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Robin Problem Involving the $p(x)$-Laplacian Operator Without Ambrosetti-Rabinowizt Condition

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ABSTRACT: The paper deals with the following Robin problem

$$\begin{cases} -\mathcal{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|\nabla u|^{p(x)}d\sigma\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda h(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + a(x)|u|^{p(x)-2}u = 0 & \text{on } \partial\Omega. \end{cases}$$

The goal is to determine the precise positive interval of λ for which the above problem admits at least two nontrivial solutions without assuming the Ambrosetti-Rabinowitz condition. Next, we give a result on the existence of an unbounded sequence of nontrivial weak solutions by employing the Fountain Theoreom with Cerami condition.

Key Words: $p(x)$-Kirchhoff type problems, Robin boundary conditions, variational methods, Cerami condition.

Contents

1. Introduction and main results

In recent years, there has been a lot of interest in differential equations and variational problems with nonstandard $p(x)$-growth conditions. It illuminates a wide range of applications in a variety of fields, including elastic mechanics, electro-rheological fluid dynamics, and image processing [16,17].

The purpose of this paper is to study the existence of nontrivial weak solutions for Kirchhoff type equations involving the $p(x)$-Laplacian with Robin boundary conditions:

$$\begin{cases} -\mathcal{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|\nabla u|^{p(x)}d\sigma\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda h(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} + a(x)|u|^{p(x)-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary, $\frac{\partial u}{\partial \nu}$ is the outer normal derivative, λ is a nonnegative parameter, $p \in C_+(\overline{\Omega})$, $1 < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) < N$, $a \in L^\infty(\partial\Omega)$ such that $a^- := \inf_{x\in\partial\Omega} a(x) > 0$, $\mathcal{M} : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and the nonlinear term $h : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition.

Since the original work of A. Ambrosetti and P.H. Rabinowitz [4], critical point theory has become one of the most important tools for determining solutions to elliptic equations of variational type. In particular, our elliptic problem (1.1) generalizes many work, since the function $\mathcal{M}$ can be $\not\equiv 1$. The main

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ingredient to obtaining the existence of solutions for superlinear problems is the condition proposed by A. Ambrosetti and P.H. Rabinowitz ((**AR**)$-$ condition for short).

Many authors have recently studied problem (1.1) in the case when $\mathcal{M} \equiv 1$, and a plethora of results have been obtained, see for instance S. G. Deng [7], M. Allaoui et al. [3], A. Ayoujil and A. Ourraoui [5] and the references therein.
However so far, there are only few results for the case $\mathcal{M} \not\equiv 1$. For example, by means of critical point theorems, M. Allaoui [2] obtained results on existence and multiplicity of solutions for problem (1.1) in the case $\lambda = 1$, where the nonlinear term h satisfying the **(AR)**-condition.
In addition, under the **(AR)**-condition and some weaker assumptions, Afrouzi et al. in [1] proved that problem (1.1) admits two distinct weak solutions for $\lambda h(x,u) = h(x,u) + \lambda g(x)$. Their approach was based on the mountain pass theorem, the Ekeland's variational principle, and Krasnoselskii's genus theory.

To state our results, we make the subsequent hypotheses on $\mathcal{M}$ and h:
$(\mathcal{M}_0)$ There exists $m \in \mathbb{R}_+^*$ such that $\inf\limits_{t\in\mathbb{R}_+} \mathcal{M}(t) \geq m > 0$.
$(\mathcal{M}_1)$ There exists $\theta \in \left[1, \frac{N}{N-p^+}\right[$ such that for all $t \in \mathbb{R}_+$,

$$t\mathcal{M}(t) \leq \theta \widetilde{\mathcal{M}}(t),$$

where $\widetilde{\mathcal{M}}(t) = \int_0^t \mathcal{M}(\tau)d\tau$.
(H_0) There exist $C > 0$ and $s \in C_+(\overline{\Omega})$ such that

$$|h(x,t)| \leq C(1 + |t|^{s(x)-1}) \quad \text{for all } (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

(H_1) $1 < p^- \leq p^+ < s^- \leq s^+ < p^*(x) \quad$ for all $\ x \in \overline{\Omega}$,
where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

(H_2) $\liminf\limits_{|t|\to\infty} \frac{H(x,t)}{|t|^{\theta p^+}} = +\infty$ uniformly a.e $x \in \Omega$,
where $H(x,t) = \int_0^t h(x,s)ds$ and θ comes from (M_1) above.
(H_3) There exist $c_1, r_1 \geq 0$ and $l \in L^\infty(\Omega)$ with $l(x) > \frac{N}{p^-}$ and $l(x) < \frac{p^-}{p^+-p^-}$ such as

$$|H(x,t)|^{l(x)} \leq c_1 |t|^{l(x)p^-} \mathcal{F}(x,t),$$

for all $(x,t) \in \Omega \times \mathbb{R}$, $|t| \geq r_1$ and $\mathcal{F}(x,t) := \frac{1}{\theta p^+} h(x,t)t - H(x,t) \geq 0$.

Remark 1.1. 1. *A typical example for $\mathcal{M}$ is given by $\mathcal{M}(t) = a_0 + b_1 t^p$ with $p > 0, a_0 > 0$ and $a_1 \geq 0$.*

2. *The conditions $(\mathcal{M}_0)$ and $(\mathcal{M}_1)$ implies the following inequality:*

$$\widetilde{\mathcal{M}}(t) \leq \widetilde{\mathcal{M}}(1)(1 + t^\theta), \qquad \textit{for all } \ t \in \mathbb{R}_+.$$

Indeed, if $0 \leq t \leq 1$, since the function $\widetilde{\mathcal{M}}(.)$ is strictly increasing on $[0, +\infty[$, then $\widetilde{\mathcal{M}}(t) \leq \widetilde{\mathcal{M}}(1)$, and if $t \geq 1$, we consider the function $G(t) = \frac{\widetilde{\mathcal{M}}(t)}{t^\theta}$. By direct calculation, it is clear that $G(.)$ is strictly decreasing on $[1, +\infty[$, then $\widetilde{\mathcal{M}}(t) \leq \widetilde{\mathcal{M}}(1)t^\theta$.

