this case, has global in time solutions. Biler and Karch [2] have established, in the critical Lebesgue space $L^{\frac{n}{\beta}}(\mathbb{R}^n)$, the existence of both local and global solutions of Equation (1.1) with small initial data. Additionally, they have demonstrated the finite-time blowup of non-negative solutions with specific initial data that satisfy high-concentration or large-mass conditions. In the critical Besov spaces $\dot{B}^{1-\beta}_{2,q}(\mathbb{R}^2)$, it has been proved global well-posedness with small initial data of Equation (1.1) by Biler and Wu [3]. Zhai [24] has demonstrated the global existence, uniqueness, and stability of solutions with a general potential type nonlinear term in the critical Besov spaces, given that the initial data is sufficiently small. Certain aspects of these results were also extended to the fractional power bipolar type drift-diffusion system. Further information on this topic can be found in [3,19] and the relevant references cited therein.

Recently, for $1 \le \beta \le 2$, Zhao [25], Xiao and Zhang [22] prove the well-posedness of Equation (1.1) in critical Besov spaces, when $m = 1$ and $0 < m < 1$, respectively.

Inspired by some results presented in [22,25], this article aims to prove the well-posedness results of Equation (1.1) in Fourier-Besov spaces $F\dot{B}^{2-2m-\beta+\frac{n}{p'}}_{p,r}(\mathbb{R}^n)$ for $1 < \beta \le 2,\ 1 \le r \le \infty,\ 1 \le p < \infty$. In addition, we prove them in the limit cases $\beta = 1$ and $p = \infty$. To address the system (1.1), we think about the following integral equations:

$$e^{-t\Lambda^{\beta}} v_0 - \int_0^t e^{-(t-\tau)\Lambda^{\beta}} \nabla \cdot \left(v \nabla (-\Delta)^{-m} v \right) \mathrm{d}\tau, \tag{1.4}$$

where $e^{-t\Lambda^{\beta}} := \mathcal{F}^{-1}\left(e^{-t|\xi|^{\beta}} \mathcal{F} \right)$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the inverse Fourier transform, respectively. We can solve (1.4) by applying the contraction mapping argument to the following mapping:

$$\Psi(v) := e^{-t\Lambda^{\beta}} v_0 - \int_0^t e^{-(t-\tau)\Lambda^{\beta}} \nabla \cdot \left(v \nabla (-\Delta)^{-m} v \right) \mathrm{d}\tau. \tag{1.5}$$

Throughout this paper, we use $F\dot{B}^s_{p,r}$ to denote the homogeneous Fourier-Besov spaces, C will represent constants that may differ at different places, $A \lesssim B$ denotes the existence of a constant $C > 0$ such that $A \le B$, and p_0 is the conjugate of $p \in [1,\infty]$ (i.e., $\frac{1}{p} + \frac{1}{p'} = 1$).

Our first theorem is as follows:

Theorem 1.1. (Well-posedness for $1 < \beta \le 2$) *Let $n \ge 2,\ 1 < \beta \le 2$. Assume that $v_0 \in F\dot{B}^{2-2m-\beta+\frac{n}{p'}}_{p,r}(\mathbb{R}^n)$ with $1 \le p, r \le \infty$. Then we have the following results:*

1. *(For $p<\infty$) Let $1\leq p<\infty$, $0<\varepsilon<\beta-1$ and $\frac{1-\varepsilon}{2}<m<\frac{1}{2}\left(1+\frac{n}{p'}\right)$. Then there is a $T=T(v_0)>0$ such that the system (1.1) admits a unique solution $v\in\mathcal{X}_T$, where*

$$\mathcal{X}_T:=\mathcal{L}_T^{\infty}\left(F\dot{B}_{p,r}^{2-2m-\beta+\frac{n}{p'}}(\mathbb{R}^n)\right)\cap\mathcal{L}_T^{\rho_1}\left(F\dot{B}_{p,r}^{s_1}(\mathbb{R}^n)\right)\cap\mathcal{L}_T^{\rho_2}\left(F\dot{B}_{p,r}^{s_2}(\mathbb{R}^n)\right),$$

with

$$s_1=1-2m+\frac{n}{p'}+\varepsilon,\ s_2=1-2m+\frac{n}{p'}-\varepsilon,\ \rho_1=\frac{\beta}{\beta-1+\varepsilon},\ \rho_2=\frac{\beta}{\beta-1-\varepsilon}.$$

Moreover, if T^* denotes the maximal existence time of v,

   (a) If $v_0\in F\dot{B}_{p,r}^{2-2m-\beta+\frac{n}{p'}}(\mathbb{R}^n)$ is sufficiently small, then $T^*=\infty$. i.e., the solution v is global;

   (b) If $T^*<\infty$, then

$$\|v\|_{\mathcal{L}_{T^*}^{\rho_1}(F\dot{B}_{p,r}^{s_1})\cap\mathcal{L}_{T^*}^{\rho_2}(F\dot{B}_{p,r}^{s_2})}=\infty.$$

2. *(For $p=\infty$) Let $p=\infty$ and $\frac{2-\beta}{2}<m<\frac{3-\beta+n}{2}$. Suppose that $\|v_0\|_{F\dot{B}_{\infty,1}^{2-2m-\beta+n}}$ is small enough. Then the system (1.1) admits a unique solution v satisfying*

$$v\in\mathcal{L}_t^{\infty}\left(F\dot{B}_{\infty,1}^{2-2m-\beta+n}(\mathbb{R}^n)\right)\cap\mathcal{L}_t^{1}\left(F\dot{B}_{\infty,1}^{2-2m+n}(\mathbb{R}^n)\right).$$

Remark 1.2. *The results of this work remain valid if we take the Fourier-Herz space $\dot{\mathcal{B}}_r^s$ or Lei-Lin space χ^{-1} instead of Fourier-Besov space $F\dot{B}_{p,r}^s$. Indeed, $F\dot{B}_{1,r}^s=\dot{\mathcal{B}}_r^s$ and $F\dot{B}_{1,1}^{-1}=\chi^{-1}$.*
It is further worth noting that, in the special case $m=1$, Equation (1.1) becomes the generalized Keller-Segel system.

