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On the Generalized Apostol-Kolodner Differential Equation of the Second-Order

Mustapha Rachidi and Mohammed Mouniane

abstract: This paper concerns the generalized Apostol-Kolodner differential equations of the second order. Our approach is based on some matrix square root properties, the Fibonacci-Hörner decomposition of matrix powers, and its related dynamical solution. Various explicit compact formulas for the solutions of the generalized Apostol-Kolodner matrix differential equations are established. Finally, to validate the results and show their robustness, examples and applications are provided.

Key Words: Matrix differential equation of the second-order, matrix power, matrix exponential, Fibonacci-Hörner decomposition, dynamical solution, companion block matrix.

Contents

1. Introduction

Over the past few decades, higher-order linear differential matrix equations have attracted considerable attention, given their importance in many fields. More specifically, applying the higher-order linear matrix differential equations theory extends beyond mathematical studies into applied sciences and engineering. (see, for instance, [3,9,13,17,22,23,27] and references therein). Apostol (see [2]) and Kolodner (see [18]) have studied the following matrix differential equation of the second-order

$$X^{''}(t) = AX(t), \tag{1.1}$$

submitted to the initial data $X(0)$ and $X'(0)$, where $A \in \mathbb{C}^{d\times d}$ the algebra of square matrices of order $d\times d$. Apostol and Kolodner gave some properties of the solutions of Equation (1.1). In [2], Apostol proposes a process for solving equation (1.1) based on Putzer's method to calculate the exponential of matrices (see [2,24]). In [18], Kolodner studies some manageable formulas to compute $exp(tA)$, where A is a square matrix, by establishing an analogous Putzer's formula for the matrix powers A^n. Then, he extends its

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process to second-order matrix differential systems (1.1). In [8], the linear matrix differential equations of higher order have been studied under the commutativity condition of the coefficients matrices. Some results have been established for a class of matrix differential equations of higher order, and some results of Apostol and Kolodner are recovered.

In the present study, we are interested in the generalized Apostol-Kolodner linear matrix differential equation of the second-order

$$X^{''}(t) = A_0X^{'}(t) + A_1X(t), \tag{1.2}$$

where A_0 and A_1 are in $\mathbb{C}^{d\times d}$, which do not necessarily commute. In the first step, we study the Apostol-Kolodner differential equation (1.1), where we exhibit the properties of its solutions in terms of the square root of the matrices. The main purpose of the second step is to investigate the properties and solutions of the generalized Apostol-Kolodner matrix differential equation (1.2). Our main tools are the computational properties of the powers of matrices, using the linear recursive relations in the algebras of square matrices, and the analytic expressions of some real or complex linear recursive sequences. Our approach will allows us to characterize explicitly the solutions of equations (1.1) and (1.2), namely, linear solutions, combinatorial, and analytical solutions.

The content of the present study is structured as follows. Section 2 consists of studying the differential equation of Apostol-Kolodner (1.1) and establishing properties related to the principal square roots of the matrices. Section 3 is devoted to the generalized Apostol-Kolodner differential equation (1.2), where the linear recursive sequences in the algebra of square matrices of order $d \times d$ play a central role. Moreover, the commutativity conditions $A_0A_1 = A_1A_0$ allows us to obtain the combinatorial solutions of Equation (1.2). Section 4 is devoted to the use of the Fibonacci-Hörner decomposition of the matrix powers to solve Equation (1.2) employing the properties of the so-called dynamical solutions. When the commutativity condition is satisfied, the combinatorial aspect of the solution of Equation (1.2) is approached. In Section 5, the linear and analytic aspects of the solution of Equation (1.2) are studied via the Fibonacci-Hörner decomposition by considering the analytical expression of the dynamical solution. In the general setting, the solutions of Equation (1.2) are expressed in terms of operators related to the derivation $D = t\dfrac{d}{dt}$. The Fibonacci-Hörner decomposition process is also applied to the differential equation of Apostol-Kolodner (1.1). Section 6 deals with an approach to the solutions of Equation (1.2), under the commutativity condition $A_0A_1 = A_1A_0$. Throughout the previous sections, illustrative examples and applications are provided. Finally, a conclusion and perspectives are discussed.

Without loss of generality and only for simplicity purpose, we will refer to the Apostol-Kolodner differential equation (1.1) as Equation (1.1), and the generalized Apostol-Kolodner differential equation (1.2) as Equation (1.2).

2. Apostol-Kolodner equation $X^{''}(t) = AX(t)$ and matrix square root

Let $\mathbb{C}^{d\times d}$ be the algebra of square matrix of order $d \times d$. We investigate here Equation (1.1), under the following equivalent matrix differential equation of the first order $Z'(t) = BZ(t)$, where $Z(t) = (X'(t), X(t))^T$ and B is the *companion block matrix* given by $B = \begin{pmatrix} O_d & A \\ I_d & O_d \end{pmatrix}$, where $A \in \mathbb{C}^{d\times d}$, O_d and I_d are the null and the identity matrix, respectively. A direct computation implies that $B^{2n} = \begin{pmatrix} A^n & O_d \\ O_d & A^n \end{pmatrix}$ and $B^{2n+1} = \begin{pmatrix} O_d & A^{n+1} \\ A^n & O_d \end{pmatrix}$. Thus, we derive that

$$e^{tB} = \sum_{n=0}^{+\infty} \frac{t^n}{n!}B^n = \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!}B^{2n} + \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)!}B^{2n+1}.$$

Considering the initial data $X(0)$ and $X'(0)$, we derive that

$$Z(t) = e^{tB}Z(0) = \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!}B^{2n}Z(0) + \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)!}B^{2n+1}Z(0).$$

Yet, using the previous expressions of the powers B^{2n} and B^{2n+1}, we get

$$\begin{bmatrix} X'(t) \\ X(t) \end{bmatrix} = \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!} \begin{bmatrix} A^n X'(0) \\ A^n X(0) \end{bmatrix} + \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)!} \begin{bmatrix} A^{n+1} X(0) \\ A^n X'(0). \end{bmatrix}.$$

Therefore, we obtain

$$X(t) = \left(\sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!} A^n \right) X(0) + \left(\sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)!} A^n \right) X'(0).$$

In summary, we recover the following result of Apostol (see [2,8,18]).

Proposition 2.1. *The unique solution of Equation (1.1), under the prescribed initial data $X(0)$ and $X'(0)$, is given by*

$$X(t) = C_1(t)X(0) + C_2(t)X'(0),$$

for every $t \in]-\infty, +\infty[$, where

$$C_1(t) = \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!} A^n \text{ and } C_2(t) = \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)!} A^n. \tag{2.1}$$

In Proposition 2.1, the solution of Equation (1.1) is expressed by the powers of the matrix A. In the Subsection 5.3, the Fibonacci-Hörner decomposition of the powers A^n of the matrix A, will yield the analytical aspect of the functions $C_1(t)$ and $C_2(t)$.

However, as observed in [2], the expressions of $C_1(t)$ and $C_2(t)$ in (2.1) are related to the square root of the matrix A, through the matrix hyperbolic functions cosh and sinh. In general, determining the square root of a matrix A of order d ($d \geq 2$), defined as the solution of the matrix equation $X^2 = A$, is not an easy task. Several studies in the literature are devoted to the square root of matrices (see, for instance, [1,14,16] and references therein). When the matrix A has no eigenvalues on $\mathbb{R}^-$ (the closed negative real axis), there exists a unique matrix X such that $X^2 = A$ and the eigenvalues of X lies on the segment $\{z \in \mathbb{C} : -\pi/2 < arg(z) < \pi/2\}$, where $arg(z)$ is the argument of z (see, for instance, [16] and references therein). Suppose that the matrix A owns a principal matrix square root S, then the two functions $C_1(t)$ and $C_2(t)$ of Expression (2.1), can be written under the forms

$$C_1(t) = \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!} S^{2n} \text{ and } C_2(t) = \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)!} S^{2n}.$$

If the matrix A is invertible and owns a principal square root S, then S is also invertible. Therefore, we have

$$C_1(t) = \cosh(tS) \text{ and } C_2(t) = S^{-1}\sinh(tS),$$

where $\cosh(tS)$ and $\sinh(tS)$ are the hyperbolic matrix functions. Thus, from Proposition 2.1, we acquire the following result.

Proposition 2.2. *Suppose that the matrix A is invertible and owns a square root S, namely, $A = S^2$. Then, under the initial data $X(0)$ and $X'(0)$, the unique solution of Equation (1.1) is given by*

$$X(t) = \cosh(tS)X(0) + S^{-1}\sinh(tS)X'(0),$$

for every $t \in]-\infty, +\infty[$, where $\cosh(tS)$ and $\sinh(tS)$ are the matrix hyperbolic functions. Especially, if S owns simple eigenvalues $\lambda_1, \ldots, \lambda_d$, then we have

$$C_1(t) = P\,diag(\cosh(t\lambda_1), \ldots, \cosh(t\lambda_d))P^{-1}, \tag{2.2}$$

$$C_2(t) = P\,diag(\lambda_1^{-1}\sinh(t\lambda_1), \ldots, \lambda_d^{-1}\sinh(t\lambda_d))P^{-1}, \tag{2.3}$$

where P is the invertible matrix such that $S = P\,diag(\lambda_1, \ldots, \lambda_d)P^{-1}$.

Let furnish an illustrative numerical example based on the principal square root.