3. *Hypothesis (H_3), which is important to ensure the boundedness of Palais-Smale type sequences of the corresponding functional, can be found in [18].*

As it is known, the main role of utilizing the famous Ambrosetti-Rabinowitz type condition in applying critical point theory is to ensure the boundedness of the Palais–Smale type sequences of the corresponding functional. However, this condition sometimes can be very restrictive, and thus undoubtedly eliminates many nonlinearities. Indeed, there are several functions which are superlinear at infinity and at the origin but do not satisfy **(AR)**-condition. For example, when $p(x) \equiv p$, the function

$$h(x,t) = |t|^{p-2}t\ln(|t|+1),$$

does not satisfy the **(AR)**-condition, but it satisfies our conditions $(H_1)-(H_3)$.

Now, we present the main results of this paper.

Theorem 1.2. *Suppose that* $(\mathcal{M}_0), (\mathcal{M}_1), (H_0), (H_1), (H_2)$ *and* (H_3) *hold. Then there exists a positive constant* λ^* *such that the problem* (1.1) *admits at least two distinct weak solutions in* $W^{1,p(x)}(\Omega)$ *for each* $\lambda \in]0, \lambda^*[$.

Theorem 1.3. *Suppose that* $(\mathcal{M}_0), (\mathcal{M}_1), (H_0), (H_1), (H_2)$ *and* (H_3) *hold. If*
(H_4) $h(x,-t) = -h(x,t)$ *for all* $(x,t) \in \Omega \times \mathbb{R}$, *then for any* $\lambda > 0$, *the problem* (1.1) *has infinitely many solutions* (u_n) *such that* $\psi_\lambda(u_n) \to +\infty$ *as* $n \to +\infty$, *where* ψ_λ *will be defined in* (2.2).

2. Preliminaries

To study problem (1.1), we need the following preliminary results. For the details we refer to [7,9,10,13] and the references therein.

For $p \in C_+(\overline{\Omega}) := \{p \in C(\overline{\Omega}) : p^- := \inf_{x \in \overline{\Omega}} p(x) > 1\}$, we designate the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty\Big\}$$

equipped with the Luxemburg norm $|u|_{p(x)} = \inf\Big\{\tau > 0 : \int_\Omega |\frac{u(x)}{\tau}|^{p(x)}dx \le 1\Big\}$.

Proposition 2.1. *[11] Let* $\rho(u) = \int_\Omega |u|^{p(x)}dx$. *For* $u \in L^{p(x)}(\Omega)$, $(u_n) \subset L^{p(x)}(\Omega)$ *and* $\alpha > 0$, *we have*

1. *For* $u \neq 0$, $|u|_{p(x)} = \alpha \Longleftrightarrow \rho\left(\frac{u}{\alpha}\right) = 1$;
2. $|u|_{p(x)} < 1\,(=1, >1) \Longleftrightarrow \rho(u) < 1\,(=1, >1)$;
3. $|u|_{p(x)} > 1 \Longrightarrow |u|^{p^-}_{p(x)} \le \rho(u) \le |u|^{p^+}_{p(x)}$;
4. $|u|_{p(x)} < 1 \Longrightarrow |u|^{p^+}_{p(x)} \le \rho(u) \le |u|^{p^-}_{p(x)}$;
5. $\lim_{n\to+\infty} |u_n - u|_{p(x)} = 0 \Leftrightarrow \lim_{n\to+\infty} \rho(u_n - u) = 0$

The variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\Big\},$$

endowed with the norm $\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}$.
With these norms, the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces [11].

Now, let us introduce a norm which will be used later. Let $a \in L^\infty(\partial\Omega)$ with $a^- = \inf_{x\in\partial\Omega} a(x) > 0$ and for any $u \in W^{1,p(x)}$, define

$$\|u\|_a = \inf\Big\{\tau > 0 : \int_\Omega |\frac{\nabla u}{\tau}|^{p(x)}dx + \int_{\partial\Omega} a(x)|\frac{u}{\tau}|^{p(x)}d\sigma \le 1\Big\}.$$

It follows from of [7, Theorem 2.1] that $\|.\|_a$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to the standard norm $\|u\|_{W^{1,p(x)}(\Omega)}$.

On $W^{1,p(x)}(\Omega)$, we define the modular $\rho_a : W^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho_a(u) = \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} a(x)|u|^{p(x)} d\sigma.$$

The norm $\|.\|_a$ and the modular ρ_a have the following connection.

Proposition 2.2. *[7] For $u \in W^{1,p(x)}(\Omega)$, $(u_n) \subset W^{1,p(x)}(\Omega)$ and $\alpha > 0$, we have*

1. *For $u \neq 0$, $\|u\|_a = \alpha \Longleftrightarrow \rho_a(\frac{u}{\alpha}) = 1$;*
2. *$\|u\|_a < 1\, (= 1, > 1) \Longleftrightarrow \rho_a(u) < 1\, (= 1, > 1)$;*
3. *$\|u\|_a > 1 \Longrightarrow \|u\|_a^{p^-} \leq \rho_a(u) \leq \|u\|_a^{p^+}$;*
4. *$\|u\|_a < 1 \Longrightarrow \|u\|_a^{p^+} \leq \rho_a(u) \leq \|u\|_a^{p^-}$;*
5. *$\lim_{n\to+\infty} \|u_n\|_a = 0 \Leftrightarrow \lim_{n\to+\infty} \rho_a(u_n) = 0$ and $\lim_{n\to+\infty} \|u_n\|_a = +\infty \Leftrightarrow \lim_{n\to+\infty} \rho_a(u_n) = +\infty$.*

Proposition 2.3. *[11] If $r \in C_+(\overline{\Omega})$ and $r(x) \leq p^*(x)$ for $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{r(x)}(\Omega)$ is continuous. In particular, if $r(x) < p^*(x)$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ is compact.*