Corresponding to Theorem 1.1, in the case $\beta=1$, We get the following theorem:

Theorem 1.3. (Well-posedness for $\beta=1$) *Let $n\geq 2$, $\beta=1$. Assume that $v_0\in F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}(\mathbb{R}^n)$ with $1<p\leq\infty$. Then we have the following results:*

1. *(For $p<\infty$) Let $1<p<\infty$ and $\frac{1}{2}<m<\frac{1}{2}\left(1+\frac{n}{p'}\right)$. Suppose that $\|v_0\|_{F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}}$ is small enough. Then the system (1.1) admits a unique solution v satisfying*

$$v\in\mathcal{L}_t^{\infty}\left(F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}(\mathbb{R}^n)\right).$$

2. *(For $p=\infty$) Let $p=\infty$ and $\frac{1}{2}<m<1+\frac{n}{2}$. Suppose that $\|v_0\|_{F\dot{B}_{\infty,1}^{1-2m+n}}$ is small enough. Then the system (1.1) admits a unique solution v satisfying*

$$v\in\mathcal{L}_t^{\infty}\left(F\dot{B}_{\infty,1}^{1-2m+n}(\mathbb{R}^n)\right)\cap\mathcal{L}_t^{1}\left(F\dot{B}_{\infty,1}^{2-2m+n}(\mathbb{R}^n)\right).$$

2. Preliminaries

This section introduces some basic knowledge of Littlewood-Paley theory and Fourier-Besov spaces and reviews some lemmas that are pertinent to our purposes.

We start by recall the Littlewood-Paley decomposition (see [1] for more details). Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$ be a smooth radial function such that

$$\begin{gathered}0\leq\varphi\leq 1,\\ \operatorname{supp}\varphi\subset\left\{\xi\in\mathbb{R}^n:\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\right\},\\ \sum_{j\in Z}\varphi\left(2^{-j}\xi\right)=1,\qquad\text{for all }\xi\neq 0,\end{gathered}$$

and we denote $\varphi_j(\xi)=\varphi(2^{-j}\xi)$. Then for every $u\in\mathcal{S}'(\mathbb{R}^n)$, we define the frequency localization operators for all $j\in\mathbb{Z}$, as follows

$$\dot{\Delta}_j u=\mathcal{F}^{-1}\varphi_j * u \qquad \text{and} \qquad \dot{S}_j u=\sum_{k\le j-1}\dot{\Delta}_k u, \tag{2.1}$$

with $\mathcal{F}^{-1}$ the inverse Fourier transform. Here, we observe that the almost orthogonality property of the Littlewood-Paley decomposition is satisfied, i.e. for any $u,v\in\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$,

$$\begin{aligned}\dot{\Delta}_i\dot{\Delta}_j u&=0, \qquad \text{if} \quad |i-j|\ge 2,\\ \dot{\Delta}_i\left(\dot{S}_{j-1}u\dot{\Delta}_j v\right)&=0, \qquad \text{if} \quad |i-j|\ge 5,\end{aligned}$$

where $\mathcal{P}$ is the set of all polynomials on $\mathbb{R}^n$.

Throughout the paper, the following Bony paraproduct decomposition will be used:

$$uv=\dot{T}_u v+\dot{T}_v u+\dot{R}(u,v), \tag{2.2}$$

with

$$\dot{T}_u v=\sum_j \dot{S}_{j-1}u\dot{\Delta}_j v, \quad \dot{R}(u,v)=\sum_j\sum_{|j-l|\le 1}\dot{\Delta}_j u\dot{\Delta}_l v.$$

With the decomposition stated above, the homogeneous Fourier-Besov space can be defined as follows:

Definition 2.1. *[21] For $s\in\mathbb{R}, 1\le p,r\le\infty$ and $u\in\mathcal{S}'(\mathbb{R}^n)$, set*

$$\|u\|_{F\dot{B}^s_{p,r}}:=\begin{cases}\left(\sum_{j\in Z}2^{jsr}\left\|\widehat{\dot{\Delta}_j u}\right\|^r_{L^p}\right)^{1/r} & \text{for} \quad r<\infty,\\ \sup_{j\in Z}2^{js}\left\|\widehat{\dot{\Delta}_j u}\right\|_{L^p} & \text{for} \quad q=\infty.\end{cases}$$

Then the homogeneous Fourier-Besov space $F\dot{B}^s_{p,r}(\mathbb{R}^n)$ is defined by

$$F\dot{B}^s_{p,r}(\mathbb{R}^n)=\left\{u\in\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}:\|u\|_{F\dot{B}^s_{p,r}}<\infty\right\}.$$

Definition 2.2. *[21] For $s\in\mathbb{R}$, $\;0<T\le\infty$ and $1\le p,r,\rho\le\infty$. We define the mixed time-space $\mathcal{L}^\rho_T\left(F\dot{B}^s_{p,r}(\mathbb{R}^n)\right)$ as the completion of $\mathcal{C}([0,T];\mathcal{S}(\mathbb{R}^n))$ by the norm*

$$\|u\|_{\mathcal{L}^\rho_T(F\dot{B}^s_{p,r})}:=\left(\sum_{j\in\mathbb{Z}}2^{jsr}\left(\int_0^T\|\Delta_j u(\cdot,t)\|^\rho_{L^p}\,\mathrm{d}t\right)^{\frac{r}{\rho}}\right)^{\frac{1}{r}}<\infty,$$

with the standard modification if $r=\infty$ or $\rho=\infty$. For simplicity, we use $\|u\|_{\mathcal{L}^\rho_t(F\dot{B}^s_{p,r})}$ instead of $\|u\|_{\mathcal{L}^\rho_\infty(F\dot{B}^s_{p,r})}$.