Example 2.3. *Let consider Equation (1.1), where its related matrix A is given by $A = \begin{pmatrix} 14 & -5 \\ 10 & -1 \end{pmatrix}$. A direct verification shows that $A = S^2$, where $S = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$. Therefore, Proposition 2.2 implies that the solution of Equation (1.1) takes the form $X(t) = \cosh(tS)X(0) + S^{-1}\sinh(tS)X'(0)$. On the other hand, since $S = P\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}P^{-1}$, where $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $S^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$, a straightforward computation permits us to show that the unique solution of Equation (1.1), defined by the matrix A, is expressed as follows*

$$X(t) = C_1(t)X(0) + C_2(t)X'(0),$$

where

$$C_1(t) = \begin{pmatrix} 2\cosh(3t) - \cosh(2t) & \cosh(2t) - \cosh(3t) \\ 2(\cosh(3t) - \cosh(2t)) & 2\cosh(2t) - \cosh(3t) \end{pmatrix},$$

and

$$C_2(t) = \frac{1}{6}\begin{pmatrix} 6\sinh(3t) - 5\sinh(2t) & 3\sinh(3t) + 5\sinh(2t) \\ 6\sinh(3t) - 7\sinh(2t) & 3\sinh(3t) + 7\sinh(2t) \end{pmatrix}.$$

Proposition 2.2 enables to determine the analytic formulas of the solution of Equation (1.1), through the spectral aspect of the matrix square root S of A. And Example 2.3 shows that if the eigenvalues of the principal square matrix S of A, are real numbers, then the unique solution of Equation (1.1) are expressed in terms of the hyperbolic functions cosh and sinh. In the following example, we illustrate a case when one of the eigenvalues of S owns an imaginary part that is not null.

Example 2.4. *Let consider Equation (1.1), whose related matrix A is $\begin{pmatrix} 22 & -13 \\ 26 & -17 \end{pmatrix}$. A direct verification implies that $A = S^2$, where $S = \begin{pmatrix} 6-2i & -3+2i \\ 6-4i & -3+4i \end{pmatrix}$ is the principal square root of A, with $S = P\begin{pmatrix} 3 & 0 \\ 0 & 2i \end{pmatrix}P^{-1}$, where $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $S^{-1} = \begin{pmatrix} \frac{2}{3} + \frac{1}{2}i & -\frac{1}{3} - \frac{1}{2}i \\ \frac{2}{3} + i & -\frac{1}{3} - i \end{pmatrix}$. Thus, a long straightforward computation permits to show that the unique solution of Equation (1.1) is determined by*

$$X(t) = C_1(t)X(0) + C_2(t)X'(0),$$

where

$$C_1(t) = \begin{pmatrix} 2\cosh(3t) - \cos(2t) & -\cosh(3t) + \cos(2t) \\ 2\cosh(3t) - 2\cos(2t) & -\cosh(3t) + 2\cos(2t) \end{pmatrix},$$

and

$$C_2(t) = \begin{pmatrix} \frac{2}{3}\sinh(3t) - \frac{1}{2}\sin(2t) & -\frac{1}{3}\sinh(3t) + \frac{1}{2}\sin(2t) \\ \frac{2}{3}\sinh(3t) - \sin(2t) & \frac{1}{3}\sinh(3t) + \sin(2t) \end{pmatrix}.$$

Example 2.4 shows that the principal matrix square root S of A owns an eigenvalue, whose imaginary part is not null. Therefore, the solution $X(t)$, of Equation (1.1) is expressed with the aid of the trigonometric functions and hyperbolic functions. For the general setting, this fact can be also deducted from Expressions (2.2) and (2.3) of $C_1(t)$ and $C_2(t)$, respectively.

There are several characterizations of the existence and uniqueness of the principal square root of a given matrix (see, for example, [1,14,16] and references therein). An Hermitian matrix A of $\mathbb{C}^{d\times d}$, namely, $A^* = A$, where $A^* = \overline{A}^T = (\overline{a}_{i,j})_{1\le i,\, j\le d}$, is called positive semi-definite, if $x^*Ax \ge 0$ for all $x \in \mathbb{C}^d$, and it is positive definite, if $x^*Ax > 0$, for all $x \in \mathbb{C}^d$. It was established in [14, Theorem 4.3], that for every positive semidefinite Hermitian matrix A and a given integer $k \ge 1$, there exists a unique positive semidefinite Hermitian matrix S such that $S^k = A$. Moreover, the matrix S is real if A is real. Since the eigenvalues of the positive semidefinite (respectively, positive definite) Hermitian matrix A are all nonnegative (respectively, positive), then for $k = 2$, the unique positive Hermitian matrix S satisfying $S^2 = A$, representing the square root of A, whose eigenvalues are also all nonnegative (respectively, positive). Thus, for a positive semidefinite Hermitian matrix A, the unique positive semidefinite Hermitian matrix S satisfying $S^2 = A$ represents the principal square matrix of A. Therefore, we have the following result.

Proposition 2.5. *Suppose that A is a positive definite Hermitian matrix and consider its unique positive Hermitian principal square root S, namely, $S^2 = A$. Then, the unique solution of Equation (1.1), under the prescribed data $X(0)$ and $X'(0)$, is as follows $X(t) = \cosh(tS)X(0) + S^{-1}\sinh(tS)X'(0)$, for every $t \in]-\infty, +\infty[$, where $\cosh(tS)$ and $\sinh(tS)$ are the hyperbolic matrix functions.*

Remark 2.6. *Let $X(t) = (x_1(t), \cdots, x_d(t))^T$ be the solution of Equation (1.1) defined by the matrix A and the initial data $X(0) = (x_1(0), \cdots, x_d(0))^T$. Then, a long straightforward computation allows us to obtain the explicit formulas for the the functions $x_1(t), \cdots, x_d(t)$, for every $t \in \mathbb{R}$.*

Equation (1.1) will also be dealt with through two other approaches in Subsection 5.3. More precisely, the linear and analytical approaches considered for the general case (1.2), in Section 4 and Subsections 5.1 and 5.2, will also be applied to the Apostol-Kolodner equation (1.1).

3. Solutions of Equation (1.2) by recursiveness for $A_0 \neq O_d$ and $A_1 \neq O_d$

3.1. Recursiveness and powers of companion 2×2 block matrices with entries in $\mathbb{C}^{d\times d}$

Let A_0, A_1 and S_0, S_1 be fixed matrices in $\mathbb{C}^{d\times d}$ such that $A_0 \neq O_d$ and $A_1 \neq O_d$, which does not necessarily commute. Let $\{Y_n\}_{n\geq 0}$ be the matrix sequence defined by $Y_n = S_n$ for $n = 0$, 1 and

$$Y_{n+1} = A_0Y_n + A_1Y_{n-1}, \quad \text{for every} \quad n \geq 1. \tag{3.1}$$

Let $\{Y_{n,s}\}_{n\geq 0}$ $(0 \leq s \leq 1)$ be the two special sequences of type (3.1), defined by

$$Y_{n+1,s} = A_0Y_{n,s} + A_1Y_{n-1,s}, \ \text{ for } \ n \geq 1, \tag{3.2}$$

where the initial values are $Y_{s,s} = I_d$ and $Y_{n,s} = O_d$ if $0 \leq n \neq s \leq 1$. Let $B \in \mathbb{C}^{2d\times 2d}$ be the companion block matrix

$$B = \begin{pmatrix} A_0 & A_1 \\ I_d & O_d \end{pmatrix}. \tag{3.3}$$

For exhibiting the powers B^n, we will use the recent technique of calculating the powers of the usual companion matrix (see [12]). That is, for $n = 0$, we show that

$$B^0 = \begin{pmatrix} I_d & O_d \\ O_d & I_d \end{pmatrix} = \begin{pmatrix} Y_{1,1} & O_d \\ O_d & Y_{0,0} \end{pmatrix}.$$

For $n = 1$, we have $Y_{2,1} = A_0Y_{1,1} + A_1Y_{0,1} = A_0$ and $Y_{2,0} = A_0Y_{1,0} + A_1Y_{0,0} = A_1$. Therefore, we have $B^1 = \begin{pmatrix} Y_{2,1} & Y_{2,0} \\ Y_{1,1} & Y_{1,0} \end{pmatrix} = \begin{pmatrix} A_0 & A_1 \\ I_d & O_d \end{pmatrix}$. Then, by an induction process, we get the following proposition.

Proposition 3.1. *Let A_0, A_1 be in $\mathbb{C}^{d\times d}$ such that $A_0 \neq O_d$, $A_1 \neq O_d$ and the sequence $Y_{n,s}$ as in (3.2). Then, we have*

$$B^n = \begin{pmatrix} Y_{n+1,1} & Y_{n+1,0} \\ Y_{n,1} & Y_{n,0} \end{pmatrix}, \text{ for every } n \geq 0. \tag{3.4}$$

It is essential to observe that in Proposition 3.1, the two matrices A_0 and A_1 do not necessarily commute. When the commutativity condition, $A_0A_1 = A_1A_0$, is satisfied, a similar straightforward computation as in [7,21] permits us to establish the combinatorial expression of Y_n, the general term of (3.1), as follows

$$Y_n = \rho(n, 2)W_0 + \rho(n-1, 2)W_1, \text{ for every } \ n \geq 2, \tag{3.5}$$

where $W_0 = A_1S_0 + A_0S_1$, $W_1 = A_1S_1$ and

$$\rho(n,2) = \sum_{k_0+2k_1=n-2} \frac{(k_0+k_1)!}{k_0!k_1!} A_0^{k_0} A_1^{k_1}, \tag{3.6}$$

for every $n \geq 2$, with $\rho(2,2) = I_d$ and $\rho(n,2) = O_d$ for $n \leq 1$ (see [5,7,20,21]). Moreover, application of Formulas (3.5) and (3.6) to the sequences $\{Y_{n,0}\}_{n\geq 0}$ and $\{Y_{n,1}\}_{n\geq 0}$ implies that

$$Y_{n,0} = A_1\rho(n,2) \text{ for } n \geq 2 \text{ and } Y_{n,1} = A_0\rho(n,2) + A_1\rho(n-1,2) \text{ for } n \geq 2, \tag{3.7}$$

where $\rho(n,2)$ is as in (3.6). This allows us to establish the combinatorial expression of the powers of B the matrix (3.3).