Let $J_a : W^{1,p(x)}(\Omega) \to \left(W^{1,p(x)}(\Omega)\right)^*$ be defined by

$$\langle J_a(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \; dx + \int_{\partial\Omega} a(x)|u|^{p(x)-2}uv \; d\sigma, \quad \text{for all } u, v \in W^{1,p(x)}(\Omega). \tag{2.1}$$

Here $\left(W^{1,p(x)}(\Omega)\right)^*$ denotes the dual space of $W_0^{1,\mathcal{H}}(\Omega)$. Then, we have that

Proposition 2.4. *[12, Proposition 2.2]*

1. *J_a is a continuous, bounded and strictly monotone operator.*
2. *J_a is a mapping of type (S_+).*
3. *J_a is a homeomorphism.*

From now on, we denote by $X = W^{1,p(x)}$ and $X^* = (W^{1,p(x)})^*$ the dual space. We notice that problem (1.1) has a variational structure, in fact, the weak solutions of (1.1) are exactly the critical points of the Euler-Lagrange functional $\psi_\lambda : X \to \mathbb{R}$, given by

$$\psi_\lambda(u) = \phi(u) - \lambda\varphi(u), \tag{2.2}$$

where

$$\phi(u) = \widetilde{\mathcal{M}}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} \; dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|u|^{p(x)} \; d\sigma\right) \quad \text{and} \quad \varphi(u) = \int_\Omega H(x,u)dx.$$

Then, it follows from assumption (H_0) that $\psi_\lambda \in C^1(X, \mathbb{R})$, and its Fréchet derivative is

$$\begin{aligned}\langle \psi_\lambda'(u), v\rangle = \mathcal{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)} \; dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|u|^{p(x)} \; d\sigma\right)\left(\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v \; dx\right.\\ \left. + \int_{\partial\Omega} a(x)|u|^{p(x)-2}uv \; d\sigma\right) - \lambda \int_\Omega h(x,u)v \; dx,\end{aligned}$$

for all $u, v \in X$.

Let $u \in X$. We say that u is a **weak solution** of the problem (1.1) if

$$\mathcal{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|u|^{p(x)}d\sigma\right)\left(\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx + \int_{\partial\Omega} a(x)|u|^{p(x)-2}uv d\sigma\right) - \lambda \int_\Omega h(x,u)v dx = 0,$$

for all $v \in X$.

Next we give the definition of the Cerami condition which was introduced by G. Cerami in [6].

Definition 2.5. *Let $(X, \|.\|)$ be a real Banach space and $J \in C^1(X, \mathbb{R})$. Given $c \in \mathbb{R}$, we say that J satisfies the Cerami condition (we denote $(C_c)-$ condition) in X, if any sequence $(u_n) \subset X$ such that $(J(u_n))$ is bounded and $\|J'(u_n)\|(1 + \|u_n\|) \to 0$ as $n \to +\infty$ has a strong convergent subsequence in X. If this condition is satisfied at every level $c \in \mathbb{R}$, then, we say that J satisfies (C)-condition.*

Remark 2.6. *It is clear from the above definition that if J satisfies the (PS)-condition, then it satisfies the (C)-condition. However, there are functionals that satisfy the (C)-condition but do not satisfy the condition (PS)-condition (see [6]). Consequently, the (C)-condition is weaker than the (PS)-condition.*

To prove Theorem 1.2, we will use the following Theorem.

Theorem 2.7. *[14, Theorem 2.6] Let X be a real Banach space, $A, B : X \to \mathbb{R}$ be two continuously Gateaus differentiable functionals such that A is bounded from below and $A(0) = B(0) = 0$. Let $\eta > 0$ be fixed, and it is assumed that for each*

$$\lambda \in \Gamma_0 := \left]0, \frac{\eta}{\sup\limits_{u \in A^{-1}(]-\infty,\eta[)} B(u)}\right[,$$

the functional $J_\lambda = A - \lambda B$ satisfies the (C)-condition for all $\lambda > 0$ and is unbounded from below. Then, for each $\lambda \in \Gamma_0$, the functional J_λ admits two distinct critical points.

In order to prove Theorem 1.3, we use the following Fountain Theorem.
Let X be a real, reflexive, and Banach space, it is known [19] that for a separable and reflexive Banach space there exist $\{e_j\}_{j\in\mathbb{N}} \subset X$ and $\{e_j^*\}_{j\in\mathbb{N}} \subset X^*$ such that

$$X = \overline{span\{e_j : j = 1, 2, ...\}}, \quad X^* = \overline{span\{e_j^* : j = 1, 2, ...\}},$$

and $\langle e_i^*, e_j\rangle = 1$ if $i = j$, $\langle e_i^*, e_j\rangle = 0$ if $i \neq j$.
We denote $X_j = span\{e_j\}$, $Y_k = \bigoplus_{j=1}^k X_j$ and $Z_k = \overline{\bigoplus_{j=k}^{+\infty} X_j}$.

Theorem 2.8. *[20] Assume that X is a real reflexive Banach space, and let $J : X \to \mathbb{R}$ be an even functional of class $C^1(X, \mathbb{R})$ and satisfies $(C)-$ condition . For every $k \in \mathbb{N}$, there exists $\gamma_k > \eta_k > 0$ such that*
(A_1) $b_k := \inf\{J(u) : u \in Z_k, \|u\| = \eta_k\} \to +\infty$ as $k \to +\infty$,
(A_2) $c_k := \max\{J(u) : u \in Y_k, \|u\| = \gamma_k\} \leq 0$.
Then, J has a sequence of critical values tending to $+\infty$.

3. Proofs of main results

First of all, we begin by showing that (C)-condition holds.

