Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–15.
ISSN-0037-8712 in press
doi:10.5269/bspm.66526

Generalized Solutions for Time ψ-Fractional Evolution Equations

Abdelmjid Benmerrous, Lalla saadia Chadli, Abdelaziz Moujahid, M'hamed Elomari and Said Melliani

ABSTRACT: This paper, focuses on the fractional system of semilinear evolution equations with initial data is singular generalized functions, the fractional derivative ${}_0^cD_\psi^\alpha$ is $\psi-$Caputo derivative of order α, $1 < \alpha \leq 2$, which we will prove to be inside Colombeau algebra. The notion of $\psi-$Cosine family is introduced and demonstrated in Colombeau algebra. Using Banach's fixed point theorem and Laplace transforms, we gave the integral solution of the problem. In Colombeau's algebra, The existence and uniqueness of the solution are demonstrated using the Gronwall lemma.

Key Words: Distributions, Colombeau algebra, $\psi-$Caputo derivative, Laplace transforms, mild solution.

Contents

1. Introduction

Colombeau proposed the best approach for solving the issues that Schwartz theory of distributions is concerned with (1984, 1985) [9,10]. He created a generalized function sequential differential algebra. $\mathcal{G}(\mathbb{R})$, it includes the distribution space $D'(\mathbb{R})$ as a subspace. Colombeau's idea of generalized functions really generalizes the notion of Schwartz distributions, these novel Colombeau generalized functions can be distinguished in the same manner as distributions can, but with regard to multiplication and other nonlinear operations. It is notable that the results of these operations are always represented as Colombeau generalized functions in this algebra. These new generalized functions are closely connected to distributions in the sense that their description may be viewed as a natural extension of Schwartz's distribution concept.

The fractional evolution equations have gained significant attention due to their ability to describe complex phenomena in various fields, ranging from physics and engineering to biology and finance. One of the primary motivations behind studying fractional evolution equations is their capability to capture non-local and memory-dependent behavior, which cannot be adequately modeled by classical differential equations. These equations incorporate fractional derivatives, allowing for the incorporation of long-term memory effects and non-local interactions into the mathematical model. By considering fractional evolution equations, researchers aim to develop a deeper understanding of intricate dynamics and improve predictions in systems exhibiting anomalous diffusion, viscoelasticity, or power-law decay. Furthermore, the study of fractional evolution equations also contributes to the advancement of mathematical analysis, numerical methods, and the development of novel tools for solving and simulating these equations efficiently.

2010 *Mathematics Subject Classification:* 46F05, 35G25, 35Q55.
Submitted January 01, 2023. Published June 07, 2023

Our main goal is to look at the next abstract fractional semi-linear problem

$$\begin{cases} {}_0^cD_\psi^\alpha u(x,t) = Au(x,t) + F\left(t, u(x,t)\right), & t \in [0,T] \\ u(x,0) = a_0(x) \end{cases} \tag{1.1}$$

Where a_0 is singular generalized functions, ${}_0^cD_\psi^\alpha$ is $\psi-$Caputo derivative of order α, $1 < \alpha \le 2$, which we will prove to be inside Colombeau algebra, F satisfies L^∞ logarithmic type and A is an operator defined from the Colombeau algebra into itself. Our goal will be to give a systematic and general treatment of (1.1) from the standpoint of existence, uniqueness, and smoothness of solutions and we presented the definition of the generalized ψ-cosine family this principle is used to prove the foregoing. The pioneering work on (1.1) (for the normal and caputo fractional derivatives in Colombeau algebra), was done by A.Benmerrous and Al in [4] [2], and our development follows his approach. Our results extend those of [24] [4] [2] in several respects. First, we allow for a more general linear term A, in that we assume A is the infinitesimal generator of an arbitrary strongly continuous cosine family. Second, we analyze various hypotheses on the nonlinear term F, some of which are more general than found in [4,2]. Third, it is demonstrated that distribution solutions to some classes of such equations exist.

The paper is structured as follows, in section 2 we mention some notions of Colombeau's algebra, in section 3 we will prove the existence of $\psi-$Caputo derivative of order α in Colombeau algebra, in section 4, in the first part we clarify the expression of generalized ψ-cosine family by gave the integral solution of the problem, in section 5, we demonstrated the existence and uniqueness of the mild solution of the problem.

2. Preliminaries

Here we list some notations and formulas to be used later. The elements of Colombeau algebras $\mathcal{G}$ are equivalence classes of regularizations, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ε. Therefore, for any set X, the family of sequences $(u_\varepsilon)_{\varepsilon\in[0;1]}$ of elements of a set X will be denoted by $X^{[0;1]}$, such sequences will also be called nets and simply written as u_ε.
Let $\mathcal{D}(\mathbb{R}^n)$ be the space of all smooth functions $\varphi : \quad \mathbb{R}^n \longrightarrow \mathbb{C}$ with compact support.
For $q \in \mathbb{N}$ we denote:

$$\mathcal{A}_q(\mathbb{R}^n) = \left\{\varphi \in \mathcal{D}\left(\mathbb{R}^n\right) / \int \varphi(x)dx = 1 and \int x^\alpha \varphi(x)dx = 0 \text{ for } 1 \le \alpha \le q\right\}.$$

The elements of the set $\mathcal{A}_q$ are called test functions.

It is obvious that $\mathcal{A}_1 \supset \mathcal{A}_2 \dots$. Colombeau in his books has proved that the sets $\mathcal{A}_k$ are non empty for all $k \in \mathbb{N}$.

For $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ and $\epsilon > 0$ it is denoted as $\varphi_\epsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)$ for $\varphi \in \mathcal{D}\left(\mathbb{R}^n\right)$ and $\check{\varphi}(x) = \varphi(-x)$.

We denote by:

$$\mathcal{E}\left(\mathbb{R}^n\right) = \left\{u : \mathcal{A}_1 \times \mathbb{R}^n \to \mathbb{C}/ \text{ with } u(\varphi, x) \text{ is } \mathcal{C}^\infty \text{ to the second variable } x\right\},$$

$$u\left(\varphi_\varepsilon, x\right) = u_\varepsilon(x) \quad \forall \varphi \in \mathcal{A}_1,$$

$$\mathcal{E}_M\left(\mathbb{R}^n\right) = \left\{\left(u_\varepsilon\right)_{\varepsilon>0} \subset \mathcal{E}\left(\mathbb{R}^n\right)/\forall K \subset \mathbb{R}^n, \forall a \in \mathbb{N}, \exists N \in \mathbb{N} \text{ such that } \sup_{x\in K}\left\|D^\alpha u_\varepsilon(x)\right\| = \mathcal{O}\left(\varepsilon^{-N}\right) \text{ as } \varepsilon \to 0\right\},$$

$$\mathcal{N}\left(\mathbb{R}^n\right) = \left\{\left(u_\varepsilon\right)_{\varepsilon>0} \in \mathcal{E}\left(\mathbb{R}^n\right)/\forall K \subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}, \forall p \in \mathbb{N} \text{ such that } \sup_{x\in K}\left\|D^\alpha u_\varepsilon(x)\right\| = \mathcal{O}\left(\varepsilon^{p}\right) \text{ as } \varepsilon \to 0\right\},$$

The generalized functions of Colombeau are elements of the quotient algebra $\mathcal{G}\left(\mathbb{R}^n\right) = \mathcal{E}_M\left[\mathbb{R}^n\right]/\mathcal{N}\left[\mathbb{R}^n\right]$, where the elements of the set $\mathcal{E}_M\left(\mathbb{R}^n\right)$ are moderate while the elements of the set $\mathcal{N}\left(\mathbb{R}^n\right)$ are negligible.

The meaning of the term 'association' in $\mathcal{G}(\mathbb{R})$ is given with the next two definitions.