Corollary 3.2. *Let A_0, A_1 be in $\mathbb{C}^{d\times d}$ such that $A_0 \neq O_d$, $A_1 \neq O_d$ and $A_0A_1 = A_1A_0$. Then, we have*

$$B^n = \begin{pmatrix} A_0\rho(n+1,2) + A_1\rho(n,2) & A_0\rho(n+1,2) \\ A_0\rho(n,2) + A_1\rho(n-1,2) & A_0\rho(n,2) \end{pmatrix}, \text{ for every } n \geq 2,$$

where $\rho(n,2)$ is as in (3.6).

Proposition 3.1 and Corollary 3.2 will play a vital role in the sequel, for exhibiting some properties of Equation (1.2).

3.2. Solving Equation (1.2) by recursiveness process

Now, we are interested in studying the solutions of Equation (1.2), using the linear matrix recursiveness (3.1). Consider Equation (1.2), whose solution X belonging to $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{d\times d})$, is subjected to the initial data $X(0)$ and $X^{'}(0)$. Set $Z(t) = (X^{'}(t), X(t))^T$ $(t \in \mathbb{R})$ and $Z(0) = (X^{'}(0), X(0))^T$. A standard computation shows that Equation (1.2) is reduced to the usual matrix differential equation

$$Z'(t) = BZ(t), \tag{3.8}$$

where B is the companion block matrix (3.3). It is well known that the solution of Equation (3.8) is given by $Z(t) = e^{tB}Z(0)$, where the formula of e^{tB} is derived from on the computation of the powers B^n, in terms of A_0, A_1. Formula (3.4) implies that, we have

$$Z(t) = \sum_{n=0}^{+\infty} \left[\begin{array}{c} Y_{n+1,1}X'(0) + Y_{n+1,0}X(0) \\ Y_{n,1}X'(0) + Y_{n,0}X(0) \end{array} \right] \frac{t^n}{n!}.$$

Hence, we have $X(t) = \left[\sum_{n=0}^{+\infty} Y_{n,1}\frac{t^n}{n!}\right] X'(0) + \left[\sum_{n=0}^{+\infty} Y_{n,0}\frac{t^n}{n!}\right] X(0)$. Thus, the solutions of Equation (1.2) are formulated as in the following theorem.

Theorem 3.3. *Let A_0 and A_1 be in $\mathbb{C}^{d\times d}$. Then, under initial data $X(0)$ and $X'(0)$, the unique solution of Equation (1.2) is given by*

$$X(t) = C_1(t)X(0) + C_2(t)X^{'}(0),$$

with

$$C_1(t) = \sum_{n=0}^{+\infty} Y_{n,0}\frac{t^n}{n!} \text{ and } C_2(t) = \sum_{n=0}^{+\infty} Y_{n,1}\frac{t^n}{n!},$$

where $\{Y_{n,0}\}_{n\geq 0}$ and $\{Y_{n,1}\}_{n\geq 0}$ are the two recursive matrices sequences defined by Expression (3.2).

Once again, in Theorem 3.3 the commutativity condition $A_0A_1 = A_1A_0$ is not considered. When the commutativity condition $A_0A_1 = A_1A_0$ is fulfilled, we can apply the results of Theorem 3.3 to succeed in making the combinatorial solutions for Equation (1.2). Indeed, taking into account the combinatorial Expression (3.7) of the two sequences $\{Y_{n,j}\}_{n\geq 0}$ $(0 \leq j \leq 1)$ defined by Expression (3.2), we show that $C_1(t) = Y_{0,0} + Y_{1,0}\frac{t}{1!} + \sum_{n=2}^{+\infty} Y_{n,0}\frac{t^n}{n!} = I_d + A_1\sum_{n=2}^{+\infty}\rho(n,2)\frac{t^n}{n!}$, and $C_2(t) = \sum_{n=0}^{+\infty} Y_{n,1}\frac{t^n}{n!} = Y_{1,0} + Y_{1,1}\frac{t}{1!} + \sum_{n=2}^{+\infty} Y_{n,1}\frac{t^n}{n!} = I_d t + A_0\sum_{n=2}^{+\infty}\rho(n,2)\frac{t^n}{n!} + A_1\sum_{n=2}^{+\infty}\rho(n-1,2)\frac{t^n}{n!}$. We observe that $g_0(t) = \sum_{n=0}^{+\infty}\rho(n,2)\frac{t^n}{n!}$ and $g_1(t) = \sum_{n=0}^{+\infty}\rho(n-1,2)\frac{t^n}{n!}$ are nothing else but the exponential generating functions of the matrix recursive sequences $\{\rho(n,2)\}_{n\geq 0}$ and $\{\rho(n-1,2)\}_{n\geq 0}$, where $\rho(n,2)$ is defined by Expression (3.6). Thus, we have the following characterization of the combinatorial solutions of Equation (1.2).

Proposition 3.4. *Consider Equation (1.2), where $A_0,\ A_1 \in \mathbb{C}^{d\times d}$ satisfying $A_0A_1 = A_1A_0$. Then, submitted to the prescribed initial data $X(0)$ and $X'(0)$, its unique solution $X(t)$ is expressed as follows*

$$X(t) = [I_d + A_1 g_0(t)]\, X(0) + [I_d t + A_0 g_0(t) + A_1 g_1(t)]\, X'(0),$$

where $g_0(t)$ and $g_1(t)$ are nothing else but the exponential generating functions of the matrices sequences $\{\rho(n,2)\}_{n\geq 0}$ and $\{\rho(n-1,2)\}_{n\geq 0}$, respectively.

4. Dynamical solution approach and solutions of Equation (1.2)

For reason of simplicity and without loss of generality, we supposes that $A_0 \neq O_d$ and $A_1 \neq O_d$ in Subsections 4.1 and 4.2.

4.1. Fibonacci-Hörner process and dynamical solution for solving Equation (1.2)

Let A be a matrix of $\mathbb{C}^{d\times d}$ and $R(z) = z^r - a_0z^{r-1} - \cdots - a_{r-1}$, with $a_{r-1} \neq 0$, be a polynomial such that $R(A) = O_d$ (the zero matrix of $\mathbb{C}^{d\times d}$). The Fibonacci-Hörner decomposition of the powers A^n is expressed as follows

$$A^n = u_nW_0 + u_{n-1}W_1 + \cdots + u_{n-r+1}W_{r-1},\ \text{ for every }\ n \geq 0, \tag{4.1}$$

where

$$W_0 = I_d\,;\ W_i = A^i - a_0A^{i-1} - \cdots - a_{i-1}I_r,\ \text{for } i = 1, \ldots, r-1, \tag{4.2}$$

and the sequence $\{u_n\}_{n\geq -r+1}$ is defined by

$$u_n = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n} \frac{(k_0+k_1+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} a_0^{k_0}a_1^{k_1}\cdots a_{r-1}^{k_{r-1}}, \tag{4.3}$$

for every $n \geq -r+1$, with initial values $u_0 = 1$ and $u_{-j} = 0$ for $1 \leq j \leq r-1$ (for more details, see [4,7,19,20,21]). Moreover, it was established in [4,20,26] that the sequence $\{u_n\}_{n\geq 0}$ satisfies the following linear recurrence relation of order r

$$u_{n+1} = a_0u_n + a_1u_{n-1} + \cdots + a_{r-1}u_{n-r+1},\ \text{for every } n \geq 0. \tag{4.4}$$

The set of matrices $\{W_0\ ,\ W_1\ ,\cdots\ ,\ W_{r-1}\}$ is called the *Fibonacci-Hörner system* of the powers decomposition of A. This system is obtained from the Fibonacci combinatorial process and the fact that each matrix W_j $(0 \leq j \leq r-1)$ verifies $W_j = h_j(A)$, where the $h_j(z)$ $(1 \leq j \leq r-1)$ are the Hörner polynomials associated to $R(z)$, namely, $h_0(z) = 1, h_1(z) = z - a_0, \ldots, h_j(z) = zh_{j-1}(z) - a_{j-1}, \ldots, h_{r-1}(z) = zh_{r-2}(z) - a_{r-1} = R(z)$.

Let B be the companion block matrix (3.3) related to Equation (1.2), namely, $B = \begin{pmatrix} A_0 & A_1 \\ I_d & O_d \end{pmatrix}$, where the A_0 and A_1 are in $\mathbb{C}^{d\times d}$. Consider the polynomial $R(z) = z^r - a_0z^{r-1} - \cdots - a_{r-2}z - a_{r-1}$, of degree $r \leq 2d$ such that $R(B) = O_{2d}$. For reason of simplicity and seek of generality, we consider in the sequel a polynomial

$$R(z) = z^r - a_0z^{r-1} - \cdots - a_{r-2}z - a_{r-1},$$

with $a_{r-1} \neq 0$, such that $R(B) = O_{2d}$. Applying the Fibonacci-Hörner decomposition for computing the powers B^n, we can formulate the following lemma.

Lemma 4.1. *Let consider Equation (1.2), where $A_0 \neq O_d$, $A_1 \neq O_d$ and B the associated companion block matrix (3.3). Let $R(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-2}z - a_{r-1}$ be the polynomial such that $R(B) = O_{2d}$. Then, we have*

$$B^n = u_n W_0 + u_{n-1}W_1 + \cdots + u_{n-r+1}W_{r-1}, \quad \textit{for every} \;\; n \geq 0, \tag{4.5}$$

where

$$W_0 = I_{2d}\,,\; W_i = B^i - a_0 B^{i-1} - \cdots - a_{i-1}I_{2d}, \tag{4.6}$$

and the sequence $\{u_n\}_{n\geq -r+1}$ is given by

$$u_n = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n} \frac{(k_0+k_1+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} a_0^{k_0} a_1^{k_1} \cdots a_{r-1}^{k_{r-1}}, \tag{4.7}$$

for every $n \geq 0$, with $u_0 = 1$ and $u_{-j} = 0$ for $1 \leq j \leq r-1$.