Lemma 3.1. *If assumptions $(\mathcal{M}_0), (\mathcal{M}_1), (H_0), (H_1), (H_2)$ and (H_3) hold, then the functional ψ_λ satisfies the (C)-condition for all $\lambda > 0$.*

Proof. Let $(u_n) \subset X$ be a Cerami sequence for ψ_λ, namely,

$$(\psi_\lambda(u_n)) \text{ is bounded and } \quad \|\psi'_\lambda(u_n)\|_{X^*}(1 + \|u_n\|_a) \to 0, \tag{3.1}$$

which imply that

$$\sup|\psi_\lambda(u_n)| \le M \text{ and } \quad \langle \psi'_\lambda(u_n), u_n\rangle = \circ(1), \tag{3.2}$$

where $\lim\limits_{n\to+\infty} \circ(1) = 0$ and $M > 0$.

We need to prove the boundedness of the sequence (u_n) in X. To this end, assume the contrary that the sequence (u_n) is unbounded in X. Without loss of generality, we can assume that $\|u_n\|_a > 1$.

By virtue of $(\mathcal{M}_1)$ and (H_3), for n large enough,

$$\begin{aligned} M + 1 &\ge \psi_\lambda(u_n) - \frac{1}{\theta p^+}\langle \psi'_\lambda(u_n), u_n\rangle \\ &= \widetilde{\mathcal{M}}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|u_n|^{p(x)}d\sigma\right) - \lambda\int_\Omega H(x, u_n)dx \\ &+ \frac{\lambda}{\theta p^+}\int_\Omega h(x, u_n)u_n dx - \frac{1}{\theta p^+}\mathcal{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|u_n|^{p(x)}d\sigma\right) \times \\ &\left(\int_\Omega |\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}d\sigma\right) \\ &\ge \lambda\int_\Omega \mathcal{F}(x, u_n)\, dx. \end{aligned} \tag{3.3}$$

Using $(\mathcal{M}_0)$ and $(\mathcal{M}_1)$, it follows

$$\begin{aligned} M &\ge \psi_\lambda(u_n) \\ &= \widetilde{\mathcal{M}}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|u_n|^{p(x)}\, d\sigma\right) - \lambda\int_\Omega H(x, u_n)\, dx \\ &\ge \frac{1}{\theta}\mathcal{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|u_n|^{p(x)}d\sigma\right)\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right. \\ &\left.+ \int_{\partial\Omega} \frac{a(x)}{p(x)}|u_n|^{p(x)}d\sigma\right) - \lambda\int_\Omega H(x, u_n)\, dx \\ &\ge \frac{m}{\theta p^+}\left(\int_\Omega |\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\, d\sigma\right) - \lambda\int_\Omega H(x, u_n)\, dx \\ &\ge \frac{m}{\theta p^+}\|u_n\|_a^{p^-} - \lambda\int_\Omega H(x, u_n)\, dx. \end{aligned} \tag{3.4}$$

Since $\|u_n\|_a \to +\infty$ as $n \to +\infty$, we deduce that

$$\int_\Omega H(x, u_n)dx \ge \frac{m}{\lambda\theta p^+}\|u_n\|_a^{p^-} - \frac{M}{\lambda} \to +\infty \quad \text{ as } n \to +\infty. \tag{3.5}$$

Furthermore, since $\widetilde{\mathcal{M}}(.)$ is strictly increasing on $[0, +\infty[$, we have

$$\begin{aligned} \psi_\lambda(u_n) &= \widetilde{\mathcal{M}}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|u_n|^{p(x)}\, d\sigma\right) - \lambda\int_\Omega H(x, u_n)\, dx \\ &\le \widetilde{\mathcal{M}}\left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\, d\sigma)\right) - \lambda\int_\Omega H(x, u_n)\, dx. \end{aligned}$$

Then, we obtain

$$\psi_\lambda(u_n) + \lambda\int_\Omega H(x, u_n)\, dx \le \widetilde{\mathcal{M}}\left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\, d\sigma)\right). \tag{3.6}$$

In view of condition (H_2), there exists a constant T_1 such that

$$H(x,t) > |t|^{\theta p^+} \quad \text{for all } \ x \in \Omega \ \text{ and } \ |t| > T_1.$$

Since $H(x,.)$ is continuous on $[-T_1, T_1]$, there exists a constant $C_0 > 0$ such that

$$|H(x,t)| \leq C_0 \quad \text{for all } \ (x,t) \in \Omega \times [-T_1, T_1].$$

Then, a real number K can be chosen such that $H(x,t) \geq K$ for all $(x,t) \in \Omega \times \mathbb{R}$.
Hence,

$$\frac{H(x,u_n) - K}{\widetilde{\mathcal{M}}\left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\, d\sigma)\right)} \geq 0, \tag{3.7}$$

for all $(x,n) \in \Omega \times \mathbb{N}$.
Put $\beta_n = \frac{u_n}{\|u_n\|_a}$, so $\|\beta_n\|_a = 1$. Up to subsequences, for some $\beta \in X$, we have

$$\begin{aligned} \beta_n &\rightharpoonup \beta \qquad \text{in } X, \\ \beta_n &\to \beta \quad \text{ in } L^{s(x)}(\Omega), \\ \beta_n(x) &\to \beta(x) \quad \text{ a.e. in } \Omega, \end{aligned} \tag{3.8}$$

where s is given in hypothesis (H_0).
Define the set $\Omega_0 = \{x \in \Omega : \beta(x) \neq 0\}$. Since $\|u_n\|_a \to +\infty$ as $n \to +\infty$, we have

$$|u_n(x)| = |\beta_n(x)| \|u_n\|_a \to +\infty,$$

for any $x \in \Omega_0$.
By (H_2) and Remark (1.1), for all $x \in \Omega_0$, we have

$$\begin{aligned} &\lim_{n\to+\infty} \frac{H(x,u_n)}{\widetilde{\mathcal{M}}\left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\, d\sigma)\right)} \\ &\geq \lim_{n\to+\infty} \frac{H(x,u_n)}{\widetilde{\mathcal{M}}(1)\left(1 + \left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\, dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\, d\sigma)\right)^\theta\right)} \\ &\geq \lim_{n\to+\infty} \frac{H(x,u_n)}{\widetilde{\mathcal{M}}(1)\left(\|u_n\|_a^{\theta p^+} + \frac{1}{(p^-)^\theta}\|u_n\|_a^{\theta p^+}\right)} \\ &\geq \lim_{n\to+\infty} \frac{H(x,u_n)}{\widetilde{\mathcal{M}}(1)\left(1 + \frac{1}{(p^-)^\theta}\right)\|u_n\|_a^{\theta p^+}} \\ &\geq \lim_{n\to+\infty} \frac{H(x,u_n)}{\widetilde{\mathcal{M}}(1)\left(1 + \frac{1}{(p^-)^\theta}\right)|u_n(x)|^{\theta p^+}} |\beta_n(x)|^{\theta p^+} \\ &= +\infty, \end{aligned} \tag{3.9}$$