We notice that we have $F\dot{B}^s_{1,r}=\dot{\mathcal{B}}^s_r$ and $F\dot{B}^s_{1,1}=\chi^s$, where $\dot{\mathcal{B}}^s_r$ and χ^s are the Fourier-Herz space [6] and the Lei-Lin space [16], respectively.

Due to Minkowski's inequality, we have

$$\begin{aligned}\mathcal{L}^\rho_T\left(F\dot{B}^s_{p,r}(\mathbb{R}^n)\right)&\hookrightarrow L^\rho_T\left(F\dot{B}^s_{p,r}(\mathbb{R}^n)\right), \quad \text{if} \;\; \rho\ge r,\\ L^\rho_T\left(F\dot{B}^s_{p,r}(\mathbb{R}^n)\right)&\hookrightarrow \mathcal{L}^\rho_T\left(F\dot{B}^s_{p,r}(\mathbb{R}^n)\right), \quad \text{if} \;\; \rho\le r,\end{aligned}$$

where $\|u\|_{L^\rho_T(F\dot{B}^s_{p,r})}:=\left(\int_0^T\|u(\cdot,t)\|^\rho_{F\dot{B}^s_{p,r}}\,\mathrm{d}t\right)^{\frac{1}{\rho}}$.

Lemma 2.3. *[12]*

1. **(Bernstein's inequality)** *For any multiindex β and $1 \leq r \leq p \leq \infty$ the following inequality is valid:*
$$\operatorname{supp} \hat{u} \subset \left\{|\xi| \leqslant A2^j\right\} \;\; \Rightarrow \;\; \left\|(i\xi)^{\beta} \hat{u}\right\|_{L^r} \leqslant C2^{j|\beta|+nj\left(\frac{1}{r}-\frac{1}{p}\right)} \|\hat{u}\|_{L^p}. \tag{2.3}$$

2. **(Young's inequality)** *Let $p, q, r \in [1, \infty]$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then for $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ we have*
$$\|f * g\|_{L^r} = \|f\|_{L^p} \|g\|_{L^q}. \tag{2.4}$$

Lemma 2.4. *[8] Let g be a homogeneous smooth function on $\mathbb{R}^n \setminus \{0\}$ of degree m. Then for every $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the operator $g(D)$ is continuous from $F\dot{B}^{s}_{p,q}(\mathbb{R}^n)$ to $F\dot{B}^{s-m}_{p,q}(\mathbb{R}^n)$.*

3. Well-posedness for $1 < \beta \leq 2$: Proof of Theorem 1.1

In this section, we establish well-posedness of the system (1.1) in critical Fourier-Besov spaces $F\dot{B}^{2-2m-\beta+\frac{n}{p'}}_{p,r}(\mathbb{R}^n)$ with $1 < \beta \leq 2$, and $1 \leq p, r \leq \infty$.

3.1. The case $p < \infty$

We first consider the fractional power dissipative equation,
$$\begin{cases} \partial_t v + \Lambda^{\beta} v = f & \text{in } \mathbb{R}^n \times (0, \infty), \\ v(x, 0) = v_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{3.1}$$
for which we give the following result:

Proposition 3.1. *[23] Let $0 < T \leq \infty$, $s \in \mathbb{R}$ and $1 \leq p, r, \gamma \leq \infty$. Assume that $v_0 \in F\dot{B}^{s}_{p,r}(\mathbb{R}^n)$ and $f \in \mathcal{L}^{\gamma}_T\left(F\dot{B}^{s+\frac{\beta}{\gamma}-\beta}_{p,r}(\mathbb{R}^n)\right)$. Then (3.1) has a unique solution v such that for any $\gamma \leq \rho \leq \infty$, we have*
$$\|v\|_{\mathcal{L}^{\rho}_T\left(F\dot{B}^{s+\frac{\beta}{\rho}}_{p,r}\right)} \leq C\left(\|v_0\|_{F\dot{B}^{s}_{p,r}} + \|f\|_{\mathcal{L}^{\gamma}_T\left(F\dot{B}^{s+\frac{\beta}{\gamma}-\beta}_{p,r}\right)}\right). \tag{3.2}$$

Next, we get the following key bilinear estimate.

Lemma 3.2. *Let $s > 0$, $T \in (0, \infty]$, $\varepsilon > max\{0, -2m+1\}$, $2m+\varepsilon-1 > 0$ and $1 \leq \rho, \rho_1, \rho_2, r \leq \infty$ with $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$. There holds*
$$\begin{aligned} \left\|v\nabla(-\Delta)^{-m} w\right\|_{\mathcal{L}^{\rho}_T(F\dot{B}^{s}_{p,r})} \lesssim\; & \|v\|_{\mathcal{L}^{\rho_1}_T(F\dot{B}^{s+\varepsilon}_{p,r}) \cap \mathcal{L}^{\rho_2}_T\left(F\dot{B}^{1-2m+\frac{n}{p'}-\varepsilon}_{p,r}\right)} \\ & \times \|w\|_{\mathcal{L}^{\rho_1}_T(F\dot{B}^{s+\varepsilon}_{p,r}) \cap \mathcal{L}^{\rho_2}_T\left(F\dot{B}^{1-2m+\frac{n}{p'}-\varepsilon}_{p,r}\right)}. \end{aligned} \tag{3.3}$$

Proof. Using the following paraproduct decomposition due to J. M. Bony [4], for fixed j,
$$\dot{\Delta}_j\left(v\nabla(-\Delta)^{-m} w\right) := J^j_1 + J^j_2 + J^j_3, \tag{3.4}$$
where
$$\begin{aligned} J^j_1 &:= \sum_{|l-j|\leq 4} \dot{\Delta}_j\left(\dot{\Delta}_l v \nabla(-\Delta)^{-m} \dot{S}_{l-1} w\right), \\ J^j_2 &:= \sum_{|l-j|\leq 4} \dot{\Delta}_j\left(\dot{S}_{l-1} v \nabla(-\Delta)^{-m} \dot{\Delta}_l w\right), \\ J^j_3 &:= \sum_{l\geq j-2} \sum_{|l-l'|\leq 1} \dot{\Delta}_j\left(\dot{\Delta}_l v \nabla(-\Delta)^{-m} \dot{\Delta}_{l'} w\right). \end{aligned}$$