Starting with expression $e^{tA} = \sum_{n\geq 0} \frac{t^n}{n!} B^n$, a direct computation, using Expressions (4.5) and (4.7), permits to have $e^{tB} = \displaystyle\sum_{n=0}^{+\infty} \frac{t^n}{n!} \sum_{j=0}^{r-1} u_{n-j} W_{n-j} = \sum_{j=0}^{r-1} \left[\sum_{n=0}^{+\infty} u_{n-j} \frac{t^n}{n!} \right] W_j$. Therefore, we obtain

$$e^{tB} = \sum_{j=0}^{r-1} \varphi_j(t) W_j \text{ where } \varphi_j(t) = \sum_{n=0}^{+\infty} u_n \frac{t^{n+j}}{(n+j)!}. \tag{4.8}$$

Note that e^{tB} is an absolutely convergent series and its radius of convergence is $R = +\infty$, then the series $\varphi_j(t) = \sum_{n=0}^{+\infty} u_n \frac{t^{n+j}}{(n+j)!}$ are convergent, with the same radius of convergence. Moreover, for every $1 \leq j \leq r-1$, we can verify that $\dfrac{d\varphi_j}{dt}(t) = \varphi_{j-1}(t)$, and by induction we show that $\dfrac{d^k\varphi_j}{dt^k}(t) = \varphi_{j-k}(t)$ for every k $(0 \leq k \leq j)$. Hence, we have

$$\frac{d^{r-j-1}\varphi_{r-1}}{dt^{r-j-1}}(t) = \varphi_{r-1-(r-j-1)}(t) = \varphi_j(t).$$

In summary, we can formulate the following lemma related to [4, Proposition 2.1].

Lemma 4.2. *Let B the companion block matrix (3.3), where $A_0 \neq O_d$ and $A_1 \neq O_d$. Let $R(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-2}z - a_{r-1}$ be the polynomial such that $R(B) = O_{2d}$. Then, we have $e^{tB} = \displaystyle\sum_{j=0}^{r-1} \varphi^{(r-j-1)}(t) W_j$, where*

$$\varphi(t) = \sum_{n=0}^{+\infty} u_n \frac{t^{n+r-1}}{(n+r-1)!}, \tag{4.9}$$

with $\varphi^{(k)}(t) = \dfrac{d^k\varphi}{dt^k}(t)$.

Expression (4.4) shows that $\varphi(t)$ defined by (4.9) satisfies the following ordinary differential equation $y^{(r)}(t) = a_0 y^{(r-1)}(t) + a_1 y^{(r-2)}(t) + \cdots + a_{r-1}y(t)$. Moreover, Expression (4.9) shows that $\varphi^{(k)}(0) = 0$ for $k = 0, 1, \ldots, r-2$ and $\varphi^{(r-1)}(0) = 1$. Hence, the function $\varphi(t)$ is nothing else but the dynamical solution of the preceding differential equation (see [25] [28]). And Expression (4.8) reveals that the functions φ_j appearing in the Fibonacci-Hörner decomposition of e^{tB} are the elements of the fundamental system of solutions $\{\varphi(t), \varphi'(t), \ldots, \varphi^{(r-2)}(t), \varphi^{(r-1)}(t)\}$ of the previous differential equation.

Lemmas 4.1 and 4.2 are very useful for solving Equation (1.2). To this aim, we express each element of the Fibonacci-Hörner system $\{W_j\}_{0\leq j\leq r-1}$ related to B the companion block matrix (3.3), in terms of $\{Y_{n,s}\}_{n\geq 0}$ $(0 \leq s \leq 1)$ of $\mathbb{C}^{d\times d}$, the two sequences (3.2). It was shown in Expression (3.4) of Proposition 3.1 that the powers of the matrix B can be expressed using the two sequences (3.2). Therefore, for W_j $(1 \leq j \leq r-1)$ the matrices given by (4.2), we have the following lemma.

Lemma 4.3. *The Fibonacci-Hörner system $\mathcal{H} = \{W_i\}_{0\leq i\leq r-1}$ associated to (3.3), the matrix B is described as follows $W_0 = I_{2d}$ and $W_i = \begin{pmatrix} W_{1,1}^{(i,1)} & W_{1,2}^{(i,0)} \\ W_{2,1}^{(i,1)} & W_{2,2}^{(i,0)} \end{pmatrix}$ is such that $W_{1,1}^{(i,1)} = Y_{i+1,1} - a_0Y_{i,1} - \cdots - a_{i-1}Y_{1,1}$, $W_{1,2}^{(i,1)} = Y_{i+1,0} - a_0Y_{i,0} - \cdots - a_{i-1}Y_{1,0}$,$W_{2,1}^{(i,1)} = Y_{i,1} - a_0Y_{i-1,1} - \cdots - a_{i-1}Y_{0,1}$ and $W_{2,2}^{(i,1)} = Y_{i,0} - a_0Y_{i-1,0} - \cdots - a_{i-1}Y_{0,0}$, where the $\{Y_{n,s}\}_{n\geq 0}$ $(0 \leq s \leq 1)$ are the matrix sequences (3.2).*

Especially, for the matrix $W_0 = \begin{pmatrix} W_{1,1}^{(0,1)} & W_{1,2}^{(0,0)} \\ W_{2,1}^{(0,1)} & W_{2,2}^{(0,0)}, \end{pmatrix}$, we have $W_{1,1}^{(0,1)} = Y_{1,1} = I_d$, $W_{1,2}^{(0,1)} = Y_{1,0} = O_d$, $W_{2,1}^{(0,1)} = Y_{0,1} = O_d$, $W_{2,2}^{(0,1)} = Y_{0,0} = I_d$, namely, $W_0 = I_{2d}$. Since the solution $X(t)$ of Equation (1.2) is derived from $Z(t) = e^{tB}Z(0)$, where $Z(t) = (X'(t), X(t))^T$, then Expression (4.8) shows that

$$\begin{bmatrix} X'(t) \\ X(t) \end{bmatrix} = \varphi_0(t) \begin{bmatrix} X'(0) \\ X(0) \end{bmatrix} + \sum_{i=1}^{r-1} \varphi_i(t) \left(W_i \begin{bmatrix} X'(0) \\ X(0) \end{bmatrix} \right).$$

On the other hand, for every $1 \leq i \leq r-1$, we have

$$W_iZ(0) = \begin{pmatrix} W_{1,1}^{(i,1)}X'(0) + W_{1,2}^{(i,0)}X(0) \\ W_{2,1}^{(i,1)}X'(0) + W_{2,2}^{(i,0)}X(0) \end{pmatrix}.$$

Therefore, we obtains

$$X(t) = \varphi_0(t)X(0) + \sum_{i=1}^{r-1} \varphi_i(t) \left[W_{2,1}^{(i,1)}X'(0) + W_{2,2}^{(i,0)}X(0) \right],$$

or equivalently

$$X(t) = \left[\sum_{i=0}^{r-1} \varphi_i(t)W_{2,1}^{(i,1)} \right] X'(0) + \left[\sum_{i=0}^{r-1} \varphi_i(t)W_{2,2}^{(i,0)} \right] X(0),$$

where the $W_{2,1}^{i,s}$, $W_{2,2}^{i,s}$ $(s = 0,\, 1)$ are described in Lemma 4.3. Taking into account Lemma 4.2, we arrive at the following result which aim toward solving the generalized Apostol-Kolodner equation using a linear process and dynamical solution.

Theorem 4.4. *Let consider Equation (1.2), where $A_0 \neq O_d$, $A_1 \neq O_d$ and B its related companion block matrix (3.3). Let $R(z) = z^r - a_0z^{r-1} - \cdots - a_{r-2}z - a_{r-1}$, be the polynomial such that $R(B) = O_{2d}$. Then, the solution of Equation (1.2) is given by*

$$X(t) = \left[\sum_{i=0}^{r-1} \varphi_i(t)W_{2,1}^{(i,1)} \right] X'(0) + \left[\sum_{i=0}^{r-1} \varphi_i(t)W_{2,2}^{(i,0)} \right] X(0),$$

where the $W_{n,s}^{(i,k)}$ are the matrices given in Lemma 4.3 and the $\varphi_j(t) = \sum_{n=0}^{+\infty} u_n \dfrac{t^{n+j}}{(n+j)!}$. Moreover, we have

$$X(t) = \left[\sum_{i=0}^{r-1} \varphi^{(r-i-1)}(t)W_{2,1}^{(i,1)} \right] X'(0) + \left[\sum_{i=0}^{r-1} \varphi^{(r-i-1)}(t)W_{2,2}^{(i,0)} \right] X(0),$$

where $\varphi(t)$ the dynamical solution is as in (4.9) with $\varphi^{(k)}(t) = \dfrac{d^k\varphi}{dt^k}(t)$.

4.2. Study of a particular case of Equation (1.2)

Let illustrate the process of the previous subsection and Theorem 4.4 by studying the following particular case. Consider Equation (1.2), namely, $X''(t) = A_0X'(t) + A_1X(t)$, where A_0 and A_1 are in $\mathbb{C}^{2\times 2}$ and the solution $X \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^{2\times 2})$ is subjected to the initial data $X(0)$ and $X'(0)$. Set $Z(t) = (X^{'}(t), X(t))^T$ $(t \in \mathbb{R})$ and $Z(0) = (X^{'}(0), X(0))^T$. Let B be the matrix (3.3), namely, $B = \begin{pmatrix} A_0 & A_1 \\ I_2 & O_2 \end{pmatrix}$.