Definition 2.1. *Generalized functions $f, g \in \mathcal{G}(\mathbb{R})$ are said to be associated, denoted $f \approx g$, if for each representative $f(\varphi_\varepsilon, x)$ and $g(\varphi_\varepsilon, x)$ and arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}$ such that for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R})$, we have:*

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \| f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x) \| \psi(x) dx = 0.$$

Definition 2.2. *Generalized functions $f \in \mathcal{G}(\mathbb{R})$ is said to admit some as $u \in \mathcal{D}^{'}(\mathbb{R})$ 'associated distribution', denoted $f \approx u$, if for each representative $f(\varphi_\varepsilon, x)$ of f and any $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}$ such that for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R})$, we have:*

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} f(\varphi_\varepsilon, x) \psi(x) dx = \langle u, \psi \rangle.$$

3. $\psi-$Fractional Derivative in $\mathcal{G}$

Let $(f_\epsilon(t))_\epsilon$ be a representative of a Colombeau generalized function $f(t) \in \mathcal{G}(\mathbb{R}^+)$ and let $n-1 < \alpha < n$, $\psi \in \mathcal{C}^n(\mathbb{R}^+)$ be an increasing function with $\psi'(t) \neq 0 \quad$ for all $t \in \mathbb{R}^+$.

The ψ-Caputo fractional derivative of $(f_\epsilon(t))_\epsilon$, is defined by

$$ {}_0^cD_\psi^\alpha f_\epsilon(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (\psi(t) - \psi(s))^{n-\alpha-1} f_\epsilon^{[n]}(s) \psi'(s) ds, & n-1 < \alpha < n, \\ f_\epsilon^{(n)}(t) = f_\epsilon^{[n]}(t) = (\frac{1}{\psi'(t)} \frac{\mathrm{d}}{dt})^n f_\epsilon(t), & \alpha = n, \end{cases} \tag{3.1}$$

$n \in \mathbb{N}, \epsilon \in (0,1)$.

Lemma 3.1. *Let $(f_\epsilon(t))_\epsilon$ be a representative of $f(t) \in \mathcal{G}(\mathbb{R}^+)$. Then, for every $\alpha > 0$, $\sup_{t\in[0,T]} \mid {}_0^cD_\psi^\alpha f_\epsilon(t) \mid$ has a moderate bound.*

Proof. Fix $\epsilon \in (0,1)$.
Let $n-1 < \alpha < n, \quad n \in \mathbb{N}$.

Then,
$\sup_{t\in[0,T]} \mid {}_0^cD_\psi^\alpha f_\epsilon(t) \mid \leq \frac{1}{\Gamma(n-\alpha)} \sup_{t\in[0,T]} \int_0^t \mid (\psi(t) - \psi(s))^{n-\alpha-1} f_\epsilon^{[n]}(s) \psi'(s) \mid ds$

$$= \frac{1}{\Gamma(n-\alpha)} \sup_{s\in[0,T]} \mid f_\epsilon^{[n]}(s) \mid \sup_{t\in[0,T]} \mid \frac{(\psi(t) - \psi(0))^{n-\alpha}}{n-\alpha} \mid$$

$$\leq \frac{1}{\Gamma(n-\alpha)} \frac{T^{n-\alpha}}{n-\alpha} \sup_{s\in[0,T]} \mid f_\epsilon^{[n]}(s) \mid .$$

Since $f(t) \in \mathcal{G}([0, +\infty))$, as a result $\sup_{s\in[0,T]} \mid f_\epsilon^{[n]}(s) \mid$ has a moderate bound.

Thus, $\exists M \in \mathbb{N}$, such that

$$\sup_{t\in[0,T]} \mid {}_0^cD_\psi^\alpha f_\epsilon(t) \mid = \mathcal{O}\left(\varepsilon^{-M}\right), \quad \varepsilon \to 0.$$

Then, $\sup_{t\in[0,T]} \mid {}_0^cD_\psi^\alpha f_\epsilon(t) \mid$ has a moderate bound, $\forall \alpha > 0$.
□

Lemma 3.2. *Let $(f_{1\epsilon}(t))_\epsilon$, $(f_{2\epsilon}(t))_\epsilon$ be two distinct representatives of $f(t) \in \mathcal{G}(\mathbb{R}^+)$. Then, for every $\alpha > 0$, $\sup_{t\in[0,T]} \mid {}_0^cD_\psi^\alpha f_{1\epsilon}(t) - {}_0^cD_\psi^\alpha f_{2\epsilon}(t) \mid$ is negligible.*

Proof. Fix $\epsilon \in (0,1)$.
Let $n-1<\alpha<n, \quad n \in \mathbb{N}$.
Then,

$$\sup_{t\in[0,T]} \mid {}_0^cD_\psi^\alpha f_{1\epsilon}(t) - {}_0^cD_\psi^\alpha f_{2\epsilon}(t) \mid \leq \frac{1}{\Gamma(n-\alpha)} \frac{T^{n-\alpha}}{n-\alpha} \sup_{s\in[0,T]} \mid f_{1\epsilon}^{[n]}(s) - f_{2\epsilon}^{[n]}(s) \mid .$$

Since $(f_{1\epsilon}(t))_\epsilon$ and $(f_{2\epsilon}(t))_\epsilon$ represent the same Colombeau generalized function $f(t)$, so $\sup_{s\in[0,T]} \mid f_{1\epsilon}^{[n]}(s) - f_{2\epsilon}^{[n]}(s) \mid$ is negligible, then for all p$\in \mathbb{N}$

$$\sup_{t\in[0,T]} \mid {}_0^cD_\psi^\alpha f_{1\epsilon}(t) - {}_0^cD_\psi^\alpha f_{2\epsilon}(t) \mid = \mathcal{O}\left(\varepsilon^{-p}\right), \quad \varepsilon \to 0.$$

Therefore, $\sup_{t\in[0,T]} \mid {}_0^cD_\psi^\alpha f_{1\epsilon}(t) - {}_0^cD_\psi^\alpha f_{2\epsilon}(t) \mid$ is negligible. □

We may now initiate the ψ-Caputo fractional derivative of a Colombeau generalized function on $\mathbb{R}^+$ after establishing the first two lemmas.

Definition 3.1. *Let $f(t) \in \mathcal{G}(\mathbb{R}^+)$ be a Colombeau function on $\mathbb{R}^+$.*
The $\psi-$Caputo fractional derivative of $f(t)$, using the notation
${}_0^cD_\psi^\alpha f(t) = \left[\left({}_0^cD_\psi^\alpha f_\epsilon(t)\right)_\epsilon\right]$, *$\alpha>0$, is the component of $\mathcal{G}(\mathbb{R}^+)$ satisfying (3.1).*

Remark 3.2. *For $\alpha \in (n-1,n), \quad n \in \mathbb{N}$.*
The first derivative of $(\mathrm{d}/\mathrm{dt}){}_0^cD_\psi^\alpha f_\epsilon(t)$ is
$(\mathrm{d}/\mathrm{dt}){}_0^cD_\psi^\alpha f_\epsilon(t) = \frac{1}{\Gamma(1-\alpha)}\left[\int_0^t \left(\frac{\psi\prime(s)}{(\psi(t)-\psi(s))^{\alpha+1-n}} f_\varepsilon^{[n+1]}(s)\right)\mathrm{d}s + \frac{\psi\prime(0)}{(\psi(t)-\psi(0))^{\alpha+1-n}} f_\epsilon^{[m]}(0)\right]$ *and it is not defined in zero, unless $f_\epsilon^{[m]}(0)=0$.*

Theorem 3.3. *Let $f(t) \in \mathcal{G}$ be a Colombeau generalized function. The $\psi-$Caputo fractional derivative ${}_0^cD_\psi^\alpha f(t)$ is a Colombeau generalized function, if $f_\varepsilon^{[n]}(0) = f_\epsilon^{[n+1]}(0) = f_\epsilon^{[n+2]}\ (0) = \cdots = 0$.*