So we can write,

$$\|v\nabla(-\Delta)^{-m}w\|_{\mathcal{L}^{\rho}_T(F\dot{B}^s_{p,r})} \lesssim \left\{\sum_{j\in\mathbb{Z}} 2^{sjr}\left\|\widehat{J^j_1}\right\|^r_{L^{\rho}_T(L^p)}\right\}^{1/r} + \left\{\sum_{j\in\mathbb{Z}} 2^{sjr}\left\|\widehat{J^j_2}\right\|^r_{L^{\rho}_T(L^p)}\right\}^{1/r} + \left\{\sum_{j\in\mathbb{Z}} 2^{sjr}\left\|\widehat{J^j_3}\right\|^r_{L^{\rho}_T(L^p)}\right\}^{1/r}. \tag{3.5}$$

We estimate the above three terms one by one. First, applying Young's inequality (2.4), Holder's inequality, Bernstein's inequality (2.3) and Lemma 2.4, when $\varepsilon > 0$, one has

$$\begin{aligned}
\left\|\widehat{J^j_1}\right\|_{L^{\rho}_T(L^p)} &\lesssim \sum_{|l-j|\le 4}\left\|\dot{\Delta}_l v\nabla\widehat{(-\Delta)^{-m}}\dot{S}_{l-1}w\right\|_{L^{\rho}_T(L^p)}\\
&\lesssim \sum_{|l-j|\le 4}\left\|\widehat{\dot{\Delta}_l v} * \nabla\widehat{(-\Delta)^{-m}}\dot{S}_{l-1}w\right\|_{L^{\rho}_T(L^p)}\\
&\lesssim \sum_{|l-j|\le 4}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L^{\rho_1}_T(L^p)}\sum_{k\le l-2}\left\|\nabla\widehat{(-\Delta)^{-m}}\dot{\Delta}_k w\right\|_{L^{\rho_2}_T(L^1)}\\
&\lesssim \sum_{|l-j|\le 4}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L^{\rho_1}_T(L^p)}\sum_{k\le l-2}2^{k\left(1-2m+\frac{n}{p'}\right)}\left\|\widehat{\dot{\Delta}_k w}\right\|_{L^{\rho_2}_T(L^p)}\\
&\lesssim \sum_{|l-j|\le 4}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L^{\rho_1}_T(L^p)}\left(\sum_{k\le l-2}2^{\varepsilon k r'}\right)^{1/r'}\|w\|_{\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{1-2m+\frac{n}{p'}-\varepsilon}_{p,r}\right)}\\
&\lesssim \sum_{|l-j|\le 4}2^{\varepsilon l}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L^{\rho_1}_T(L^p)}\|w\|_{\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{1-2m+\frac{n}{p'}-\varepsilon}_{p,r}\right)}.
\end{aligned}$$

Multiplying by 2^{sj}, and taking l^r-norm of both sides in the above estimate, we get

$$\left\{\sum_{j\in\mathbb{Z}} 2^{sjr}\left\|\widehat{J^j_1}\right\|^r_{L^{\rho}_T(L^p)}\right\}^{1/r} \lesssim \|v\|_{\mathcal{L}^{\rho_1}_T(F\dot{B}^{s+\varepsilon}_{p,r})}\|w\|_{\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{1-2m+\frac{n}{p'}-\varepsilon}_{p,r}\right)}. \tag{3.6}$$

Similarly, when $2m+\varepsilon-1>0$, we prove that

$$\left\{\sum_{j\in\mathbb{Z}} 2^{sjr}\left\|\widehat{J^j_2}\right\|^r_{L^{\rho}_T(L^p)}\right\}^{1/r} \lesssim \|w\|_{\mathcal{L}^{\rho_1}_T(F\dot{B}^{s+\varepsilon}_{p,r})}\|v\|_{\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{1-2m+\frac{n}{p'}-\varepsilon}_{p,r}\right)}. \tag{3.7}$$

Now for the third term, using Young's inequality (2.4), Holder's inequality, Bernstein's inequality (2.3) and Lemma 2.4, one has

$$
\begin{aligned}
\left\|\widehat{J_3^j}\right\|_{L^\rho_T(L^p)} &\lesssim \sum_{l\geq j-2}\sum_{|l-l'|\leq 1}\left\|\dot{\Delta}_l v\nabla\widehat{(-\Delta)^{-m}}\dot{\Delta}_{l'} w\right\|_{L^\rho_T(L^p)}\\
&\lesssim \sum_{l\geq j-2}\sum_{|l-l'|\leq 1}\left\|\widehat{\dot{\Delta}_l v} * \nabla\widehat{(-\Delta)^{-m}}\dot{\Delta}_{l'} w\right\|_{L^\rho_T(L^p)}\\
&\lesssim \sum_{l\geq j-2}\sum_{|l-l'|\leq 1}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L^{\rho_1}_T(L^p)}\left\|\nabla\widehat{(-\Delta)^{-m}}\dot{\Delta}_{l'} w\right\|_{L^{\rho_2}_T(L^1)}\\
&\lesssim \sum_{l\geq j-2}\sum_{|l-l'|\leq 1}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L^{\rho_1}_T(L^p)} 2^{l'\left(1-2m+\frac{n}{p'}\right)}\left\|\widehat{\dot{\Delta}_{l'} w}\right\|_{L^{\rho_2}_T(L^p)}\\
&\lesssim \sum_{|l-j|\leq 4} 2^{\varepsilon l}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L^{\rho_1}_T(L^p)}\|w\|_{\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{1-2m+\frac{n}{p'}-\varepsilon}_{p,r}\right)}.
\end{aligned}
$$