Thus, $|\Omega_0| = 0$. In fact, suppose by contradiction that $|\Omega_0| \neq 0$. Uisng (3.5), (3.6), (3.9) and Fatou's lemma, we get

$$
\begin{aligned}
\frac{1}{\lambda} &= \liminf_{n\to+\infty} \frac{\int_\Omega H(x,u_n)\ dx}{\psi_\lambda(u_n) + \lambda\int_\Omega H(x,u_n)\ dx} \\
&\geq \liminf_{n\to+\infty} \int_\Omega \frac{H(x,u_n)}{\widetilde{\mathcal{M}}\left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\ dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\ d\sigma)\right)} dx \\
&\geq \liminf_{n\to+\infty} \int_{\Omega_0} \frac{H(x,u_n)}{\widetilde{\mathcal{M}}\left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\ dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\ d\sigma)\right)} dx \\
&- \limsup_{n\to+\infty} \int_{\Omega_0} \frac{K}{\widetilde{\mathcal{M}}\left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\ dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\ d\sigma)\right)} dx \\
&= \liminf_{n\to+\infty} \int_{\Omega_0} \frac{H(x,u_n)-K}{\widetilde{\mathcal{M}}\left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\ dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\ d\sigma)\right)} dx \\
&\geq \int_{\Omega_0} \liminf_{n\to+\infty} \frac{H(x,u_n)-K}{\widetilde{\mathcal{M}}\left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\ dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\ d\sigma)\right)} dx \\
&\geq \int_{\Omega_0} \liminf_{n\to+\infty} \frac{H(x,u_n(x))}{\widetilde{\mathcal{M}}\left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\ dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\ d\sigma)\right)} dx \\
&- \int_{\Omega_0} \limsup_{n\to+\infty} \frac{K}{\widetilde{\mathcal{M}}\left(\frac{1}{p^-}(\int_\Omega |\nabla u_n|^{p(x)}\ dx + \int_{\partial\Omega} a(x)|u_n|^{p(x)}\ d\sigma)\right)} dx \\
&= +\infty,
\end{aligned}
$$

which is a contradiction. Therefore, $\beta(x) = 0$ for a.e. $x \in \Omega$.
From (3.4) and (3.8), respectively, we can deduce that

$$\beta_n \to 0 \quad \text{in } L^{s(x)}(\Omega) \quad \text{and } \beta_n(x) \to 0 \quad \text{a.e. in } \Omega, \tag{3.10}$$

and

$$0 < \frac{m}{\theta\lambda p^+} \leq \limsup_{n\to+\infty} \int_\Omega \frac{|H(x,u_n)|}{\|u_n\|_a^{p^-}}\ dx. \tag{3.11}$$

Using (H_0) and (H_1), we get

$$
\begin{aligned}
\int_{\{0\leq|u_n(x)|\leq r_1\}} \frac{|H(x,u_n)|}{\|u_n\|_a^{p^-}}\ dx &\leq C\int_{\{0\leq|u_n(x)|\leq r_1\}} \frac{|u_n| + \frac{1}{s(x)}|u_n|^{s(x)}}{\|u_n\|_a^{p^-}}\ dx \\
&\leq C\frac{|u_n|_1}{\|u_n\|_a^{p^-}} + \frac{C}{s^-}\int_{\{0\leq|u_n(x)|\leq r_1\}} |u_n|^{s(x)-p^-}|\beta_n|^{p^-}\ dx \\
&\leq CC_0\frac{\|u_n\|_a}{\|u_n\|_a^{p^-}} + \frac{Cr_1^{s-p^-}}{s^-}|\beta_n|_{p^-}^{p^-} \\
&\leq \frac{CC_0}{\|u_n\|_a^{p^--1}} + \frac{CC_1r_1^{s-p^-}}{s^-}\|\beta_n\|_a^{p^-} \\
&\to 0, \quad \text{as} \quad n\to+\infty,
\end{aligned}
\tag{3.12}
$$

where $C_0, C_1 > 0$ and s is either s^+ or s^-.
Put $l'(x) = \frac{l(x)}{l(x)-1}$. Since $l \in L^\infty(\Omega)$ with $l(x) > \frac{N}{p^-}$ and $l(x) < \frac{p^-}{p^+-p^-}$, it follows that