Proof. Let $n-1<\alpha<n, \quad n \in \mathbb{N}$.
In Lemma 1, we proved that $\sup_{t\in[0,T]} \mid {}_0^cD_\psi^\alpha f_\epsilon(t) \mid$ has a moderate limit for indefinite Colombeau generalized function. To get a moderate limit for the initial derivative $(\mathrm{d}/\mathrm{d}\ t){}_0^cD_\psi^\alpha f_\epsilon(t)$ we utilize the expression acquired in Remark 1 and for $f_\epsilon^{[n]}(0)=0$, we obtain

$$(\mathrm{d}/\mathrm{dt}){}_0^cD_\psi^\alpha f_\epsilon(t) = (1/\Gamma(1-\alpha))\int_0^t \left(\frac{\psi\prime(S)}{(\psi(t)-\psi(s))^{\alpha+1-n}} f_\varepsilon^{[n+1]}(s)\right)\mathrm{d}s.$$

.
Now, in the same way as in Lemma 1 we acquires a moderate limit for $\sup_{t\in[0,T]} \mid (\mathrm{d}/\mathrm{dt}){}_0^cD_\psi^\alpha f_\epsilon(t) \mid$.
Using the conditions, higher-order derivatives can be estimated similarly. $f_\epsilon^{[n]}(0) = f_\epsilon^{[n+1]}(0) = f_\epsilon^{[n+2]}(0) = \cdots = 0$.
Finally, if $f_\epsilon^{[n]}(0)=0$, therefore, it follows that for each $\alpha>0$, all derivatives of ${}_0^cD_\psi^\alpha f_\epsilon(t)$ have moderate representations. □

Definition 3.4. *Let $(f_\epsilon(t))_\epsilon$ be a representative of $ft) \in \mathcal{G}(\mathbb{R}^+)$.*
The regularized $\psi-$Caputo fractional derivative of $(f_\epsilon(t))_\epsilon$, is given by

$${}_0^c\tilde{D}_\psi^\alpha f_\epsilon(t) = \begin{cases} \left({}_0^cD_\psi^\alpha f_\epsilon(t) * \varphi_\varepsilon\right)(t), & n-1<\alpha<n \\ f_\epsilon^{(n)}(t) = f_\epsilon^{[n]}(t) = \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{dt}\right)^n f_\epsilon(t), & \alpha = n, \end{cases} \tag{3.2}$$

$n \in \mathbb{N}, \epsilon \in (0,1)$.

where ${}_0^cD_\psi^\alpha f_\epsilon(t)$ is provided by (3.1).

The convolution in (3.2) is $\left({}_0^cD_\psi^\alpha f_\epsilon(t) * \varphi_\epsilon\right)(t) = \int_0^\infty {}_0^cD_\psi^\alpha f_\epsilon(t)\varphi_\epsilon(t-s)\mathrm{d}s$.

4. Generalized ψ-cosine family

4.1. The notion

Let $(X, \|.\|)$ denote a Banach space, and $\mathcal{L}(X)$ denote the space of all linear continuous mappings. Before we define the generalized $\psi-$cosine family, we will state that an application from $\mathcal{G} \longrightarrow \mathcal{G}$ must be linear.

Definition 4.1. *Let X be a locally convex space with a semi-norm familly $(q_i)_{i \in I}$.*

We define $\mathcal{E}_M$ by the set of $(y_\varepsilon)_\varepsilon \subset X$ such that there exist $n \in \mathbb{N}$ and for all $i \in I \subset \mathbb{N}$, $q_i(y_\varepsilon) = \mathcal{O}_{\epsilon \to 0}(\varepsilon^{-n})$

And
$\mathcal{N}(X)$ by $(y_\varepsilon)_\varepsilon \subset X$ such that for all $m \in \mathbb{N}$ and for all $i \in I \subset \mathbb{N}$, $q_i(y_\varepsilon) = \mathcal{O}_{\epsilon \to 0}(\varepsilon^n)$.

Then the Colombeau generalized function type by:

$$\overline{X} = \mathcal{E}_M(X)/\mathcal{N}(X)$$

Initially, using a provided family $(A_\varepsilon)_{\varepsilon \in [0,1]}$ of maps $A\varepsilon : X \longrightarrow X$ we want to see if we can define a map $A : \overline{X} \longrightarrow \overline{X}$, $A_\varepsilon \in \mathcal{L}(X)$.

The next lemma expresses the basic requirement:

Lemma 4.1. *Let $(A_\epsilon)_\epsilon$ represent a family of maps $A_\epsilon : X \longrightarrow X$.*

For each $(x_\epsilon)_\epsilon \in \mathcal{E}_M(X)$ and $(y_\epsilon)_\epsilon \in \mathcal{N}(X)$, suppose that:
1) $(A_\epsilon x_\epsilon)_\varepsilon \in \mathcal{E}_M(X)$
2) $(A_\epsilon(x_\epsilon + y_\epsilon))_\epsilon - (A_\epsilon x_\epsilon)_\epsilon \in \mathcal{N}(X)$

So

$$A : \begin{cases} \overline{X} \longrightarrow \overline{X} \\ x = [x_\epsilon] \longmapsto Ax = [A_\epsilon x_\epsilon] \end{cases}$$

is clearly stated.

Proof. The first attribute reveals that the class $[(A_\epsilon x_\epsilon)_\epsilon] \in \overline{X}$.
Let $x_\epsilon + y_\epsilon$ should serve as another example of $x = [x_\epsilon]$, we have from the second property:

$$(A_\epsilon(x_\epsilon + y_\epsilon))_\epsilon - (A_\epsilon x_\epsilon)_\epsilon \in \mathcal{N}(X)$$

and

$$[(A_\epsilon(x_\epsilon + y_\epsilon))_\epsilon] = [(A_\epsilon x_\epsilon))_\epsilon] \text{ in } \overline{X}$$

So A is well defined. □

We shall now introduce the idea of the generalized ψ-cosine family(Convolution-type cosine family).

Definition 4.2.

$$E_{M,\psi}\left(\mathbb{R}^+, \mathcal{L}(X)\right) := \{C_{\psi,\varepsilon} : \mathbb{R}^+ \to \mathcal{L}(X), \varepsilon \in]0,1[/\forall T > 0, \exists a \in \mathbb{R} \ \text{ such that } \sup_{t \in [0,T]} \|C_{\psi,\epsilon}(t)\| = \mathcal{O}(\varepsilon^a) \ \text{ as } \ \varepsilon \to 0\} \tag{4.1}$$

$$N_\psi(\mathbb{R}^+, \mathcal{C}(X)) := \{ N_\varepsilon : [0, +\infty[\to \mathcal{L}(X), \varepsilon \in]0,1[/\forall T > 0, \forall b \in \mathbb{R} \ \text{ such that } \sup_{t \in [0,T]} \|N_\varepsilon(t)\| = \mathcal{O}\left(\varepsilon^b\right) \ \text{ as } \ \varepsilon \to 0\} \tag{4.2}$$

Proposition 4.3. *$N_\psi(\mathbb{R}^+,\mathcal{L}(X))$ is an ideal of $E_{M,\psi}(\mathbb{R}^+,\mathcal{L}(X))$ and $E_{M,\psi}(\mathbb{R}^+,\mathcal{L}(X))$ is an algebra with respect to composition.*

Proof. Let $(S_{\psi,\epsilon}(t))_\epsilon \in E_{M,\psi}(\mathbb{R}^+,\mathcal{L}(X))$, $(N_\epsilon(t))_\epsilon \in N_\psi(\mathbb{R}^+,\mathcal{L}(X))$.