Multiplying by 2^{sj}, and taking l^r-norm of both sides in the above estimate, when $s>0$, we get

$$
\left\{\sum_{j\in\mathbb{Z}} 2^{sjr}\left\|\widehat{J_3^j}\right\|^r_{L^\rho_T(L^p)}\right\}^{1/r} \lesssim \|v\|_{\mathcal{L}^{\rho_1}_T\left(F\dot{B}^{s+\varepsilon}_{p,r}\right)}\|w\|_{\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{1-2m+\frac{n}{p'}-\varepsilon}_{p,r}\right)}. \tag{3.8}
$$

Combining the estimates (3.5), (3.6), (3.7) and (3.8), we get the inequality (3.3). □

We can now start to prove the first assertion of Theorem 1.1. For $t\in[0,T]$, we define the following map:

$$
\Psi : v(t)\to e^{-t\Lambda^\beta}v_0-\int_0^t e^{-(t-\tau)\Lambda^\beta}\nabla\cdot\left(v\nabla(-\Delta)^{-m}v\right)\mathrm{d}\tau, \tag{3.9}
$$

in the metric space:

$$
\mathcal{E}_T := \left\{v : \|v\|_{\mathcal{L}^{\rho_1}_T\left(F\dot{B}^{s_1}_{p,r}\right)\cap\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{s_2}_{p,r}\right)}\leq\eta,\quad d(v,w):=\|v-w\|_{\mathcal{L}^{\rho_1}_T\left(F\dot{B}^{s_1}_{p,r}\right)\cap\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{s_2}_{p,r}\right)}\right\},
$$

with

$$
s_1=1-2m+\frac{n}{p'}+\varepsilon,\ \ s_2=1-2m+\frac{n}{p'}-\varepsilon,\ \ \rho_1=\frac{\beta}{\beta-1+\varepsilon},\ \ \rho_2=\frac{\beta}{\beta-1-\varepsilon},\ \ 0<\varepsilon<\beta-1.
$$

Using Proposition 3.1 and Lemma 3.2 by choosing $\gamma=\rho=\frac{\beta}{2\beta-2}$, for any $v,w\in\mathcal{E}_T$, we obtain

$$
\begin{aligned}
\|\Psi(v)\|_{\mathcal{L}^{\rho_1}_T\left(F\dot{B}^{s_1}_{p,r}\right)\cap\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{s_2}_{p,r}\right)} &\lesssim \|e^{-t\Lambda^\beta}v_0\|_{\mathcal{L}^{\rho_1}_T\left(F\dot{B}^{s_1}_{p,r}\right)\cap\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{s_2}_{p,r}\right)}+\left\|v\nabla(-\Delta)^{-m}v\right\|_{\mathcal{L}^{\frac{\beta}{2\beta-2}}_T\left(F\dot{B}^{1-2m+\frac{n}{p'}}_{p,r}\right)}\\
&\lesssim \|v_0\|_{F\dot{B}^{2-2m-\beta+\frac{n}{p'}}_{p,r}}+\|v\|^2_{\mathcal{L}^{\rho_1}_T\left(F\dot{B}^{s_1}_{p,r}\right)\cap\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{s_2}_{p,r}\right)},
\end{aligned} \tag{3.10}
$$

and

$$
\begin{aligned}
d\left(\Psi(v),\Psi(w)\right) &\leq \left(\|v\|_{\mathcal{L}^{\rho_1}_T\left(F\dot{B}^{s_1}_{p,r}\right)\cap\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{s_2}_{p,r}\right)}+\|w\|_{\mathcal{L}^{\rho_1}_T\left(F\dot{B}^{s_1}_{p,r}\right)\cap\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{s_2}_{p,r}\right)}\right)\|v-w\|_{\mathcal{L}^{\rho_1}_T\left(F\dot{B}^{s_1}_{p,r}\right)\cap\mathcal{L}^{\rho_2}_T\left(F\dot{B}^{s_2}_{p,r}\right)}\\
&\lesssim \eta d(v,w).
\end{aligned} \tag{3.11}
$$

Using the standard contraction mapping argument ([14]) with these two estimates (3.10) and (3.11), we can demonstrate that if T is appropriately small, then Ψ is a contraction mapping from $(\mathcal{E}_T,d)$ into itself, and here, we omit the details. Hence there is $v \in \mathcal{E}_T$ such that $\Psi(v) = v$, which is a unique solution of Equation (1.1). Furthermore, Proposition 3.1 has given us,

$$\begin{aligned} \|v\|_{\mathcal{L}^\infty_T\left(F\dot{B}^{2-2m-\beta+\frac{n}{p'}}_{p,r}\right)} &\lesssim \|v_0\|_{F\dot{B}^{2-2m-\beta+\frac{n}{p'}}_{p,r}} + \|v\|^2_{\mathcal{L}^{\rho_1}_T(F\dot{B}^{s_1}_{p,r})\cap\mathcal{L}^{\rho_2}_T(F\dot{B}^{s_2}_{p,r})} \\ &\lesssim \|v_0\|_{F\dot{B}^{2-2m-\beta+\frac{n}{p'}}_{p,r}} + \eta^2. \end{aligned} \tag{3.12}$$

Thus

$$v \in \mathcal{X}_T = \mathcal{L}^\infty_T\left(F\dot{B}^{2-2m-\beta+\frac{n}{p'}}_{p,r}(\mathbb{R}^n)\right) \cap \mathcal{L}^{\rho_1}_T\left(F\dot{B}^{s_1}_{p,r}(\mathbb{R}^n)\right) \cap \mathcal{L}^{\rho_2}_T\left(F\dot{B}^{s_2}_{p,r}(\mathbb{R}^n)\right).$$

Let T^* be the maximal existence time of v in $\mathcal{X}_{T^*}$. Note that if $v_0 \in F\dot{B}^{2-2m-\beta+\frac{n}{p'}}_{p,r}(\mathbb{R}^n)$ is small enough, we can directly choose $T = \infty$ in (3.10) and (3.12), which gives that $T^* = \infty$. i.e., the solution v is global.