Suppose that the characteristic polynomial of B is given by $P(z) = z^4 - a_0z^3 - a_1z^2 - a_2z - a_3$, then the matrices in Lemma 4.3 of the Fibonacci-Hörner system W_i $(0 \le i \le 3)$, are as follows $W_0 = I_4$, $W_1 = B - a_0I_4$, $W_2 = B^2 - a_0B - a_1I_4$, $W_3 = B^3 - a_0B^2 - a_1B - a_2I_4$. A direct computation shows that

$$W_0 = \begin{pmatrix} I_2 & O_2 \\ O_2 & I_2 \end{pmatrix}, \quad W_1 = \begin{pmatrix} A_0 - a_0I_2 & A_1 \\ I_2 & -a_0I_2 \end{pmatrix}, \quad W_2 = \begin{pmatrix} A_0^2 + A_1 - a_0A_0 - a_1I_2 & A_0A_1 - a_0A_1 \\ A_0 - a_0I_2 & A_0A_1 - a_1I_2 \end{pmatrix}$$

and

$$W_3 = \begin{pmatrix} A_0^3 + A_0A_1 + A_1A_0 - a_0A_1 - a_1A_0 - a_2I_2 & A_0^2A_1 + A_1^2 - a_0A_0A_1 - a_1A_1 \\ A_0^2 + A_1 - a_0A_0 - a_1I_2 & A_0A_1 - a_0A_1 - a_2I_2 \end{pmatrix}.$$

Therefore, the matrices $W_{2,1}^{i,s}$, $W_{2,2}^{i,s}$ $(s = 0,\ 1)$ given by in Lemma 4.3 are presented under the form

$$\begin{cases} W_{2,1}^{(0,1)} = O_d,\ W_{2,2}^{(0,0)} = I_d,\ W_{2,1}^{(1,1)} = I_d,\ W_{2,2}^{(1,0)} = -a_0I_2, \\ W_{2,1}^{(2,1)} = A_0 - a_0I_d,\ W_{2,2}^{(2,0)} = A_0A_1 - a_1I_2, \\ W_{2,1}^{(3,1)} = A_0^2 + A_1 - a_0A_0 - a_1I_2,\ W_{2,2}^{(3,0)} = A_0A_1 - a_0A_1 - a_2I_2. \end{cases} \tag{4.10}$$

Now the associated dynamical solution (4.9) is given by

$$\varphi(t) = \sum_{n=0}^{+\infty} u_n \frac{t^{n+3}}{(n+3)!}, \tag{4.11}$$

where $\{u_n\}_{n\ge -2}$ is defined by

$$u_n = \sum_{k_0+2k_1+\cdots+4k_3=n} \frac{(k_0+k_1+k_2+k_3)!}{k_0!k_1!k_2!k_3!} a_0^{k_0}a_1^{k_1}a_2^{k_2}a_3^{k_3},$$

for every $n \ge 0$, with $u_0 = 1$ and $u_{-j} = 0$ for $1 \le j \le 2$. Therefore, the solution of the previous generalized Apostol-Kolodner equation is formulated as follows.

Proposition 4.5. *Consider Equation (1.2), namely, $X''(t) = A_0X'(t) + A_1X(t)$ whose solution $X \in \mathcal{C}^\infty(\mathbb{R},\ \mathbb{C}^{2\times 2})$ is subjected to the prescribed initial data $X(0)$ and $X^{'}(0)$. Let $P(z) = z^4 - a_0z^3 - a_1z^2 - a_2z - a_3$ be the characteristic polynomial of B the associated companion block matrix (3.3). Then, the solution of Equation (1.2) is given by*

$$X(t) = \left[\sum_{i=0}^{3} \varphi_i(t)W_{2,1}^{(i,1)}\right] X'(0) + \left[\sum_{i=0}^{3} \varphi_i(t)W_{2,2}^{(i,0)}\right] X(0),$$

where the matrices $W_{2,j}^{(i,s)}$ are given by (4.10) and $\varphi_j(t) = \sum_{n=0}^{+\infty} u_n \frac{t^{n+j}}{(n+j)!}$, $j = 0,\ 1,\ 2,\ 3$. Moreover, we have

$$X(t) = \left[\sum_{i=0}^{3} \varphi^{(3-i)}(t)W_{2,1}^{(i,1)}\right] X'(0) + \left[\sum_{i=0}^{3} \varphi^{(3-i)}(t)W_{2,2}^{(i,0)}\right] X(0),$$

where $\varphi(t)$ is the dynamical solution given as in (4.11), with $\varphi^{(k)}(t) = \frac{d^k\varphi}{dt^k}(t)$.

For more clarity, we consider the following illustrative numerical example.

Example 4.6. *Consider Equation (1.2), namely,* $X''(t) = A_0X'(t) + A_1X(t)$*, whose solution* $X \in \mathcal{C}^\infty(\mathbb{R},\ \mathbb{C}^{2\times 2})$ *is subjected to the prescribed initial data* $X(0)$ *and* $X^{'}(0)$*. Set* $Z(t) = (X^{'}(t), X(t))^T$ *(*$t \in \mathbb{R}$*) and* $Z(0) = (X^{'}(0), X(0))^T$*. Suppose that the characteristic polynomial of* $B = \begin{pmatrix} A_0 & A_1 \\ I_2 & O_2 \end{pmatrix}$ *is* $P(z) = det(B - zI_4) = z^4 - 2z^3 - 3z^2 + 4z + 1$*. Since* $a_0 = -2$*,* $a_1 = -3$*,* $a_2 = 4$ *and* $a_3 = 1$*, we derive that the matrices* W_i *(*$0 \le i \le 3$*) of the Fibonacci-Hörner system, are given by*

$$\begin{cases} W_0 = I_4, \quad W_1 = B - a_0I_4 = B + 2I_4, \\ W_2 = B^2 - a_0B - a_1I_4 = B^2 + 2B + 3I_4, \\ W_3 = B^3 - a_0B^2 - a_1B - a_2I_4 = B^3 + 2B^2 + 3B - 4I_4. \end{cases} \tag{4.12}$$

Then, using Expression (4.12), we can establish that the matrices $W_{2,j}^{(i,s)}$ *are*

$$\begin{cases} W_{2,1}^{(0,1)} = O_4, \quad W_{2,2}^{(0,1)} = I_4, \quad W_{2,1}^{(1,1)} = I_4, \quad W_{2,2}^{(1,0)} = 2I_4, \\ W_{2,1}^{(2,1)} = A_1 + 2I_4, \quad W_{2,2}^{(2,0)} = A_0A_1 + 2I_4, \\ W_{2,1}^{(3,1)} = A_0^2 + 2A_0 + A_1 + 3I_4, \quad W_{2,2}^{(3,0)} = A_0A_1 + 2A_1 - 4I_4. \end{cases} \tag{4.13}$$

The associated dynamical solution is given by

$$\varphi(t) = \sum_{n=0}^{+\infty} u_n \frac{t^{n+3}}{(n+3)!}, \tag{4.14}$$

where $\{u_n\}_{n\ge -2}$ *is defined by*

$$u_n = \sum_{k_0+2k_1+\cdots+4k_3=n} \frac{(k_0+k_1+k_2+k_3)!}{k_0!k_1!k_2!k_3!}(-2)^{k_0}(-3)^{k_1}4^{k_2}1^{k_3},$$

for every $n \ge 0$*, with* $u_0 = 1$ *and* $u_{-j} = 0$ *for* $1 \le j \le 2$*. Therefore, the solution of the preceding generalized matrix differential equation of Apostol-Kolodner is*

$$X(t) = \left[\sum_{i=0}^{3} \varphi^{(3-i)}(t)W_{2,1}^{(i,1)}\right] X'(0) + \left[\sum_{i=0}^{3} \varphi^{(3-i)}(t)W_{2,2}^{(i,0)}\right] X(0),$$

where $\varphi(t)$ *is the dynamical solution (4.14) and the matrices* $W_{2,j}^{(i,s)}$ *are as in (4.13).*

Theorem 4.4 shows that the solution of the generalized Apostol-Kolodner equation are obtained in terms of the linear process (3.1), (3.2) and the dynamical solution (4.9), related to matrix exponential e^{tB}. Moreover, the two matrices $A_0 \ne O_d$ and $A_1 \ne O_d$ do not necessarily commute.
When $A_0A_1 = A_1A_0$, Expression (3.7) shows that $Y_{n,0} = A_1\rho(n,2)$ and $Y_{n,1} = A_0\rho(n,2) + A_1\rho(n-1,2)$ for $n \ge 2$, where $\rho(n,2)$ is given by (3.6). Hence, we derive that $W_{i,s}$ the matrices in Lemma 4.3 take the following form

$$W_{i,0} = A_1\rho(i,2) - A_1\sum_{j=0}^{i-1} a_i\rho(i-j-1,2), \tag{4.15}$$

$$W_{i,1} = A_0\rho(i,2) + A_1\rho(i,2) - \sum_{j=0}^{i-1} a_i\left[A_0\rho(i-j-1,2) - A_1\rho(i-j-2,2)\right], \tag{4.16}$$

where $\rho(i,2) = 0$, for $i \le 1$. Expressions (4.15) and (4.16) allow us to obtain the following corollary of Theorem 4.4.

Corollary 4.7. *Suppose the data of Theorem 4.4. Then, if the commutativity conditions* $A_0A_1 = A_1A_0$ *is verified, we have*

$$X(t) = \left[\sum_{i=0}^{r-1} \varphi^{(r-1-i)}(t)W_{i,1}\right] X'(0) + \left[\varphi_0(t)I_d + \sum_{i=1}^{r-1} \varphi^{(r-1-i)}(t)W_{i,0}\right] X(0),$$

where the $W_{i,s}$ *are the matrices given by (4.15) and (4.16), and* φ *the dynamical solution is as in (4.9).*

5. Fibonacci-Hörner method and analytical approach for solving Equation (1.2): Study of the simple case

In this section, we are concerned with the analytical approach of Equations (1.1) and (1.2) based on the Fibonacci-Hörner method.