We shall solely establish the first argument, which is that:

$$(S_{\psi,\varepsilon}(t)N_\varepsilon(t))_\varepsilon , (N_\varepsilon(t)S_{\psi,\varepsilon}(t))_\varepsilon \in N_\psi(\mathbb{R}^+,\mathcal{L}(X))$$

where $S_{\psi,\varepsilon}(t)N_\varepsilon(t)$ denotes the composition.
Let $\epsilon \in]0,1[$. By (3) and (4), $\exists a\in\mathbb{R}$, $\forall b\in\mathbb{R}$ such that:

$$\|S_{\psi,\varepsilon}(t)N_\varepsilon(t)\| \leq \|S_{\psi,\varepsilon}(t)\|\,\|N_\varepsilon(t)\| = O\left(\epsilon^{a+b}\right) \quad \text{as} \quad \epsilon\to 0.$$

The same applies to :

$$\|N_\varepsilon(t)S_{\psi,\varepsilon}(t)\| \leq \|N_\varepsilon(t)\|\,\|S_{\psi,\varepsilon}(t)\| = \mathcal{O}\left(\epsilon^{a+b}\right) \quad \text{as} \quad \epsilon\to 0.$$

□

Definition 4.4. *The Colombenu type algebra define by:*

$$G(\mathbb{R}^+,\mathcal{L}(X)) = E_{M,\psi}(\mathbb{R}^+,\mathcal{L}(X))/N_\psi(\mathbb{R}^+,\mathcal{L}(X))$$

Remark 4.5. *Let $C_\psi \in G([0,+\infty[,\mathcal{L}(X))$.*
We denoted by $C_\psi = [(C_{\psi,\varepsilon})]$ with $C_{\psi,\varepsilon}\in E_{M,\psi}([0,+\infty[,\mathcal{L}(X))$.

Definition 4.6. *$C_\psi = [(C_\epsilon)]$ with $C_\epsilon \in E_{M,\psi}(\mathbb{R}^+,\mathcal{L}(E))$ say the generalized $\psi-$cosine family if:*
1) $C_\varepsilon(0) = I$.
2) ${}^{c}_{0}D^{\alpha}_{\psi}C_\psi = AC_\psi$
3) $C_\varepsilon(t)x$ is continuous in t on $\mathbb{R}^+$, $x\in X$.

Definition 4.7. *$C_\psi = [(C_{\psi,\epsilon})]$ is say the generalized cosine family associated with $S_\psi = [(S_{\psi,\varepsilon})]$ generalized sine family if $\forall\epsilon\in]0,1[$,*
we have:

$$S_{\psi,\varepsilon}(t) = \int_0^t C_{\psi,\varepsilon}(\tau)d\tau.$$

Proposition 4.8. *S=[($S_{\psi,\varepsilon} \in G(\mathbb{R}^+,\mathcal{L}(X))$)]*

Proof. Let $C_\psi = [(C_{\psi,\epsilon})]$ generalized cosine family and $t\in[0,T]$ we have:

$$S_{\psi,\varepsilon}(t) = \int_0^t C_{\psi,\varepsilon}(\tau)d\tau$$

Then,

$$\sup_{t\in[0,T]} \|S_{\psi,\varepsilon}(t)\| \leq T \sup_{t\in[0,T]} \|C_{\psi,\varepsilon}(t)\|$$

As

$$C_{\psi,\varepsilon}\in E_{M,\psi}(\mathbb{R}^+,\mathcal{L}(X)) \text{ then } S_{\psi,\varepsilon}\in E_{M,\psi}(\mathbb{R}^+,\mathcal{L}(X))$$

finally:

$$S_\psi = [(S_{\psi,\varepsilon})] \in G(\mathbb{R}^+,\mathcal{L}(X))$$

□

Proposition 4.9. *Let $C_\psi = [(C_{\psi,\varepsilon})]$, be a strongly continuous generalized cosine family with associated generalized sine family $S_\psi = [(S_{\psi,\varepsilon})]$, we have:*
(1) $S_{\psi,\varepsilon}(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in X$
(2) there exist constants M and $w \geq 0$ such that:

$$|C_{\psi,\varepsilon}(t)| \leq Me^{w|t|}$$

(3)

$$|S_{\psi,\varepsilon}(t) - S_{\psi,\varepsilon}(t')| \leq M|\int_{t'}^{t} e^{w|s|}ds| \quad \forall t, t' \in \mathbb{R}^+.$$

(4) $C_{\psi,\varepsilon}(s), S_{\psi,\varepsilon}(s), C_{\psi,\varepsilon}(t)$, and $S_{\psi,\varepsilon}(t)$ commute for all $s, t \in \mathbb{R}^+$.

Definition 4.10. *Let $X_{2,\varepsilon} := \Big\{x \in X : t \longrightarrow {}^c_0D^\alpha_\psi C_{\psi,\varepsilon}(t)x$ is continuous in $t \in \mathbb{R}^+\}$*

We use the lemma 3 the infinitesimal generator of a strongly continuous generalized cosine family $C_\psi = [(C_{\psi,\varepsilon})], t \in \mathbb{R}^+$, is the operator $A = [(A_\varepsilon)]$

$$A_\varepsilon x = {}^c_0D^\alpha_\psi C_\varepsilon(0)x$$

with:

$$D(A_\varepsilon) = X_{2,\varepsilon}$$

In the next paragraph, we will explain in detail the expression of each of $C_{\psi,\epsilon}$ and $S_{\psi,\epsilon}$.

4.2. The integral solution

Definition 4.11. *[14] Define the Mittag-Leffler function by:*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k\alpha + \beta)}.$$

Definition 4.12. *Describe the Laplace transform of a function g by*

$$\mathcal{L}(g(x))(s) = \int_0^{+\infty} e^{-sx}g(x)dx.$$

Proposition 4.13. *Let f and g two functions, we have*

$$\mathcal{L}\left((f * g)(x)\right)(s) = \mathcal{L}\left(f(x)\right)(s)\mathcal{L}\left(g(x)\right)(s).$$

Definition 4.14. *[14]*

1. *The Gamma function is given by*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \forall x > 0$$

2. *The $\mathbb{B}$ function is described by*

$$\forall x, y > 0, \quad \mathbb{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Proposition 4.15. *[14]*

1. *$\forall x, y \in \mathbb{R}^*_+ \times \mathbb{R}^*_+$, $\mathbb{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.*

2. For all $x > 0$, $\Gamma(x+1) = x\Gamma(x)$.

Definition 4.16. *The Wright type function is represented by*

$$\phi_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} = \sum_{n=0}^{\infty} \frac{(-x)^n \Gamma(\alpha(n+1)) \sin(\pi(n+1)\alpha)}{n!}$$

for $\alpha \in (0,1)$ and $x \in \mathbb{C}$.

Proposition 4.17. *The Wright function ϕ_α is a complete function with the following characteristics:*

(i) $\int_0^\infty \phi_\alpha(\theta)\theta^r d\theta = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$ for $r > -1$;
(ii) $\phi_\alpha(\theta) \geq 0$ for $\theta \geq 0$ and $\int_0^\infty \phi_\alpha(\theta) d\theta = 1$
(iii) $\int_0^\infty \phi_\alpha(\theta) e^{-z\theta} d\theta = E_\alpha(-z)$, $z \in \mathbb{C}$;
(iv) $\alpha \int_0^\infty \theta\phi_\alpha(\theta) e^{-z\theta} d\theta = E_{\alpha,\alpha}(-z)$, $z \in \mathbb{C}$.

Definition 4.18. *We proceed with the observed one-sided steady probability density in*

$$\rho_\alpha(\theta) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \theta^{-\alpha k - 1} \frac{\Gamma(\alpha k + 1)}{k!} \sin(k\pi\alpha), \quad \theta \in (0, \infty)$$

And we have,

$$\int_0^\infty e^{-\lambda\theta} \rho_\alpha(\theta) d\theta = e^{-\lambda^\alpha}, \text{ where } \alpha \in (0,1). \tag{4.3}$$

Lemma 4.2. *Let $f : \mathcal{C}(J,X) \to \mathcal{C}(J,X)$ be continuous.*

The issue (1.1) is equal to the mild equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{1}{(\psi(t) - \psi(s))^{1-\alpha}} \psi'(s)(Au(s) + F(s, u(s))ds, \quad t \in J, \tag{4.4}$$

With:
$u : D(A) \longrightarrow D(A)$ offered that the integral in 4.4 exists.

We will need the following lemma.