Now, if $T^* < \infty$. and $\|v\|_{\mathcal{L}^{\rho_1}_{T^*}(F\dot{B}^{s_1}_{p,r})\cap\mathcal{L}^{\rho_2}_{T^*}(F\dot{B}^{s_2}_{p,r})} < \infty$. Let $T_0 \in (0, T^*)$, and for $t \in [T_0, T)$, we consider the following integer equation

$$v(t) \to e^{-(t-T_0)\Lambda^\beta} v(T_0) - \int_{T_0}^t e^{-(t-\tau)\Lambda^\beta} \nabla \cdot \left(v\nabla(-\Delta)^{-m} v\right) \mathrm{d}\tau.$$

As before, we can show that

$$\begin{aligned} \|v\|_{\mathcal{L}^{\rho_1}(T_0,T^*;F\dot{B}^{s_1}_{p,r})\cap\mathcal{L}^{\rho_2}_T(T_0,T^*;F\dot{B}^{s_2}_{p,r})} &\lesssim \|v_0\|_{F\dot{B}^{2-2m-\beta+\frac{n}{p'}}_{p,r}} \\ &\quad + \|v\|^2_{\mathcal{L}^{\rho_1}(T_0,T^*;F\dot{B}^{s_1}_{p,r})\cap\mathcal{L}^{\rho_2}(T_0,T^*;F\dot{B}^{s_2}_{p,r})}. \end{aligned} \tag{3.13}$$

Using again the contraction mapping argument as in (3.10), which yields that the solution exists on $[T, T^*]$. Choosing T sufficiently close to T^*, then the solution existing on a time larger than T^*, which is a contradiction. This completes the proof of the first assertion of Theorem 1.1.

3.2. The case $p = \infty$

In this subsection, we study the limit case $p = \infty$. The following is the essential bilinear estimate.

Lemma 3.3. *Let* $1 \le \beta \le 2$ *and* $\frac{2-\beta}{2} < m < \frac{3-\beta+n}{2}$. *There holds*

$$\begin{aligned} \left\|v\nabla(-\Delta)^{-m}w\right\|_{\mathcal{L}^1_T\left(F\dot{B}^{3-2m-\beta+n}_{\infty,1}\right)} &\lesssim \|v\|_{\mathcal{L}^\infty_T\left(F\dot{B}^{2-2m-\beta+n}_{\infty,1}\right)\cup\mathcal{L}^1_T\left(F\dot{B}^{2-2m+n}_{\infty,1}\right)} \\ &\quad \times \|w\|_{\mathcal{L}^\infty_T\left(F\dot{B}^{2-2m-\beta+n}_{\infty,1}\right)\cup\mathcal{L}^1_T\left(F\dot{B}^{2-2m+n}_{\infty,1}\right)}. \end{aligned} \tag{3.14}$$

Proof. Using the decomposition (3.4) as in Lemma 3.2, and in order to estimate the three terms $J_i^j (i = 1, 2, 3)$, we use Young's inequality (2.4), Holder's inequality, Bernstein's inequality (2.3) and Lemma 2.4, as follows:

$$\begin{aligned} \left\|\widehat{J_1^j}\right\|_{L^1_T(L^\infty)} &\lesssim \sum_{|l-j|\le 4} \left\|\dot{\Delta}_l v \nabla \widehat{(-\Delta)^{-m}} \dot{S}_{l-1} w\right\|_{L^1_T(L^\infty)} \\ &\lesssim \sum_{|l-j|\le 4} \left\|\widehat{\dot{\Delta}_l v}\right\|_{L^1_T(L^\infty)} \sum_{k\le l-2} \left\|\nabla \widehat{(-\Delta)^{-m}} \dot{\Delta}_k w\right\|_{L^\infty_T(L^1)} \\ &\lesssim \sum_{|l-j|\le 4} \left\|\widehat{\dot{\Delta}_l v}\right\|_{L^1_T(L^\infty)} \sum_{k\le l-2} 2^{(\beta-1)k} 2^{-(\beta-1)k} 2^{k(1-2m+n)} \left\|\widehat{\dot{\Delta}_k w}\right\|_{L^\infty_T(L^\infty)} \\ &\lesssim \sum_{|l-j|\le 4} 2^{(\beta-1)l} 2^{-(2-2m+n)l} 2^{(2-2m+n)l} \left\|\widehat{\dot{\Delta}_l v}\right\|_{L^1_T(L^\infty)} \|w\|_{\mathcal{L}^\infty_T\left(F\dot{B}^{2-2m-\beta+n}_{\infty,1}\right)}. \\ &\lesssim 2^{(-3+2m+\beta-n)j} \sum_{|l-j|\le 4} 2^{(-3+2m+\beta-n)(l-j)} 2^{(2-2m+n)l} \left\|\widehat{\dot{\Delta}_l v}\right\|_{L^1_T(L^\infty)} \|w\|_{\mathcal{L}^\infty_T\left(F\dot{B}^{2-2m-\beta+n}_{\infty,1}\right)}. \end{aligned}$$

Multiplying by $2^{(3-2m-\beta+n)j}$, and taking l^1-norm of both sides in the above estimate, we get

$$\left\{\sum_{j\in\mathbb{Z}} 2^{(3-2m-\beta+n)j}\left\|\widehat{J_1^j}\right\|_{L_T^1(L^\infty)}\right\} \lesssim \|v\|_{\mathcal{L}_T^1(F\dot{B}_{\infty,1}^{2-2m+n})}\|w\|_{\mathcal{L}_T^\infty(F\dot{B}_{\infty,1}^{2-2m-\beta+n})}. \tag{3.15}$$

Similarly, when $2-2m-\beta<0$, we prove that

$$\left\{\sum_{j\in\mathbb{Z}} 2^{(3-2m-\beta+n)j}\left\|\widehat{J_2^j}\right\|_{L_T^1(L^\infty)}\right\} \lesssim \|w\|_{\mathcal{L}_T^1(F\dot{B}_{\infty,1}^{2-2m+n})}\|v\|_{\mathcal{L}_T^\infty(F\dot{B}_{\infty,1}^{2-2m-\beta+n})}. \tag{3.16}$$