5.1. Study of the simple case

It is known in the literature that the sequence $\{u_n\}_{n\geq -r+1}$ satisfies the linear recurrence relation (4.4) of order r (see, for instance, [4,20,26]), namely, $u_{n+1} = a_0u_n + a_1u_{n-1} + \cdots + a_{r-1}u_{n-r+1}$ for every $n \geq 0$, with initial data $u_0 = 1$ and $u_{-j} = 0$ for $1 \leq j \leq r-1$. Moreover, sequence (4.4) owns an analytical expression in terms of the roots of the (characteristic) polynomial $R(z) = z^r - a_0z^{r-1} - \cdots - a_{r-2}z - a_{r-1}$ and the initial data. More precisely, we have

$$u_n = \sum_{k=1}^{s}\left[\sum_{j=0}^{m_k-1}\beta_{kj}n^j\right]\lambda_k^n, \tag{5.1}$$

for every $n \geq -r+1$, where the λ_k $(1 \leq k \leq s)$ are the roots of the polynomial $R(z)$ of multiplicities m_k $(1 \leq k \leq s)$, respectively. The scalars β_{kj} $(1 \leq k \leq s,\ 0 \leq j \leq m_k - 1)$ are obtained by solving a generalized Vandermonde linear system of equations (see, for example, [6,11,26]). Especially, when the roots of the polynomial $R(z)$ are simple, then, the analytic Expression (5.1) takes the simple form $u_n = \sum_{k=1}^{r}\beta_k\lambda_k^n$, where the scalars β_k $(1 \leq k \leq r)$ are determined by solving an usual Vandermonde system. It was established in [5,6] that the scalars β_k $(1 \leq k \leq r)$ are obtained from the following useful lemma.

Lemma 5.1 ([5,6]). *Suppose that the roots $\lambda_1, \ldots, \lambda_r$ of $R(z) = z^r - a_1z^{r-1} - \cdots - a_{r-2}z - a_r$ $(a_r \neq 0)$ satisfy $\lambda_i \neq \lambda_j$ for $i \neq j$. Then, we have*

$$u_n = \sum_{i=1}^{r}\frac{\lambda_i^{n-1}}{R'(\lambda_i)} = \sum_{i=1}^{r}\frac{\lambda_i^{n-1}}{\prod_{k\neq i}(\lambda_i - \lambda_k)} \ \textit{for every}\ \ n \geq 1, \tag{5.2}$$

otherwise $u_0 = 1$, $u_{-j} = 0$ for $1 \leq j \leq r-1$, where $R'(z) = \frac{dR}{dz}(z)$.

The dynamical solution (4.8), namely, $\varphi(t) = \sum_{n=0}^{+\infty}u_n\frac{t^{n+r-1}}{(n+r-1)!}$ is expressed in terms of the coefficients of the recursive sequence $\{u_n\}_{n\geq -r+1}$ defined by (4.7). Therefore, using Expression (5.2) of Lemma 5.1, we can express the dynamical solution in terms of the simple roots $\lambda_1, \cdots, \lambda_r$ of the polynomial $R(z)$. That is, we have

$$\varphi(t) = \sum_{n=0}^{+\infty}u_n\frac{t^{n+r-1}}{(n+r-1)!} = \sum_{n=0}^{+\infty}\left[\sum_{i=1}^{r}\frac{\lambda_i^{n-1}}{R'(\lambda_i)}\right]\frac{t^{n+r-1}}{(n+r-1)!}.$$

Therefore, we obtain $\varphi(t) = -\sum_{n=0}^{r-2}\left[\sum_{i=1}^{r}\frac{\lambda_i^n}{\lambda_i^rR'(\lambda_i)}\right]\frac{t^n}{n!} + \sum_{i=1}^{r}\frac{1}{\lambda_i^rR'(\lambda_i)}e^{\lambda_it}$. In summary, we have established the following theorem whose goal is to solve the generalized Apostol-Kolodner equation using analytic process and dynamical solution.

Theorem 5.2. *Let consider Equation (1.2), where $A_0 \neq O_d$, $A_1 \neq O_d$ and B the associated companion block matrix (3.3). Let $R(z) = z^r - a_0z^{r-1} - \cdots - a_{r-2}z - a_{r-1}$ be the polynomial such that $R(B) = O_{2d}$. Then, the solution is given by*

$$X(t) = \left[\sum_{i=0}^{r-1}\varphi^{(r-1-i)}(t)W_{2,1}\right]X'(0) + \left[\varphi_0(t)I_d + \sum_{i=1}^{r-1}\varphi^{(r-1-i)}(t)W_{2,2}\right]X(0),$$

where the $W_{n,s}$ $(1 \leq n,\, s \leq 2)$ *are the matrices in Lemma 4.3 and* φ *is the dynamical solution* $\varphi(t) = -H_r(t) + \sum_{i=1}^{r} \frac{1}{\lambda_i^r R'(\lambda_i)} e^{\lambda_i t}$ *such that* $H_r(t)$ *is the polynomial* $H_r(t) = \sum_{n=0}^{r-2} \left[\sum_{i=1}^{r} \frac{\lambda_i^n}{\lambda_i^r R'(\lambda_i)} \right] \frac{t^n}{n!}$.

Note that the expression $\varphi_0(t) = \sum_{n=0}^{+\infty} u_n \frac{t^n}{n!}$ can be obtained by a straightforward computation, using the formula (5.2), as follows $\varphi_0(t) = \sum_{i=1}^{r} \frac{1}{\lambda_i R'(\lambda_i)} e^{\lambda_i t}$. The preceding formula can be also deduced from (4.9), utilizing from the fact that $\varphi_0(t) = \varphi^{(r-1)}(t)$.

Let illustrate the preceding result of Theorem 5.2 by the following numerical example.

Example 5.3. *Consider the Equation (1.2), studied in Example 4.6, namely,* $X''(t) = A_0 X'(t) + A_1 X(t)$, *whose solution* $X \in \mathcal{C}^\infty(\mathbb{R},\, \mathbb{C}^{2\times 2})$ *is subjected to the prescribed initial data* $X(0)$ *and* $X'(0)$. *Set* $Z(t) = (X'(t), X(t))^T$ $(t \in \mathbb{R})$ *and* $Z(0) = (X'(0), X(0))^T$. *Suppose that the characteristic polynomial of* $B = \begin{pmatrix} A_0 & A_1 \\ I_2 & O_2 \end{pmatrix}$ *is* $P(z) = z^4 - 2z^3 - 3z^2 + 4z + 1$. *Hence, we have* $a_0 = -2$, $a_1 = -3$, $a_2 = 4$ *and* $a_3 = 1$. *It was established by a straightforward computation, using Expression (4.12), the entries* $W_{2,j}^{(i,s)}$ $(s = 0$ *or* 1 *and* $0 \leq i \leq 3)$ *of the* W_i, *are given by (4.13), namely,*

$$\begin{cases} W_{2,1}^{(0,1)} = O_d, \quad W_{2,2}^{(0,1)} = I_d, \quad W_{2,1}^{(1,1)} = I_d, \quad W_{2,2}^{(1,0)} = 2I_d, \\ W_{2,1}^{(2,1)} = A_1 + 2I_d, \quad W_{2,2}^{(2,0)} = A_0 A_1 + 2I_d, \\ W_{2,1}^{(3,1)} = A_0^2 + 2A_0 + A_1 + 3I_d, \quad W_{2,2}^{(3,0)} = A_0 A_1 + 2A_1 - 4I_d. \end{cases}$$

On the other hand, using any numerical computing software we can show that the roots of the polynomial $P(z)$ *are simple, and thus we have* $P(z) = \prod_{i=1}^{4} (z - \lambda_i)$, *where* $\lambda_1 \approx -1.49550$, $\lambda_2 \approx -0.21968$, $\lambda_3 \approx 1.21968$, $\lambda_4 \approx 2.49550$. *The associated dynamical solution is given by*

$$\varphi(t) = -H_4(t) + \sum_{i=1}^{4} \frac{1}{\lambda_i^4 P'(\lambda_i)} e^{\lambda_i}, \tag{5.3}$$

where $H_4(t) = \sum_{i=1}^{4} \frac{1}{\lambda_i^r P'(\lambda_i)} + \left[\sum_{i=1}^{4} \frac{\lambda_i}{\lambda_i^r P'(\lambda_i)} \right] \frac{t}{1!} + \left[\sum_{i=1}^{4} \frac{\lambda_i^2}{\lambda_i^r P'(\lambda_i)} \right] \frac{t^2}{2!}$. *Therefore, the solution of the preceding generalized matrix differential equation of Apostol-Kolodner is,*

$$X(t) = \left[\sum_{i=0}^{3} \varphi^{(3-i)}(t) W_{2,1}^{(i,1)} \right] X'(0) + \left[\sum_{i=0}^{3} \varphi^{(3-i)}(t) W_{2,2}^{(i,0)} \right] X(0),$$

where $\varphi(t)$ *is the dynamical solution (5.3) and the the matrices* $W_{2,j}^{(i,s)}$ $(s = 0$ *or* 1 *and* $0 \leq i \leq 3)$ *are as in (4.13).*