Lemma 4.3. *For all $\alpha \in]n-1, n]$ $n \in \mathbb{N}$ and $s > 0$, and let $\phi \in \mathcal{C}^n(\mathbb{R}^+)$ be an increasing function with $\phi'(t) \neq 0$ for all $t \in \mathbb{R}^+$. We have,*
1) $s^{\alpha-1}(s^\alpha - A)^{-1} = \mathcal{L}\Big(\int_0^\infty \rho_\alpha(\theta) T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right) d\theta\Big)(s)$,

2) $(s^\alpha - A)^{-1} X(s) = \mathcal{L}\Big(\Big(\int_0^\tau \int_0^\infty \alpha\rho_\alpha(\theta) \frac{(\phi(\tau)-\phi(s))^{\alpha-1}}{\theta^\alpha} T\left(\frac{(\phi(\tau)-\phi(s))^\alpha}{\theta^\alpha}\right)$
$u(s)\phi'(s) d\theta ds\Big)\Big)(s)$.

With,
$X(s) = \int_0^\infty e^{-\lambda(\phi(s)-\phi(0))} u(s)\phi'(s) ds$.

Proof. 1) For $s > 0$,
$s^{\alpha-1}(s^\alpha - A)^{-1} = s^{\alpha-1} \int_0^\infty e^{-s^\alpha} T(\tau) d\tau = \alpha \int_0^\infty (s\hat{t})^{\alpha-1} e^{-(s\hat{t})^\alpha} T(\hat{t}^\alpha) dt$
Where $\{T\}_{t\geq 0}$ is $C_0-semigroup$ defined by
$Ax = \lim_{t\longrightarrow 0^+} \frac{T(t)x - x}{x}$ *and* $(\lambda^\alpha I - A)^{-1} x = \int_0^\infty exp(-\lambda^\alpha t) T(t) x dt$
Putting $\hat{t} = \phi(t) - \phi(0)$, we have

$= \alpha \int_0^\infty s^{\alpha-1}(\phi(t)-\phi(0))^{\alpha-1} e^{-(s(\phi(t)-\phi(0))^\alpha} \times T\left((\phi(t)-\phi(0))^\alpha\right)\psi'(t)dt$

$= \int_0^\infty -\frac{1}{s}\frac{d}{dt}\left(e^{-(s(\phi(t)-\phi(0)))^\alpha}\right) T\left((\phi(t)-\phi(0))^\alpha\right)dt.$

Using (4), we get

$= \int_0^\infty \int_0^\infty \theta\rho_\alpha(\theta) e^{-s(\phi(t)-\phi(0))^\theta} T\left((\phi(t)-\phi(0))^\alpha\right)\psi'(t)d\theta dt$

$= \int_0^\infty e^{-s(\phi(t)-\phi(0))}\left(\int_0^\infty \rho_\alpha(\theta) T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right)d\theta\right)\psi'(t)dt$

$= \mathcal{L}\big(\int_0^\infty \rho_\alpha(\theta) T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right)d\theta\big)(s).$

2) For $s>0$,

$$\begin{aligned}
(s^\alpha - A)^{-1} X(s) &= \int_0^\infty e^{-s^\alpha \tau} T(\tau) X(s) d\tau \\
&= \alpha \int_0^\infty \hat{\tau}^{\alpha-1} e^{-(s\tau))^\alpha} T\left(\hat{\tau}^\alpha\right) X(s) d\tau \\
& Where \quad \{T\}_{t\geq 0} \quad is \quad C_0 - semigroup \quad defined \quad by \\
& Ax = \lim_{t\longrightarrow 0^+} \frac{T(t)x - x}{x} \quad and \\
& (\lambda^\alpha I - A)^{-1}x = \int_0^\infty exp(-\lambda^\alpha t) T(t) x dt \\
& Putting \quad \hat{t} = \phi(t)-\phi(0), \quad we \quad have \\
&= \int_0^\infty \alpha(\phi(\tau)-\phi(0))^{\alpha-1} e^{-(s(\phi(\tau)-\phi(0)))^\alpha} \\
& \times T\left((\phi(\tau)-\phi(0))^\alpha\right)\phi'(\tau) X(s) d\tau \\
&= \int_0^\infty \int_0^\infty \alpha(\phi(\tau)-\phi(0))^{\alpha-1} e^{-(s(\phi(\tau)-\phi(0)))^\alpha} \\
& T\left((\phi(\tau)-\phi(0))^\alpha\right) \times e^{-(\lambda(\phi(r)-\phi(0)))} u(r)\psi'(r)\phi'(\tau) dr d\tau,
\end{aligned}$$

Using (4), we get

$= \int_0^\infty \int_0^\infty \int_0^\infty \alpha(\phi(\tau)-\phi(0))^{\alpha-1}\rho_\alpha(\theta) e^{-s(\phi(\tau)-\phi(0))\theta'} T\left((\phi(\tau)-\phi(0))^\alpha\right)$
$\times e^{-s(\phi(r)-\phi(0))} u(r)\phi'(r)\phi'(\tau) d\theta dr d\tau$

$= \int_0^\infty \int_0^\infty \int_0^\infty \alpha e^{-s(\phi(\tau)+\phi(r)-2\phi(0))} \frac{(\phi(\tau)-\phi(0))^{\alpha-1}}{\theta^\alpha}\rho_\alpha(\theta)$
$\times T\left(\frac{(\phi(\tau)-\phi(0))^\alpha}{\theta^\alpha}\right) u(r)\phi'(r)\phi'(\tau) d\theta dr d\tau$

$= \int_0^\infty \int_t^\infty \int_0^\infty \alpha e^{-s(\phi(\tau)-\phi(0))}\rho_\alpha(\theta) \frac{(\phi(t)-\phi(0))^{\alpha-1}}{\theta^\alpha} T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right)$
$u\left(\phi^{-1}(\phi(\tau)-\phi(t)+\phi(0))\right)$
$\phi'(\tau)\phi'(t) d\theta d\tau dt$

$= \int_0^\infty \int_0^\tau \int_0^\infty \alpha e^{-s(\phi(\tau)-\phi(0))}\rho_\alpha(\theta) \frac{(\phi(t)-\phi(0))^{\alpha-1}}{\theta^\alpha} T\left(\frac{(\phi(t)-\phi(0))^\alpha}{\theta^\alpha}\right)$
$u\left(\phi^{-1}(\phi(\tau)-\phi(t)+\phi(0))\right)\phi'(\tau)$
$\phi'(t) d\theta dt d\tau$

$= \int_0^\infty e^{-s(\phi(\tau)-\phi(0))}\Big(\int_0^\tau \int_0^\infty \alpha\rho_\alpha(\theta) \frac{(\phi(\tau)-\phi(r))^{\alpha-1}}{\theta^\alpha} T\left(\frac{(\phi(\tau)-\phi(r))^\alpha}{\theta^\alpha}\right)$
$u(r)\phi'(r) d\theta dr\Big) \times \phi'(\tau) d\tau.$

$= \mathcal{L}\Big(\Big(\int_0^\tau \int_0^\infty \alpha\rho_\alpha(\theta) \frac{(\phi(\tau)-\phi(r))^{\alpha-1}}{\theta^\alpha} T\left(\frac{(\phi(\tau)-\phi(r))^\alpha}{\theta^\alpha}\right)$
$u(r)\phi'(r)d\theta dr\Big)\Big)(s)$

□

Proposition 4.19. *If*

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)(Au(s)+F(s,u(s))ds,$$

holds, then we have

$u(t) = E(t)u_0 + \alpha\int_0^t E(t)(\psi(t)-\psi(s))^{\alpha-1}\theta^{-\alpha}F(s,u(s))\psi'(s)ds.$

With,
$E(t) = \int_0^\infty \phi_a(\theta)T\left((\psi(t)-\psi(0))^\alpha\theta^{-\alpha}\right)d\theta$