For the third term J_3^j, one has

$$\begin{aligned}
\left\|\widehat{J_3^j}\right\|_{L_T^1(L^\infty)} &\lesssim \sum_{l\geq j-2}\sum_{|l-l'|\leq 1}\left\|\dot{\Delta}_l v\nabla\widehat{(-\Delta)^{-m}}\dot{\Delta}_{l'}w\right\|_{L_T^1(L^\infty)}\\
&\lesssim \sum_{l\geq j-2}\sum_{|l-l'|\leq 1}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L_T^\infty(L^\infty)} 2^{l'}2^{-l'}2^{l'(1-2m+n)}\left\|\widehat{\dot{\Delta}_{l'}w}\right\|_{L_T^1(L^\infty)}\\
&\lesssim \sum_{l\geq j-2} 2^{-l}2^{-(3-2m-\beta+n)l}2^{(3-2m-\beta+n)l}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L_T^\infty(L^\infty)}\|w\|_{\mathcal{L}_T^1(F\dot{B}_{\infty,1}^{2-2m+n})}.\\
&\lesssim 2^{(-3+2m+\beta-n)j}\sum_{l\geq j-2}2^{(-3+2m+\beta-n)(l-j)}2^{(2-2m-\beta+n)l}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L_T^\infty(L^\infty)}\|w\|_{\mathcal{L}_T^1(F\dot{B}_{\infty,1}^{2-2m-\beta+n})}.
\end{aligned}$$

Multiplying by $2^{(3-2m-\beta+n)j}$, and taking l^1-norm of both sides in the above estimate, when $-3+2m+\beta-n<0$, we get

$$\left\{\sum_{j\in\mathbb{Z}} 2^{(3-2m-\beta+n)j}\left\|\widehat{J_3^j}\right\|_{L_T^1(L^\infty)}\right\} \lesssim \|w\|_{\mathcal{L}_T^1(F\dot{B}_{\infty,1}^{2-2m+n})}\|v\|_{\mathcal{L}_T^\infty(F\dot{B}_{\infty,1}^{2-2m-\beta+n})}. \tag{3.17}$$

Combining the estimates (3.15), (3.16) and (3.17), we obtain the inequality (3.14). This completes the proof of Lemma 3.3. □

We consider the resolution space $\mathcal{L}_t^\infty(F\dot{B}_{\infty,1}^{2-2m-\beta+n}(\mathbb{R}^n))\cap\mathcal{L}_t^1(F\dot{B}_{\infty,1}^{2-2m+n}(\mathbb{R}^n))$, in order to demonstrate the second assertion of Theorem 1.1. Returning to the mapping (3.9), and according to Proposition 3.1 with $\gamma=1$, Lemma 3.3, we have

$$\begin{aligned}
\|\Psi(v)\|_{\mathcal{L}_t^\infty(F\dot{B}_{\infty,1}^{2-2m-\beta+n})\cap\mathcal{L}_t^1(F\dot{B}_{\infty,1}^{2-2m+n})} &\lesssim \|e^{-t\Lambda^\beta}v_0\|_{\mathcal{L}_t^\infty(F\dot{B}_{\infty,1}^{2-2m-\beta+n})\cap\mathcal{L}_t^1(F\dot{B}_{\infty,1}^{2-2m+n})}\\
&\quad+\left\|v\nabla(-\Delta)^{-m}v\right\|_{\mathcal{L}_T^1(F\dot{B}_{\infty,1}^{3-2m-\beta+n})}\\
&\lesssim \|v_0\|_{F\dot{B}_{\infty,1}^{2-2m-\beta+n}}+\|v\|^2_{\mathcal{L}_t^\infty(F\dot{B}_{\infty,1}^{2-2m-\beta+n})\cap\mathcal{L}_t^1(F\dot{B}_{\infty,1}^{2-2m+n})}.
\end{aligned} \tag{3.18}$$

Due to the standard contraction mapping argument as in Subsection 3.1, if $\|v_0\|_{F\dot{B}_{\infty,1}^{2-2m-\beta+n}}$ is small enough, we can prove that Equation (1.1) has a unique solution v satisfying

$$v\in\mathcal{L}_t^\infty(F\dot{B}_{\infty,1}^{2-2m-\beta+n}(\mathbb{R}^n))\cap\mathcal{L}_t^1(F\dot{B}_{\infty,1}^{2-2m+n}(\mathbb{R}^n)).$$

The proof of Theorem 1.1 is complete.

4. Well-posedness for $\beta=1$: Proof of Theorem 1.3

In this section, we will establish the global well-posedness for the system (1.1) in the limit cases $\beta=1$, with initial data in critical Fourier-Besov paces $F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}(\mathbb{R}^n)$ with $1<p\leq\infty$.

4.1. The case $p<\infty$

In this case, the crucial estimate is the following:

Lemma 4.1. *Let $1<p<\infty$ and $\frac{1}{2}<m<\frac{1}{2}\left(1+\frac{n}{p'}\right)$. There holds*

$$\left\|v\nabla(-\Delta)^{-m}w\right\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}\right)}\lesssim\|v\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}\right)}\times\|w\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}\right)}. \tag{4.1}$$

Proof. We get the estimates of the terms $J_i^j(i=1,2,3)$ by making a slight modification to the proof of Lemmas 3.2 and 3.3, as follows:

$$\begin{aligned}\left\|\widehat{J_1^j}\right\|_{L_t^\infty(L^p)}&\lesssim\sum_{|l-j|\le 4}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L_t^\infty(L^p)}\sum_{k\le l-2}2^{k\left(1-2m+\frac{n}{p'}\right)}\left\|\widehat{\dot{\Delta}_k w}\right\|_{L_t^\infty(L^p)}\\&\lesssim\sum_{|l-j|\le 4}2^{-(1-2m+\frac{n}{p'})l}2^{(1-2m+\frac{n}{p'})l}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L_t^\infty(L^p)}\|w\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}\right)}.\end{aligned}$$