5.2. Study of the general setting

Let consider the dynamical solution as in (4.9), namely, $\varphi(t) = \sum_{n=0}^{+\infty} u_n \frac{t^{n+r-1}}{(n+r-1)!}$, where the sequence $\{u_n\}_{n \geq -r+1}$ satisfies the linear recursive equation (4.4), with initial data $u_0 = 1$ and $u_{-j} = 0$ for $1 \leq j \leq r-1$. In the general setting, the analytical expression of the sequence $\{u_n\}_{n \geq -r+1}$ is expressed as follows, $u_n = \sum_{k=1}^{s} \left[\sum_{j=0}^{m_k - 1} \beta_k^{(j)} n^j \right] \lambda_k^n$, for every $n \geq -r+1$, where the λ_k $(1 \leq k \leq s)$ are the roots of the polynomial $R(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-2} z - a_{r-1}$, of multiplicities m_k $(1 \leq k \leq s)$, respectively. The scalars β_{kj} ($1 \leq k \leq s$, $0 \leq j \leq m_k - 1$) are computed by solving a generalized Vandermonde linear system of equations (see, for example, [6,11,26]). For every $n \geq 0$, we set $u_n = \sum_{k=1}^{s} Q_k(n)\lambda_k^n$, where $Q_k(n) = \sum_{j=0}^{m_k-1} \beta_{kj} n^j$. Now, we substitute the former analytic expression of u_n in Expression (4.9) of the dynamic solution. Then, we have

$$\varphi(t) = \sum_{n=0}^{+\infty} u_n \frac{t^{n+r-1}}{(n+r-1)!} = \sum_{k=1}^{s}\sum_{n=0}^{+\infty} Q_k(n)\lambda_k^n \frac{t^{n+r-1}}{(n+r-1)!}.$$

Hence, the former function $\varphi(t)$ can be written under the form $\varphi(t) = \sum_{k=1}^{s} \psi_k(t)$, where

$$\psi_k(t) = \sum_{n=0}^{+\infty}\sum_{j=0}^{m_k-1} \beta_k^{(j)} n^j \frac{t^{n+r-1}}{(n+r-1)!}\lambda_k^n.$$

Then, we have

$$\psi_k(t) = \sum_{j=0}^{m_k-1} \frac{\beta_k^{(j)}}{\lambda_k^{r-1}} \sum_{n=r-1}^{+\infty}\sum_{p=0}^{j} (-1)^{j-p}\binom{j}{p} n^p (r-1)^{j-p}\frac{t^n}{n!}\lambda_k^n.$$

Let consider the derivation $D = t\dfrac{d}{dt}$, then the function $\psi_k(t)$ takes the form

$$\psi_k(t) = \sum_{j=0}^{m_k-1}\sum_{p=0}^{j} (-1)^{j-p}\binom{j}{p}(r-1)^{j-p}\frac{\beta_k^{(j)}}{\lambda_k^{r+p-1}} D^p\left[\sum_{n=r-1}^{+\infty}\frac{(\lambda_k t)^n}{n!}\right].$$

From the identity $\sum_{h=0}^{j}\sum_{i=0}^{p} z_{k,i} = \sum_{i=0}^{p}\sum_{h=i}^{j} z_{h,i}$, we derive that

$$\psi_k(t) = \left[\sum_{p=0}^{m_k-1}\frac{1}{\lambda_k^{r+p-1}}\left(\sum_{j=p}^{m_k-1}(-1)^{j-p}\binom{j}{p}(r-1)^{j-p}\beta_k^{(j)}\right) D^p\right]\Omega_k(t),$$

where $\Omega_k(t) = \sum_{n=r-1}^{+\infty}\dfrac{(\lambda_k t)^n}{n!}$. Therefore, the dynamical solution can be written under the form $\varphi(t) = \sum_{k=1}^{s}\Gamma_k(D)\Omega_k(t)$, where $\Gamma_k(D)$ is the differential operator given by

$$\Gamma_k(D) = \sum_{p=0}^{m_k-1}\frac{1}{\lambda_k^{r+p-1}}\left(\sum_{j=p}^{m_k-1}(-1)^{j-p}\binom{j}{p}(r-1)^{j-p}\beta_k^{(j)}\right) D^p. \tag{5.4}$$

Now, we can formulate the analytic solution of Equation (1.2) when the minimal polynomial of the matrix B owns some roots of multiplicities $m_k \geq 2$.

Theorem 5.4. *Let consider Equation (1.2), where $A_0 \neq O_d$ and $A_1 \neq O_d$ and B its related companion block matrix (3.3). Let $R(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-2}z - a_{r-1} = \prod_{k=1}^{s}(z-\lambda_k)^{m_k}$, with $m_k \geq 1$, be the polynomial such that $P(B) = O_{2d}$. Then, the solution of Equation (1.2) is given by*

$$X(t) = \left[\sum_{i=0}^{r-1}\varphi^{(r-i-1)}(t)W_{2,1}^{(i,1)}\right]X'(0) + \left[\sum_{i=0}^{r-1}\varphi^{(r-i-1)}(t)W_{2,2}^{(i,0)}\right]X(0),$$

where $W_{2,s}$ ($1 \leq s \leq 2$) are the matrices described in Lemma 4.3 and $\varphi(t)$ the dynamical solution is $\varphi(t) = \sum_{k=1}^{s}\Gamma_k(D)\Omega_k(t)$, with $\Omega_k(t) = \sum_{n=r-1}^{+\infty}\dfrac{(\lambda_k t)^n}{n!}$, $D = t\dfrac{d}{dt}$ and $\Gamma_k(D)$ is the differential operator given by Expression (5.4).

Let $E_k(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \ldots + \frac{1}{k!}x^k$ be the exponential polynomial of degree k. For $r \geq 3$, we show that the function $\Omega_k(t) = \sum_{n=r-1}^{+\infty} \frac{(\lambda_k t)^n}{n!}$ can takes the form

$$\Omega_k(t) = \sum_{n=r-1}^{+\infty} \frac{(\lambda_k t)^n}{n!} = -E_{r-2}(\lambda_k t) + \sum_{n=0}^{+\infty} \frac{(\lambda_k t)^n}{n!} = -E_{r-2}(\lambda_k t) + e^{\lambda_k t}.$$

Therefore, the dynamical solution can be written under the form

$$\varphi(t) = \sum_{k=1}^{s} \Gamma_k(D)\Omega_k(t) = -\sum_{k=1}^{s} \Gamma_k(D)E_{r-2}(\lambda_k t) + \sum_{k=1}^{s} \Gamma_k(D)e^{\lambda_k t},$$

where $D = t\frac{d}{dt}$ and $\Gamma_k(D)$ is the differential operator given by Expression (5.4).

Corollary 5.5. *Consider the Equation (1.2), where $A_0 \neq O_d$, $A_1 \neq O_d$ and B its related companion block matrix (3.3). Let $R(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-2}z - a_{r-1} = \prod_{k=1}^{s}(z-\lambda_k)^{m_k}$, with $m_k \geq 1$, be the polynomial such that $R(B) = O_{2d}$. Then, the solution of Equation (1.2) is given by*

$$X(t) = \left[\sum_{i=0}^{r-1} \varphi^{(r-i-1)}(t) W_{2,1}^{(i,1)}\right] X'(0) + \left[\sum_{i=0}^{r-1} \varphi^{(r-i-1)}(t) W_{2,2}^{(i,0)}\right] X(0),$$

where $W_{2,s}^{i,j}$ $(s = 1,2,\ j = 0,1)$ are the matrices in Lemma 4.3 and $\varphi(t)$ is the dynamical solution given by

$$\varphi(t) = -\sum_{k=1}^{s} \Gamma_k(D)E_{r-2}(\lambda_k t) + \sum_{k=1}^{s} \Gamma_k(D)e^{\lambda_k t},$$

with $D = t\frac{d}{dt}$ and $\Gamma_k(D)$ is the differential operator given by Expression (5.4), namely,

$$\Gamma_k(D) = \sum_{p=0}^{m_k-1} \frac{1}{\lambda_k^{r+p-1}} \left(\sum_{j=p}^{m_k-1} (-1)^{j-p} \binom{j}{p} (r-1)^{j-p} \beta_k^{(j)}\right) D^p.$$

We illustrate Theorem 5.4 and Corollary 5.5 with the following numerical example.

Example 5.6. *Consider the Equation (1.2), where $A_0 \neq O_d$, $A_1 \neq O_d$ and B its related companion block matrix (3.3). Let $R(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-2}z - a_{r-1}$ be the polynomial such that $R(B) = O_{2d}$. Suppose that $R(z) = (z-\lambda_1)^2 \prod_{k=2}^{r-1}(z-\lambda_k)$, where $\lambda_1 \neq \lambda_k$, for $2 \leq k \leq r-1$, and $\lambda_k \neq \lambda_j$, for $2 \leq j \neq k \leq r-1$. Thus, the analytic expression of $\{u_n\}_{n \geq -r+1}$ is a recursive sequence (4.7), can be written under the form, $u_n = Q_1(n)\lambda_1^n + \sum_{k=2}^{r-1} \beta_k \lambda_k^n$, for every $n \geq 0$, where $Q_1(n) = \alpha_1 + \alpha_2 n$. Therefore, the dynamical solution can be written under the form $\varphi(t) = \Delta_1(t) + \sum_{k=2}^{r-1} \Omega_k(t)$, such that $\Delta_1(t) = \Gamma_1(D)\Omega_1(t)$ with $D = t\frac{d}{dt}$, $\Gamma_1(D) = \alpha_1 + \frac{\alpha_2}{\lambda_1}D$ and $\Omega_k(t) = \sum_{n=r-1}^{+\infty} \frac{(\lambda_k t)^n}{n!} = -E_{r-2}(\lambda_k t) + \sum_{n=0}^{+\infty} \frac{(\lambda_k t)^n}{n!} = -E_{r-2}(\lambda_k t) + e^{\lambda_k t}$, where $E_{r-2}(z)$ is the exponential polynomial of degree $r-2$. On the other side, we have*

$$\Delta_1(t) = \Gamma_1(D)\Omega_1(t) = \sum_{n=0}^{+\infty} Q_1(n) \frac{t^{n+r-1}}{(n+r-1)!} \lambda_1^n = \sum_{n=r-1}^{+\infty} \frac{Q_1(n-r+1)}{\lambda_1^{r-1}} \frac{(\lambda_1 t)^n}{n!}.$$