Proof. Since $u(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(\psi(t)-\psi(s))^{1-\alpha}}\psi'(s)(Au(s)+F(s,u(s)))ds$, using the Laplace transform, we obtain

$$\begin{aligned}
\mathcal{L}\big(u(t)\big)(s) &= \mathcal{L}\big(u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(\psi(t)-\psi(\tau))^{1-\alpha}}\psi'(\tau)(Au(s)+F(s,u(s)))d\tau\big)(s)\\
&= \mathcal{L}(u_0)(s) + \mathcal{L}\big(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(\psi(t)-\psi(\tau))^{1-\alpha}}\psi'(\tau)(Au(s)+F(s,u(s)))d\tau\big)(s)\\
&= \frac{u_0}{s} + \mathcal{L}\big(\frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(\psi(t)-\psi(\tau))^{1-\alpha}}\psi'(\tau)(Au(s)+F(s,u(s)))d\tau\big)(s)\\
&= \frac{u_0}{s} + \frac{1}{s^\alpha}A\int_0^\infty e^{-\lambda(\psi(s)-\psi(0))}u(s)\psi'(s)ds + \frac{1}{s^\alpha}A\int_0^\infty e^{-\lambda(\psi(s)-\psi(0))}F(s,u(s))\psi'(s)ds
\end{aligned}$$

We can deduce
$\mathcal{L}\big(u(t)\big)(s) = s^{\alpha-1}\left(s^\alpha - A\right)^{-1}u_0 + \left(s^\alpha - A\right)^{-1}X(s).$
Now, use the lemma 4.3, then
$\mathcal{L}\big(u(t)\big)(s) = \mathcal{L}\big(\int_0^\infty \rho_\alpha(\theta)T\left(\frac{(\psi(t)-\psi(0))^\alpha}{\theta^\alpha}\right)d\theta\big)(s)u_0 +$
$\mathcal{L}\Big(\Big(\int_0^\tau\int_0^\infty \alpha\rho_\alpha(\theta)\frac{(\psi(\tau)-\psi(s))^{\alpha-1}}{\theta^\alpha}T\left(\frac{(\psi(\tau)-\psi(s))^\alpha}{\theta^\alpha}\right)$
$F(s,u(s))\psi'(s)d\theta ds\Big)\Big)(s)$

We can now invert the Laplace transform to obtain the result.

$\forall x \in X$, characterize operators $S_{\psi,\alpha}(t,s)$ and $T_{\psi,\alpha}(t,s)$ by

$$S_{\psi,\alpha}(t,s)x = \int_0^\infty \phi_\alpha(\theta)T\left((\psi(t)-\psi(s))^\alpha\theta\right)ud\theta$$

And

$$T_{\psi,\alpha}(t,s)x = \alpha\int_0^\infty \theta\phi_a(\theta)T\left((\psi(t)-\psi(s))^\alpha\theta\right)ud\theta$$

for $0 \le s \le t \le T$. □

Lemma 4.4. *$S_{\psi,\alpha}$ and $T_{\psi,\alpha}$ provide the following characteristics :*

(i) The operators $S_{\psi,\alpha}(t,s)$ and $T_{\psi,\alpha}(t,s)$ are strongly continuous for all $t \geq s \geq 0$, that is, for every $x \in X$ and $0 \leq s \leq t_1 < t_2 \leq T$ we have

$$| S_{\psi,\alpha}(t_2,s)x - S_{\psi,\alpha}(t_1,s)x\| \to 0 \ \text{and} \ | T_{\psi,\alpha}(t_2,s)x - T_{\psi,\alpha}(t_1,s)x\| \to 0$$

as $t_1 \to t_2$.

(ii) For any fixed $t \geq s \geq 0, S_{\psi,\alpha}(t,s)$ and $T_{\psi,\alpha}(t,s)$ are bounded linear operators with

$$| S_{\psi,\alpha}(t,s)(x)\| \leq M\|x\| \ \text{and} \ \|T_{\psi,\alpha}(t,s)(x)\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\| = \frac{M}{\Gamma(\alpha)}\|x\|$$

for all $x \in X$.

5. Existence and Uniqueness of the Solution in colombeau algebra $\mathcal{G}$

In this section consider the following problem:

$$\begin{cases} {}_0^cD_\psi^\alpha u(x,t) + Au(x,t) = F(t,u(x,t)), & t \in [0,T] \\ u(0,x) = a_0(x) \end{cases} \tag{5.1}$$

with $a_0(x) \in D'(\mathbb{R}^n)$.

Now we will transform the problem in the Colombeau algebra by using section 2, we have:

$$\begin{cases} {}_0^cD_\psi^\alpha u_\varepsilon(t,x) + A_\varepsilon u_\varepsilon(t,x) = F_\varepsilon(t,u_\varepsilon(t,x)) & x \in \mathbb{R}^n, \quad t \geq 0 \\ \quad u_\varepsilon(0,x) = a_{0,\varepsilon}(x) \end{cases}$$

with $a_{0,\varepsilon}(x)$ is the regularizetion of $a_0(x)$, and as stated definition (4.6) $A = [(A_\varepsilon)]$ is the infinitesimal generator of generalized $\psi-$cosine family $C_\psi = [(C_{\psi,\varepsilon})]$.

Definition 5.1. *Let $f \in \mathcal{G}[\mathbb{R}^n]$, f is said L^∞ logarithmic type if it has a representative $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M[\mathbb{R}^n]$ such that*

$$\|f_\varepsilon\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\log(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0$$

Theorem 5.2. *Let ∇F_ε is L^∞ log-type and the generalized operators $S_{\psi,\alpha}$ and $T_{\psi,\alpha}$ verify the Lemma (4.4). Then the problem (5.1) has a unique solution in $\mathcal{G}(\mathbb{R}^+ \times \mathbb{R}^n)$.*

Proof. **Existence results**

The integral solution of the problem (4.3) is:

$$u_\varepsilon(t) = \int_0^\infty \phi_{\varepsilon,a}(\theta)T\left((\psi_\varepsilon(t)-\psi_\varepsilon(0))^\alpha\theta\right)u_{\varepsilon,0}d\theta$$
$$+\alpha\int_0^t\int_0^\infty \phi_{\varepsilon,a}(\theta)(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}T\left((\psi_\varepsilon(t)-\psi_\varepsilon(0))^\alpha\theta\right)f_\varepsilon(s,u_\varepsilon(s))\psi'_\varepsilon(s)d\theta ds.$$
$$= S_{\psi,\alpha\epsilon}(t,s)u_{\varepsilon,0} + \int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}T_{\psi,\alpha\epsilon}(t,s)f_\varepsilon(s,u_\varepsilon(s))\psi'_\varepsilon ds$$

Which implies that:

$$\|u_\varepsilon(t,.)\| \leq \|S_{\psi,\alpha\epsilon}(t,0)u_{\varepsilon,0}\| + \int_0^t \left\|(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}T^\alpha_{\psi,\alpha\epsilon}(t,s)f_\varepsilon(s,u_\varepsilon(s))\psi'_\varepsilon(s)\right\| ds$$
$$\leq M\|u_{\varepsilon,0}\| + \int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\left\|T^\alpha_{\psi,\varepsilon}(t,s)f_\varepsilon(s,u_\varepsilon(s))\right\|\psi'_\varepsilon(s)ds$$
$$\leq M\|u_{\varepsilon,0}\| + \frac{M}{\Gamma(\alpha)}\int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\|f_\varepsilon(s,u_\varepsilon(s))\|\psi'_\varepsilon(s)ds$$

The first approximation of F_ε :

$$F_\varepsilon(s,u_\varepsilon(s,.)) = F_\varepsilon(s,0) + \nabla F_\varepsilon u_\varepsilon(s,.) + N_\varepsilon(s)$$

with $N_\varepsilon \in \mathcal{N}(\mathbb{R}^+)$

Then

$\|u_\varepsilon(t,.)\| \leq M\|u_{\varepsilon,0}\| + \frac{M}{\Gamma(\alpha)}\int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\|F_\varepsilon(s,0)\|\psi'_\varepsilon(s)ds$
$+\frac{M}{\Gamma(\alpha)}\|\nabla F_\varepsilon\|\int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\|u_\varepsilon(s,.)\|\psi'_\varepsilon(s)ds$
$+\frac{M}{\Gamma(\alpha)}\int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\|N_\varepsilon\|\psi'_\varepsilon(s)ds.$