Thus, we have

$$\left\{\sum_{j\in\mathbb{Z}}2^{(1-2m+\frac{n}{p'})j}\left\|\widehat{J_1^j}\right\|_{L_T^1(L^\infty)}\right\}\lesssim\|v\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+\frac{n}{p'}}\right)}\|w\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}\right)}. \tag{4.2}$$

Similarly, when $m>\frac{1}{2}$, there holds

$$\begin{aligned}\left\|\widehat{J_2^j}\right\|_{L_t^\infty(L^p)}&\lesssim\sum_{|l-j|\le 4}\sum_{k\le l-2}2^{k\frac{n}{p'}}\left\|\widehat{\dot{\Delta}_k v}\right\|_{L_t^\infty(L^p)}2^{l(1-2m)}\left\|\widehat{\dot{\Delta}_l w}\right\|_{L_t^\infty(L^p)}\\&\lesssim\sum_{|l-j|\le 4}2^{-(1-2m+\frac{n}{p'})l}2^{(1-2m+\frac{n}{p'})l}\left\|\widehat{\dot{\Delta}_l w}\right\|_{L_t^\infty(L^p)}\|v\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}\right)},\end{aligned}$$

therefor

$$\left\{\sum_{j\in\mathbb{Z}}2^{(1-2m+\frac{n}{p'})j}\left\|\widehat{J_2^j}\right\|_{L_T^1(L^\infty)}\right\}\lesssim\|w\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+\frac{n}{p'}}\right)}\|v\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+\frac{n}{p'}}\right)}, \tag{4.3}$$

and for the the last term, one has

$$\begin{aligned}\left\|\widehat{J_3^j}\right\|_{L_t^\infty(L^p)}&\lesssim\sum_{l\ge j-2}\sum_{|l-l'|\le 1}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L_t^\infty(L^p)}2^{l'\left(1-2m+\frac{n}{p'}\right)}\left\|\widehat{\dot{\Delta}_{l'} w}\right\|_{L_t^\infty(L^p)}\\&\lesssim\sum_{l\ge j-2}2^{-(1-2m+\frac{n}{p'})l}2^{(1-2m+\frac{n}{p'})l}\left\|\widehat{\dot{\Delta}_l v}\right\|_{L_t^\infty(L^p)}\|w\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}\right)},\end{aligned}$$

then when $1-2m+\frac{n}{p'}>0$, we find

$$\left\{\sum_{j\in\mathbb{Z}}2^{(1-2m+\frac{n}{p'})j}\left\|\widehat{J_3^j}\right\|_{L_T^1(L^\infty)}\right\}\lesssim\|v\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+\frac{n}{p'}}\right)}\|w\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+\frac{n}{p'}}\right)}. \tag{4.4}$$

This completes the proof of Lemma 4.1. $\square$

We are now in a position to demonstrate the first assertion of Theorem 1.3. We consider the resolution space $\mathcal{L}_t^\infty(F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}(\mathbb{R}^n))$ and returning to the mapping (3.9). Proposition 3.1 (with $\beta=1,\ \gamma=\infty$)

and Lemma 4.1 give us

$$\begin{aligned}\|\Psi(v)\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+\frac{n}{p'}}\right)} &\lesssim \|v_0\|_{F\dot{B}_{\infty,1}^{1-2m+\frac{n}{p'}}} + \left\|v\nabla(-\Delta)^{-m}v\right\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+\frac{n}{p'}}\right)} \\ &\lesssim \|v_0\|_{F\dot{B}_{\infty,1}^{1-2m+\frac{n}{p'}}} + \|v\|^2_{\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+\frac{n}{p'}}\right)}. \end{aligned} \tag{4.5}$$

Using the standard contraction mapping argument again as before, wa can show that the system (1.1) has a unique solution v in $\mathcal{L}_t^\infty(F\dot{B}_{p,1}^{1-2m+\frac{n}{p'}}(\mathbb{R}^n))$ if $\|v_0\|_{F\dot{B}_{\infty,1}^{2-2m-\beta+n}}$ is small enough.

4.2. The case $p=\infty$

In the case $p=\infty$, the resolution space $\mathcal{L}_t^\infty(F\dot{B}_{\infty,1}^{1-2m+n}(\mathbb{R}^n))$ can't be adjusted to Equation (1.1), and thus, we move on to think about the resolution space $\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+n}(\mathbb{R}^n)\right)\cap\mathcal{L}_t^1(F\dot{B}_{\infty,1}^{2-2m+n}(\mathbb{R}^n))$. And from the mapping (3.9), Proposition 3.1 and Lemma 3.3 with $\beta=\gamma=1$, we have

$$\begin{aligned}\|\Psi(v)\|_{\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+n}\right)\cap\mathcal{L}_t^1\left(F\dot{B}_{\infty,1}^{2-2m+n}\right)} &\lesssim \|v_0\|_{F\dot{B}_{\infty,1}^{1-2m+n}} + \left\|v\nabla(-\Delta)^{-m}v\right\|_{\mathcal{L}_T^1\left(F\dot{B}_{\infty,1}^{3-2m-\beta+n}\right)} \\ &\lesssim \|v_0\|_{F\dot{B}_{\infty,1}^{1-2m+n}} + \|v\|^2_{\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+n}\right)\cap\mathcal{L}_t^1\left(F\dot{B}_{\infty,1}^{2-2m+n}\right)}. \end{aligned} \tag{4.6}$$

And as before, we apply the standard contraction mapping argument, then if $\|v_0\|_{F\dot{B}_{\infty,1}^{2-2m-\beta+n}}$ is small enough, the system (1.1) has a unique solution v in $\mathcal{L}_t^\infty\left(F\dot{B}_{\infty,1}^{1-2m+n}(\mathbb{R}^n)\right)\cap\mathcal{L}_t^1(F\dot{B}_{\infty,1}^{2-2m+n}(\mathbb{R}^n))$. This completes the proof of Theorem 1.3.

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