$$p(x) < l'(x)p^- < p^*(x).$$

On the other hand, by virtue of hypothesis (H_3), (3.3) and (3.10), we deduce

$$\begin{aligned}
\int_{\{|u_n(x)|\geq r_1\}} \frac{|H(x,u_n)|}{\|u_n\|_a^{p^-}}\,dx &\leq 2\left[\int_{\{|u_n(x)|\geq r_1\}} \left(\frac{|H(x,u_n)|}{|u_n|^{p^-}}\right)^{l(x)} dx\right]^{\frac{1}{l(x)}} \left[\int_{\{|u_n(x)|\geq r_1\}} |\beta_n|^{l'(x)p^-}\,dx\right]^{\frac{1}{l'(x)}}\\
&\leq 2c_1^{\frac{1}{l(x)}}\left[\int_{\{|u_n(x)|\geq r_1\}} \mathcal{F}(x,u_n)dx\right]^{\frac{1}{l(x)}} \left[\int_{\{|u_n(x)|\geq r_1\}} |\beta_n|^{l'(x)p^-}\,dx\right]^{\frac{1}{l'(x)}}\\
&\leq 2c_1^{\frac{1}{l(x)}}\left[\int_\Omega \mathcal{F}(x,u_n)dx\right]^{\frac{1}{l(x)}} \left[\int_\Omega |\beta_n|^{l'(x)p^-}\,dx\right]^{\frac{1}{l'(x)}}\\
&\leq 2c_1^{\frac{1}{l(x)}}\left(\frac{M+1}{\lambda}\right)^{\frac{1}{l(x)}} \left[\int_\Omega |\beta_n|^{l'(x)p^-}\,dx\right]^{\frac{1}{l'(x)}}\\
&\to 0 \quad \text{as} \quad n\to+\infty.
\end{aligned}$$

Finally, combining this with(3.12), it follows that

$$\int_\Omega \frac{|H(x,u_n)|}{\|u_n\|_a^{p^-}}\,dx = \int_{\{0\leq|u_n(x)|\leq r_1\}} \frac{|H(x,u_n)|}{\|u_n\|_a^{p^-}}\,dx + \int_{\{|u_n(x)|\geq r_1\}} \frac{|H(x,u_n)|}{\|u_n\|_a^{p^-}}\,dx \longrightarrow 0, \quad \text{as} \quad n\to+\infty,$$

which is a contradiction to (3.11). Thus, (u_n) is bounded in X.

Finally, we need to prove that any (C)-sequence has a convergent subsequence.
Let $(u_n)\subset X$ be a (C)-sequence. Then, (u_n) is bounded in X. Passing to the limit, if necessary, to a subsequence, from Proposition 2.3, we have

$$u_n \rightharpoonup u \quad \text{in } X, \quad u_n\to u \ \text{ in } L^{s(x)}(\Omega), 1\leq s(x)<p^* \quad \text{and} \quad u_n(x)\to u(x) \ \text{ a.e. } x\in\Omega. \tag{3.13}$$

It is easy to check from (H_0), (3.13) and Hölder's inequality that

$$\Big|\int_\Omega h(x,u_n)(u_n-u)\,dx\Big| \leq C|1+|u_n|^{s(x)-1}|_{s'(x)}|u_n-u|_{s(x)} \longrightarrow 0 \quad \text{as} \quad n\to+\infty, \tag{3.14}$$

where $\frac{1}{s(x)}+\frac{1}{s'(x)}=1$.
On the other hand, using $(\mathcal{M}_0)$, we obtain

$$\mathcal{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega}\frac{a(x)}{p(x)}|u_n|^{p(x)}\,d\sigma\right) \longrightarrow k\neq 0, \quad \text{as} \quad n\to+\infty. \tag{3.15}$$

Next, since $u_n\rightharpoonup u$, from (3.1), we have

$$\langle \psi'_\lambda(u_n), u_n-u\rangle \longrightarrow 0, \quad \text{as} \quad n\to+\infty. \tag{3.16}$$

Then

$$\begin{aligned}
\langle \psi'_\lambda(u_n), u_n-u\rangle =& \mathcal{M}\left(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega}\frac{a(x)}{p(x)}|u_n|^{p(x)}\,d\sigma\right)\langle J_a(u_n), u_n-u\rangle\\
&-\lambda\int_\Omega h(x,u_n)(u_n-u)\,dx\\
&\longrightarrow 0 \quad \text{as} \quad n\to+\infty,
\end{aligned}$$

where J_a is given in (2.1).
Since J_a is a mapping of type (S_+) by Proposition 2.4, we can obtain that $u_n\to u$ in X. The proof is complete. □

Proof of Theorem 1.2

Let $J_\lambda = \psi_\lambda$, $A = \phi$ and $B = \varphi$. Obviously, the functional ϕ is bounded from below. In view of the definition of ϕ and φ, we have $\phi(0) = \varphi(0) = 0$. According to Lemma 3.1, ψ_λ satisfies the (C)- condition. To apply Theorem 2.7, it suffices to check that
(a_1) the functional ψ_λ is unbounded from below,
(a_2) for given $\eta = 1$, there exists $\lambda^* > 0$ such that

$$\sup_{u\in\phi^{-1}(]-\infty,\eta[)} \varphi(u) < \frac{1}{\lambda^*}.$$

• Verification of (a_1). From (H_2), it follows that for every $k > 0$, there exists a constant T_k such that

$$H(x,t) > k|t|^{\theta p^+} \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad |t| > T_k.$$

Since $H(x,.)$ is continuous on $[-T_k, T_k]$, there exists a constant $C_0 > 0$ such that

$$|H(x,t)| \leq C_0 \quad \text{for all} \quad (x,t) \in \Omega \times [-T_k, T_k]\,.$$

Thus,

$$H(x,t) \geq k|t|^{\theta p^+} - C_0, \qquad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}. \tag{3.17}$$

Let $w \in X \setminus \{0\}$ and $l > 1$ be large enough, using Remark 1.1 and the above inequality, we obtain

$$\begin{aligned}
\psi_\lambda(lw) &= \widetilde{\mathcal{M}}\left(\int_\Omega \frac{1}{p(x)}|l\nabla w|^{p(x)}\,dx + \int_{\partial\Omega}\frac{a(x)}{p(x)}|lw|^{p(x)}\,d\sigma\right) - \lambda\int_\Omega H(x,lw)dx \\
&\leq \widetilde{\mathcal{M}}(1)\left(\int_\Omega \frac{1}{p(x)}|l\nabla w|^{p(x)}\,dx + \int_{\partial\Omega}\frac{a(x)}{p(x)}|lw|^{p(x)}\,d\sigma\right)^\theta - \lambda k l^{\theta p^+}\int_\Omega |w|^{\theta p^+}\,dx + \lambda C_0|\Omega| \\
&\leq |l|^{\theta p^+}\left(\frac{\widetilde{\mathcal{M}}(1)}{(p^-)^\theta}\left(\int_\Omega |\nabla w|^{p(x)}\,dx + \int_{\partial\Omega} a(x)|w|^{p(x)}\,d\sigma\right)^\theta - \lambda k\int_\Omega |w|^{\theta p^+}\,dx\right) + \lambda C_0|\Omega|
\end{aligned}$$