We get
$\|u_\varepsilon(t,.)\| \leq M\|u_{\varepsilon,0}\| + \frac{M}{\Gamma(\alpha+1)}\|F_\varepsilon(s,0)\|(\psi_\varepsilon(T)-\psi_\varepsilon(0))^\alpha$
$+\frac{M}{\Gamma(\alpha)}\|\nabla F_\varepsilon\|\int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\|u_\varepsilon(s,.)\|\psi'_\varepsilon(s)ds + \frac{M}{\Gamma(\alpha)}\|N_\varepsilon\|(\psi_\varepsilon(T)-\psi_\varepsilon(0))^\alpha.$

So,
$\|u_\varepsilon(t,.)\| \leq M\|u_{\varepsilon,0}\| + \frac{M}{\Gamma(\alpha+1)}\|F_\varepsilon(s,0)\|(\psi_\varepsilon(T)-\psi_\varepsilon(0))^\alpha$
$+\frac{M}{\Gamma(\alpha+1)}\|N_\varepsilon\|(\psi_\varepsilon(T)-\psi_\varepsilon(0))^\alpha$
$+\frac{M}{\Gamma(\alpha)}\|\nabla F_\varepsilon\|\int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\|u_\varepsilon(s,.)\|\psi'_\varepsilon(s)ds.$

By the Granwall's inequality

$$\begin{aligned}\|u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq \Bigg(M\|a_{\varepsilon,0}\| &+ \frac{M}{\Gamma(\alpha+1)}\|F_\varepsilon(s,0)\|(\psi_\varepsilon(T)-\psi_\varepsilon(0))^\alpha \\ &+\frac{M}{\Gamma(\alpha+1)}\|N_\varepsilon\|(\psi_\varepsilon(T)-\psi_\varepsilon(0))^\alpha\Bigg) \\ &\times \exp\left(\frac{M}{\Gamma(\alpha+1)}\|\nabla F_\varepsilon\|(\psi_\varepsilon(T)-\psi_\varepsilon(0))^\alpha\right).\end{aligned}$$

Since $\psi_\varepsilon \in G(\mathbb{R}^+)$, $a_0 \in \mathcal{G}(\mathbb{R}^n)$ $(N_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^+)$ and ∇F is $L^\infty-$ logtype there exist $K \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]}\|u_\varepsilon(t,.)\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}\left(\varepsilon^{-K}\right), \quad \varepsilon \to 0$$

Uniqueness.

Let's say there are two solutions $u_{1,\epsilon}(t,.)$, $u_{2,\epsilon}(t,.)$ to problem (5.1), consequently :

$$\begin{cases} {}_0^cD_\psi^\alpha u_{1,\epsilon}(t,x) + A_\epsilon u_{1,\epsilon}(t,x) - {}_0^cD_\psi^\alpha u_{2,\epsilon}(t,x) - A_\epsilon u_{2,\epsilon}(t,x) \\ \quad = F_\epsilon(t,u_{1,\epsilon}(t,x)) - F_\epsilon(t,u_{2,\epsilon}(t,x)) \\ \quad x\in\mathbb{R}^n, \quad t\geq 0 \\ u_{1,\epsilon}(0,x) - u_{2,\epsilon}(0,x) = N_{0,\epsilon}(x) \end{cases} \tag{5.2}$$

Then:

$$\begin{cases} {}_0^cD_\psi^\alpha\left(u_{1,\epsilon}(t,x) - u_{2,\epsilon}(t,x)\right) + A_\epsilon\left(u_{1,\epsilon}(t,x) - u_{2,\epsilon}(t,x)\right) = F_\epsilon(t,u_{1,\epsilon}(t,x)) \\ \qquad - F_\epsilon(t,u_{2,\epsilon}(t,x)) \\ \qquad x\in\mathbb{R}^n, \quad t\geq 0 \\ u_{1,\epsilon}(0,x) - u_{2,\epsilon}(0,x) = N_{0,\epsilon}(x) \end{cases} \tag{5.3}$$

With $(N_{0,\varepsilon})_\varepsilon \in \mathcal{N}(\mathbb{R}^+)$.

The integral solution of the equation (5.3) is:

$$\begin{aligned} u_\varepsilon(t) = &\int_0^\infty \phi_{\varepsilon,a}(\theta)T\left((\psi_\varepsilon(t)-\psi_\varepsilon(0))^\alpha\theta\right)N_{0,\epsilon}(x)d\theta \\ &+\alpha\int_0^t\int_0^\infty \phi_{\varepsilon,a}(\theta)(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}T\left((\psi_\varepsilon(t)-\psi_\varepsilon(0))^\alpha\theta\right) \\ &\times\left(f_\varepsilon(s,u_{1,\varepsilon}(s)) - f_\varepsilon(s,u_{2,\varepsilon}(s))\right)\psi'_\varepsilon(s)d\theta ds. \\ &= S_{\psi,\alpha\epsilon}(t,s)N_{0,\epsilon}(x) + \int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}T_{\psi,\alpha\epsilon}(t,s) \\ &\times\left(f_\varepsilon(s,u_{1,\varepsilon}(s)) - f_\varepsilon(s,u_{2,\varepsilon}(s))\right)\psi'_\varepsilon ds \end{aligned}$$

Then

$$
\begin{aligned}
&\|u_{1,\varepsilon}(t,.)-u_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq \|S_{\psi,\alpha\epsilon}(t,0)N_{0,\varepsilon}(.)\| \\
&+\int_0^t \left\|(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}T_{\psi,\alpha\epsilon}(t,s)\left(f_\varepsilon(s,u_{1,\varepsilon}(s))-f_\varepsilon(s,u_{2,\varepsilon}(s))\right)\psi'_\varepsilon(s)\right\| ds \\
&\leq M\|N_{0,\varepsilon}(.)\|+\int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\|T_{\psi,\alpha\epsilon}(t,s)\left(f_\varepsilon(s,u_{1,\varepsilon}(s))-f_\varepsilon(s,u_{2,\varepsilon}(s))\right)\| \\
&\times\psi'_\varepsilon(s)ds \\
&\leq M\|N_{0,\varepsilon}(.)\|+\frac{M}{\Gamma(\alpha)}\int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\|(f_\varepsilon(s,u_{1,\varepsilon}(s))-f_\varepsilon(s,u_{2,\varepsilon}(s)))\| \\
&\times\psi'_\varepsilon(s)ds
\end{aligned}
$$

.

The initial estimate of $F_\varepsilon\left(s,u_{1,\varepsilon}(s,.)\right)-F_\varepsilon\left(s,u_{2,\varepsilon}(s,.)\right)$ is provided by

$$F_\varepsilon\left(s,u_{1,\epsilon}(s,.)\right)-F_\varepsilon\left(s,u_{2,\epsilon}(s,.)\right)=\|\nabla F_\varepsilon\|\left(u_{1,\epsilon}(s,.)\ -u_{2,\epsilon}(s,.)\right)+N_\varepsilon(s),$$

With $(N_\varepsilon)_\varepsilon\in\mathcal{N}\left(\mathbb{R}^+\right)$.

so

$$
\begin{aligned}
&\|u_{1,\varepsilon}(t,.)-u_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq M\|N_{0,\varepsilon}(.)\| \\
&+\frac{M}{\Gamma(\alpha)}\int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\|(\|\nabla F_\varepsilon\|\left(u_{1,\epsilon}(s,.)\ -u_{2,\epsilon}(s,.)\right)+N_\varepsilon(s))\|\,\psi'_\varepsilon(s)ds \\
&\leq M\|N_{0,\varepsilon}(.)\|+\frac{M}{\Gamma(\alpha+1)}\|N_\varepsilon(s)\|\left(\psi_\varepsilon(T)-\psi_\varepsilon(0)\right)^\alpha+ \\
&\frac{M}{\Gamma(\alpha)}\int_0^t(\psi_\varepsilon(t)-\psi_\varepsilon(s))^{\alpha-1}\|\nabla F_\varepsilon\|\left(u_{1,\epsilon}(s,.)\ -u_{2,\epsilon}(s,.)\right)\psi'_\varepsilon(s)ds
\end{aligned}
$$

Using the Granwall's inequality:

$$
\begin{aligned}
\|u_{1,\varepsilon}(t,.)-u_{2,\varepsilon}(t,.)\|_{L^\infty(\mathbb{R}^n)} \leq &\left(M\|N_{0,\varepsilon}(.)\|+\frac{M}{\Gamma(\alpha+1)}\|N_\varepsilon(s)\|\left(\psi_\varepsilon(T)-\psi_\varepsilon(0)\right)^\alpha\right) \\
&\times\exp\left(\frac{M}{\Gamma(\alpha+1)}\|\nabla F_\varepsilon\|\left(\psi_\varepsilon(T)-\psi_\varepsilon(0)\right)^\alpha\right).
\end{aligned}
$$

Since
$\psi_\varepsilon\in G(\mathbb{R}^+)$, $(N_{0,\varepsilon})_\varepsilon\in\mathcal{N}(\mathbb{R}^+)$, $(N_\varepsilon)_\varepsilon\in\mathcal{N}\left(\mathbb{R}^+\right)$ and ∇F is L^∞ - logtype and for every $q\in\mathbb{N}$ such that:

$$\sup_{t\in[0,T]}\|u_{1,\epsilon}(t,.)-u_{2,\epsilon}(t,.)\|_{L^\infty}=\mathcal{O}\left(\varepsilon^q\right)\quad \varepsilon\to 0$$

□

Acknowledgments The authors thank the referees for useful comments and suggestions to improve the manuscript.

References

1. Benmerrous, A, Chadli,L. S, Moujahid, A, Elomari,M. H, and Melliani,S., "Generalized Cosine Family", Journal of Elliptic and Parabolic Equations, 8(1), pp. 367-381 (2022).
2. Benmerrous, A, Chadli,L. S, Moujahid, A, Elomari,M. H, and Melliani,S., "Generalized Fractional Cosine Family", International Journal of Difference Equations (IJDE), 18(1), pp. 11-34 (2023).
3. Benmerrous, A, Chadli,L. S, Moujahid, A, Elomari,M. H, and Melliani,S., "Generalized solutions for time $\psi-$fractional heat equation", Filomat, 37, 9327–9337 (2023).

4. Benmerrous, A, Chadli,L S, Moujahid, A, Elomari,M. H, and Melliani,S., "Generalized solution of Schrödinger equation with singular potential and initial data", Int. J. Nonlinear Anal. Appl, 13(1), pp. 3093-3101 (2022).

5. Benmerrous, A, Chadli,L. S, Moujahid, A, Elomari,M. H, and Melliani,S., "Solution of Non-homogeneous Wave Equation in Extended Colombeau Algebras", International Journal of Difference Equations (IJDE), 18(1), 107-118 (2023).

6. Benmerrous, A, Chadli,L. S, Moujahid, A, Elomari,M. H, and Melliani,S., "Solution of Schrödinger type Problem in Extended Colombeau Algebras", In 2022 8th International Conference on Optimization and Applications (ICOA), pp. 1-5 (2022, October).

7. Bourgain, J., "Global solutions of nonlinear Schrödinger equations", AMS, Colloquium Publications, vol.46 (1999).

8. Chadli,L S, Benmerrous, A, Moujahid, A, Elomari,M H, and Melliani,S., "Generalized Solution of Transport Equation", In Recent Advances in Fuzzy Sets Theory, Fractional Calculus, Dynamic Systems and Optimization, pp. 101-111 (2022).

9. Colombeau,J F., "Elementary Introduction to New Generalized Function", North Holland, Amsterdam, 1985.

10. Colombeau,J F., "New Generalized Function and Multiplication of Distribution", North Holland, Amsterdam / New York / Oxford, 1984.

11. El Mfadel, A., Melliani, S., Elomari, M., "Existence results for nonlocal Cauchy problem of nonlinear ψ Caputo type fractional differential equations via topological degree methods". Advances in the Theory of Nonlinear Analysis and its Application. 6(2), 270–279 (2022).

12. El Mfadel, A., Melliani, S., Elomari, M., "New existence results for nonlinear functional hybrid differential equations involving the ψ Caputo fractional derivative". Results in Nonlinear Analysis. 5(1), 78-86 (2022).

13. Fattorini, H O., "Ordinary differential equations in linear topological spaces", II, J. Differential Equations, 6 , 50–70 (1969).

14. Gorenflo, R, Kilbas, A A, Mainardi, F, Rogosin, S V., "Mittag-Leffler Functions, Related Topics and Applications" (Springer, New York, 2014).

15. Gorenflo, R, Mainardi, F., Fractional calculus; Integral an differential equations of fractional order, in: Fractals and Fractional Calculus in Continuum Mechanics, in: A. Carpinteri, F. Mainardi (Eds.), CISM Lecture notes, Springer Verlag, Wien and New York, 1997, pp. 223–276.

16. Jenkins, P F., "Making sense of the chest x-ray: a hands-on guide". New York (NY): Oxford University Press; 2005.

17. Kilbas, A A, Srivastava, H M, Trujillo,J J., "Theory and Applications of Fractional Differential Equations", in: Jan van Mill (Ed.), North-Holland Mathematics Studies, vol. 204, Amsterdam, Netherlands (2006).

18. Mainardi, F, Paradisi, P, Gorenflo, R., "Probability distributions generated by fractional diffusion equations, in: J. Kertesz, I. Kondor (Eds.), Econophysics: An Emerging Science", Kluwer, Dordrecht, 2000.

19. Oberguggenberger, M, "Multiplication of Distributions and Applications to Partial Differential Equations", Pitman Research Notes in Mathematics, (1992).

20. Podlubny, I, "Fractional Differential Equations", Academic Press, San Diego, (1999).

21. Segal, I, "Non-linear semi-groups", Ann. Math., 78 (1963), 339–364.

22. Stojanovic, M., "Extension of Colombeau algebra to derivatives of arbitrary order D^α, $\alpha \geq 0$. Application to ODEs and PDEs with entire and fractional derivatives". 71 (2009) 5458–5475.

23. Stojanovic, M., "Foundation of the fractional calculus in generalized function algebra". Analysis and Applications, Vol. 10, No. 4 (2012) 439–467.

24. Travis, C C, Webb, G F., "Cosine families and abstract nonlinear second order differential equations", Acta Mathematica Hungarica, 1588–2632, 1978.

25. Trujillo, J J., "On best fractional derivative to be applied in fractional modeling. The fractional Fourier transform", in: Proc. FDA08, Ankara, 5–7 November, 2008.

Abdelmjid Benmerrous,
Laboratory of Applied Mathematics and Scientific Computing,
Sultan Moulay Slimane University,
PO Box 532, Beni Mellal, 23000,
Morocco.
E-mail address: abdelmajid.benmerrous@usms.ma

and

Lalla saadia Chadli,
Laboratory of Applied Mathematics and Scientific Computing,
Sultan Moulay Slimane University,
PO Box 532, Beni Mellal, 23000,
Morocco.
E-mail address: chadli@fstbm.ac.ma

and

Abdelaziz Moujahid,
Laboratory of Applied Mathematics and Scientific Computing,
Sultan Moulay Slimane University,
PO Box 532, Beni Mellal, 23000,
Morocco.
E-mail address: moujahid.aziz.usms@gmail.com

and

M'hamed Elomari,
Laboratory of Applied Mathematics and Scientific Computing,
Sultan Moulay Slimane University,
PO Box 532, Beni Mellal, 23000,
Morocco.
E-mail address: m.elomari@usms.ma

and

Said Melliani,
Laboratory of Applied Mathematics and Scientific Computing,
Sultan Moulay Slimane University,
PO Box 532, Beni Mellal, 23000,
Morocco.
E-mail address: melliani@fstbm.ac.ma