where θ comes from $(\mathcal{M}_1)$.
As

$$\frac{\widetilde{\mathcal{M}}(1)}{(p^-)^\theta}\left(\int_\Omega |\nabla w|^{p(x)}\,dx + \int_{\partial\Omega} a(x)|w|^{p(x)}\,d\sigma\right)^\theta - \lambda k\int_\Omega |w|^{\theta p^+}\,dx < 0$$

for k large enough, we deduce

$$\psi_\lambda(lw) \to -\infty, \qquad \text{as} \quad l \to +\infty.$$

Consequently, ψ_λ is unbounded from below.
• Verification of (a_2). By virtue of (H_0) and Proposition 2.3, we get

$$\begin{aligned}
\varphi(u) &= \int_\Omega H(x,u)dx \\
&\leq C\int_\Omega \left(|u| + \frac{1}{s(x)}|u|^{s(x)}\right)\,dx \\
&\leq Cc_4\|u\|_a + \frac{1}{s^-}\max\{|u|^{s^+}_{s(x)}, |u|^{s^-}_{s(x)}\} \\
&\leq Cc_4\|u\|_a + \max\{C_s^{-s^+}, C_s^{-s^-}\}\max\{\|u\|_a^{s^+}, \|u\|_a^{s^-}\}
\end{aligned} \tag{3.18}$$

where $c_4 > 0$ and $C_s = \inf\limits_{u\in X\setminus\{0\}} \frac{\|u\|_a}{|u|_{s(x)}}$.

On the other hand, for all $u \in \phi^{-1}(]-\infty, 1[)$, according to $(\mathcal{M}_0)$, $(\mathcal{M}_1)$ and Proposition 2.2, we get

$$\begin{aligned}
\theta p^+ \geq \theta p^+ \phi(u) &= \theta p^+ \widetilde{\mathcal{M}}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|u|^{p(x)}\, d\sigma\right)\\
&\geq \frac{m\theta p^+}{\theta p^+}\int_\Omega |\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} a(x)|u|^{p(x)}\, d\sigma\\
&\geq m\max\{\|u\|_a^{p^+}, \|u\|_a^{p^-}\}\\
&\geq \min\{1,m\}\max\{\|u\|_a^{p^+}, \|u\|_a^{p^-}\}.
\end{aligned}$$

Because $\theta p^+ > 1 \geq \min\{1,m\}$, we have $\frac{\theta p^+}{\min\{1,m\}} > 1$.
Thus, we obtain

$$\|u\|_a \leq \max\left\{\left(\frac{\theta p^+}{\min\{1,m\}}\right)^{\frac{1}{p^-}}, \left(\frac{\theta p^+}{\min\{1,m\}}\right)^{\frac{1}{p^+}}\right\} = \left(\frac{\theta p^+}{\min\{1,m\}}\right)^{\frac{1}{p^-}}.$$

In view of (H_1) and (3.18), we have

$$\begin{aligned}
\sup_{u\in\phi^{-1}(]-\infty,\eta[)} \varphi(u) &\leq Cc_4\left(\frac{\theta p^+}{\min\{1,m\}}\right)^{\frac{1}{p^-}} + \max\{C_s^{-s^+}, C_s^{-s^-}\}\max\{\|u\|_a^{\frac{s^+}{p^-}}, \|u\|_a^{\frac{s^-}{p^-}}\}\\
&= Cc_4\left(\frac{\theta p^+}{\min\{1,m\}}\right)^{\frac{1}{p^-}} + \max\{C_s^{-s^+}, C_s^{-s^-}\}\left(\frac{\theta p^+}{\min\{1,m\}}\right)^{\frac{s^+}{p^-}}.
\end{aligned} \tag{3.19}$$

Let us denote

$$\lambda^* := \left(Cc_4\left(\frac{\theta p^+}{\min\{1,m\}}\right)^{\frac{1}{p^-}} + \max\{C_s^{-s^+}, C_s^{-s^-}\}\left(\frac{\theta p^+}{\min\{1,m\}}\right)^{\frac{s^+}{p^-}}\right)^{-1}.$$

Taking into account (3.19), we assert that

$$\sup_{u\in\phi^{-1}(]-\infty,\eta[)} \varphi(u) \leq \frac{1}{\lambda^*} < \frac{1}{\lambda}.$$

Finally, all assumptions of Lemma 2.7 are satisfied. Then, for all $\lambda \in]0, \lambda^*[\subset \Gamma_0$, the problem (1.1) admits at least two distinct weak solutions in X. This completes the proof.

Proof of Theorem 1.3

To prove Theorem 1.3, we need the following auxiliary lemmas.

Lemma 3.2. *([8]) For $s \in C_+(\overline{\Omega})$ such that $s(x) < p^*(x)$ for all $x \in \overline{\Omega}$. Let*

$$\delta_k = \sup\{|u|_{s(x)} : \|u\|_a = 1, u \in Z_k\}.$$

Then, $\lim\limits_{k\to+\infty} \delta_k = 0$.

Lemma 3.3. *For all $s \in C_+(\overline{\Omega})$ and $u \in L^{s(x)}(\Omega)$, there exists $y \in \Omega$ such that*

$$\int_\Omega |u|^{s(x)}\, dx = |u|_{s(x)}^{s(y)}. \tag{3.20}$$

Proof. According to Proposition 2.1, we know that

$$\rho\left(\frac{u}{|u|_{s(x)}}\right)=\int_\Omega\left(\frac{|u|}{|u|_{s(x)}}\right)^{s(x)}\,dx=1.$$

Furthermore, the mean value theorem for integrals guarantees that there exists a positive constant $s^*\in[s^-,s^+]$ such that

$$\int_\Omega\left(\frac{|u|}{|u|_{s(x)}}\right)^{s(x)}\,dx=\left(\frac{1}{|u|_{s(x)}}\right)^{s^*}\int_\Omega|u|^{s(x)}\,dx.$$

Since s is a continuous function on Ω, there exists $y\in\Omega$ such that $s(y)=s^*$. Then, we deduce the equality (3.20). $\square$

Now, we return to the proof of Theorem 1.3. To this end, based on the Fountain Theorem 2.8, we will show that the problem (1.1) possesses infinitely many of solutions with unbounded energy. Evidently, according to (H_4), ψ_λ is an even functional. By Lemma 3.1, we know that ψ_λ satisfies the (C)- condition. Then, to prove Theorem 1.3, it only remains to verify the following assertions:
(A_1) $b_k:=\inf\{J(u):u\in Z_k,\|u\|=\eta_k\}\to+\infty$ as $k\to+\infty$,
(A_2) $c_k:=\max\{J(u):u\in Y_k,\|u\|=\gamma_k\}\leq 0$.

(A_1) For any $u\in Z_k$ such that $\|u\|_a=\eta_k>1$. It follows from $(\mathcal{M}_0)$, $(\mathcal{M}_1)$, (H_0), Proposition 2.3 and Lemma 3.3 that

$$\begin{aligned}
\psi_\lambda(u)&=\widetilde{\mathcal{M}}\left(\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}\frac{a(x)}{p(x)}|u|^{p(x)}\,d\sigma\right)-\lambda\int_\Omega H(x,u)dx\\
&\geq\frac{1}{\theta}\mathcal{M}\left(\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}\frac{a(x)}{p(x)}|u|^{p(x)}\,d\sigma\right)\left(\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int_{\partial\Omega}\frac{a(x)}{p(x)}|u|^{p(x)}\,d\sigma\right)\\
&-\lambda C\int_\Omega|u|\,dx-\lambda C\int_\Omega\frac{|u|^{s(x)}}{s(x)}dx\\
&\geq\frac{m}{\theta p^+}\|u\|_a^{p^-}-\lambda c_5\|u\|_a-\frac{\lambda c_6}{s^-}|u|_{s(x)}^{s(y)}\\
&\geq\begin{cases}\frac{m}{\theta p^+}\|u\|_a^{p^-}-\lambda c_5\|u\|_a-\frac{\lambda c_6}{s^-} & \text{if } |u|_{s(x)}\leq 1\\ \frac{m}{\theta p^+}\|u\|_a^{p^-}-\lambda c_5\|u\|_a-\frac{\lambda c_6}{s^-}(\delta_k\|u\|_a)^{s^+} & \text{if } |u|_{s(x)}>1\end{cases}\\
&\geq\frac{m}{\theta p^+}\|u\|_a^{p^-}-\lambda c_5\|u\|_a-\frac{\lambda c_6}{s^-}(\delta_k\|u\|_a)^{s^+}-\frac{\lambda c_6}{s^-}\\
&\geq\eta_k^{p^-}\left(\frac{m}{\theta p^+}-\frac{\lambda c_6}{s^-}\delta_k^{s^+}\eta_k^{s^+-p^-}\right)-\lambda c_5\eta_k-\frac{\lambda c_6}{s^-}.
\end{aligned}$$

Since $p^+<s^+$ and $\delta_k\to 0$ as $k\to+\infty$, we conclude that $\eta_k\to+\infty$ as $k\to+\infty$.
Finally,

$$\psi_\lambda(u)\to+\infty \text{ as } \ k\to+\infty.$$

Hence, (A_1) holds.

(A_2) Because $Y_k=\bigoplus_{j=1}^k X_j$ is finite-dimensional space, all norms are equivalent. Then, there exists

$d_k > 0$, for all $u \in Y_k$ with $\|u\|_a$ is large enough, we obtain

$$\begin{aligned}\phi(u) &= \widetilde{\mathcal{M}}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|u|^{p(x)}\, d\sigma\right)\\ &\le \widetilde{\mathcal{M}}(1)\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx + \int_{\partial\Omega} \frac{a(x)}{p(x)}|u|^{p(x)}\, d\sigma\right)^\theta\\ &\le \frac{\widetilde{\mathcal{M}}(1)}{(p^-)^\theta}\|u\|_a^{\theta p^+}\\ &\le d_k|u|_{\theta p^+}^{\theta p^+}.\end{aligned} \tag{3.21}$$

Moreover, it follows from (H_2) that there exist $D_k > 0$ such that for every $|t| \ge D_k$, we get

$$H(x,t) \ge 2d_k|t|^{\theta p^+} \quad \text{for all } \ x \in \Omega.$$

Then,

$$H(x,t) \ge 2d_k|t|^{\theta p^+} - L_k, \quad \text{ for all } \ (x,t) \in \Omega \times \mathbb{R},$$

where $L_k = \max_{|t|\le D_k} H(x,t)$.

Combining this with (3.21), for $u \in Y_k$ such that $\|u\|_a = \gamma_k > \eta_k$, we find

$$\begin{aligned}\psi_\lambda(u) &= \phi(u) - \lambda\int_\Omega H(x,u)\, dx\\ &\le d_k|u|_{\theta p^+}^{\theta p^+} - 2d_k|u|_{\theta p^+}^{\theta p^+} + L_k|\Omega|\\ &\le -d_k|u|_{\theta p^+}^{\theta p^+} + L_k|\Omega|\\ &\le -\frac{\widetilde{\mathcal{M}}(1)}{(p^-)^\theta}\|u\|_a^{\theta p^+} + L_k|\Omega|.\end{aligned}$$

Thus, for γ_k large enough, we obtain from the above inequalities

$$c_k := \max\{\psi_\lambda(u) : u \in Y_k, \|u\|_a = \gamma_k\} \le 0.$$

This completes the proof.

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