Since $Q_1(n) = \alpha_1 + \alpha_2 n$, *we derive that*

$$\Delta_1(t) = \sum_{n=r-1}^{+\infty} \frac{\alpha_1 + \alpha_2(n-r+1)}{\lambda_1^{r-1}} \frac{(\lambda_1 t)^n}{n!} = C_1 \left[e^{\lambda_1 t} - E_{r-2}(\lambda_1 t)\right] + \frac{C_2}{\lambda_1} D \left[e^{\lambda_1 t} - E_{r-2}(\lambda_1 t)\right],$$

and a direct computation shows that

$$\Delta_1(t) = \Gamma_1(D)\Omega_1(t) = (C_1 + C_2 t)e^{\lambda_1 t} - (C_1 + \frac{C_2}{\lambda_1} D)E_{r-2}(\lambda_1 t).$$

Therefore, the associated dynamical solution is given by

$$\varphi(t) = \Delta_1(t) + \sum_{k=2}^{r-1} \beta_k e^{\lambda_k t} - \sum_{k=2}^{r-1} \beta_k E_{r-2}(\lambda_k t), \tag{5.5}$$

with

$$\Delta_1(t) = (C_1 + C_2 t)e^{\lambda_1 t} - (C_1 + \frac{C_2}{\lambda_1} D)E_{r-2}(\lambda_1 t). \tag{5.6}$$

A direct calculation shows that $\Delta_1(t) = (C_1 + C_2 t)\left[e^{\lambda_1 t} - E_{r-3}(\lambda_1 t)\right] - C_1 \frac{(\lambda_1 t)^{r-2}}{(r-2)!}$. *Therefore, the solution of Equation* (1.2) *is*

$$X(t) = \left[\sum_{i=0}^{r-1} \varphi^{(r-i-1)}(t) W_{2,1}^{(i,1)}\right] X'(0) + \left[\sum_{i=0}^{r-1} \varphi^{(r-i-1)}(t) W_{2,2}^{(i,0)}\right] X(0),$$

where $W_{2,s}^{i,j}$ $(s = 1, 2,\ j = 0, 1)$ *are the matrices described in Lemma 4.3 and* $\varphi(t)$ *is the dynamical solution given by* (5.5) *and* (5.6).

5.3. Linear and analytic approach of Equation (1.1)

This subsection is devoted to the study of the linear and analytic approaches for Equation (1.1). It was established in Proposition 2.1 that the unique solution is specified by $X(t) = C_1(t)X(0) + C_2(t)X'(0)$, for every $t \in]-\infty, +\infty[$, submitted to the initial data $X(0)$ and $X'(0)$, where $C_1(t)$ and $C_2(t)$ are as in (2.1), namely, $C_1(t) = \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!} A^n$ and $C_2(t) = \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)!} A^n$. Let $R(z) = z^r - b_0 z^{r-1} - \cdots - b_{r-1}$ be a polynomial such that $R(A) = O_d$. Then, the Fibonacci-Hörner decomposition for the powers A^n of the matrix A, is determined by Expressions (4.1) and (4.3), namely,

$$A^n = v_n U_0 + v_{n-1} U_1 + \cdots + v_{n-r+1} U_{r-1}, \quad \text{for every} \ \ n \geq 0, \tag{5.7}$$

where $U_0 = I_r$; $U_i = A^i - b_0 A^{i-1} - \cdots - b_{i-1} I_r$, for $i = 1, \ldots, r-1$, and the sequence $\{v_n\}_{n \geq -r+1}$ is defined by $v_n = \sum_{k_0 + 2k_1 + \cdots + rk_{r-1} = n} \frac{(k_0 + k_1 + \cdots + k_{r-1})!}{k_0! k_1! \cdots k_{r-1}!} b_0^{k_0} b_1^{k_1} \cdots b_{r-1}^{k_{r-1}}$, for every $n \geq -r+1$, with initial values $v_0 = 1$ and $v_{-j} = 0$ for $1 \leq j \leq r-1$ (for more details, see [4,7,20,21]). Moreover, it was established in [4,20,26] that the sequence $\{v_n\}_{n \geq -r+1}$ satisfies the following recursive relation $v_{n+1} = b_0 v_n + b_1 v_{n-1} + \cdots + b_{r-1} v_{n-r+1}$, for every $n \geq 0$. Using Expression (5.7), we show that $C_1(t) = \sum_{k=0}^{r-1} \left[\sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!} v_{n-k}\right] U_k$ and $C_2(t) = \sum_{k=0}^{r-1} \left[\sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)!} v_{n-k}\right] U_k$. For every k $(0 \leq k \leq r-1)$, we have

$$\psi_k(t) = \sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!} v_{n-k} = \sum_{n=0}^{+\infty} \frac{t^{2(n+k)}}{(2(n+k))!} v_n,\ \ \phi_k(t) = \sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2n+1)!} v_{n-k} = \sum_{n=0}^{+\infty} \frac{t^{2(n+k)+1}}{(2(n+k)+1)!} v_n.$$

If we consider

$$\psi_{r-1}(t) = \sum_{n=0}^{+\infty} \frac{t^{2(n+r-1)}}{(2(n+r-1))!} v_n \text{ and } \phi_{r-1}(t) = \sum_{n=0}^{+\infty} \frac{t^{2(n+r-1)+1}}{(2(n+r-1)+1)!} v_n, \tag{5.8}$$

then, a direct computation permits us to show that for every k $(0 \leq k \leq r-1)$, we have $\psi_k(t) = \psi_{r-1}^{(2k)}(t)$ and $\phi_k(t) = \phi_{r-1}^{(2k)}(t)$, with $\psi^{(k)}(t) = \dfrac{d^k\psi}{dt^k}(t)$ and $\phi^{(k)}(t) = \dfrac{d^k\phi}{dt^k}(t)$. Therefore, we get the following analogous result of Theorem 4.4.

Theorem 5.7. *Let consider the Equation (1.1), where the solution is submitted to initial data $X(0)$ and $X'(0)$. Let $R(z) = z^r - a_0z^{r-1} - \cdots - a_{r-2}z - a_{r-1}$, be the polynomial such that $R(A) = O_d$. Then, its unique solution is given by,*

$$X(t) = \left[\sum_{k=0}^{r-1} \psi_k(t)U_k\right] X(0) + \left[\sum_{k=0}^{r-1} \phi_k(t)U_k\right] X'(0), \tag{5.9}$$

where $U_0 = I_r$; $U_i = A^i - b_0A^{i-1} - \cdots - b_{i-1}I_r$, for $i = 1, \ldots, r-1$, $\psi_k(t) = \displaystyle\sum_{n=0}^{+\infty} \frac{t^{2(n+k)}}{(2(n+k))!}v_n$ and $\phi_k(t) = \displaystyle\sum_{n=0}^{+\infty} \frac{t^{2(n+k)+1}}{(2(n+k)+1)!}v_n$. Moreover, we have

$$X(t) = \left[\sum_{k=0}^{r-1} \psi_{r-1}^{(2k)}(t)U_k\right] X(0) + \left[\sum_{k=0}^{r-1} \phi_{r-1}^{(2k)}(t)U_k\right] X'(0), \tag{5.10}$$

where $\psi_{r-1}(t)$ and $\phi_{r-1}(t)$ are given as in (5.8), with $\psi^{(k)}(t) = \dfrac{d^k\psi}{dt^k}(t)$ and $\phi^{(k)}(t) = \dfrac{d^k\phi}{dt^k}(t)$.

We can observe that Theorem 5.7 is analogous to Theorem 4.4. Indeed, Expression (5.9) represents the Fibonacci-Hörner decomposition of the unique solution of Equation (1.1), and Expression (5.10) shows that the functions $\psi_{r-1}(t)$ and $\phi_{r-1}(t)$ represent the related dynamical solutions.

The two fundamental functions $\psi_{r-1}(t)$ and $\phi_{r-1}(t)$ considered in (5.8), are expressed in terms of the sequence $\{v_n\}_{n\geq -r+1}$ exhibited in the Fibonacci-Hörner decomposition (5.7). Since $\{v_n\}_{n\geq -r+1}$ satisfies the recursive Equation (4.4) of order r, namely, $v_{n+1} = a_0v_n + a_1v_{n-1} + \cdots + a_{r-1}v_{n-r+1}$, for every $n \geq 0$, with initial data $v_0 = 1$ and $v_{-j} = 0$ for $1 \leq j \leq r-1$, hence this sequence owns an analytical form, with the aid of the roots of the polynomial $R(z) = z^r - a_0z^{r-1} - \cdots - a_{r-2}z - a_{r-1}$ and the initial data (see, for example, [6,11,26]).

For the sake of simplicity and length of the text, we assume that the roots λ_k $(1 \leq k \leq s)$ of the polynomial $R(z) = z^r - a_0z^{r-1} - \cdots - a_{r-2}z - a_{r-1}$ are simple. In this special case, the analytic expression of the sequence $\{v_n\}_{n\geq -r+1}$, is as follows $v_n = \displaystyle\sum_{k=1}^{r-1} \beta_k\lambda_k^n$, where the scalars β_k $(1 \leq k \leq s)$ are obtained by solving a Vandermonde linear system of equations (see, for example, [6,11,26]). Here the scalars β_k $(1 \leq k \leq r)$ are obtained from the Lemma 5.1, and we have

$$v_n = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{R'(\lambda_i)} = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)} \text{ for every } n \geq 1,$$

otherwise $u_0 = 1$, $u_{-j} = 0$ for $1 \leq j \leq r-1$, where $P'(z) = \frac{dR}{dz}(z)$. Therefore, the former functions $\psi_k(t) = \sum_{n=0}^{+\infty} \dfrac{t^{2n}}{(2n)!}v_{n-k} = \sum_{n=0}^{+\infty} \dfrac{t^{2(n+k)}}{(2(n+k))!}v_n$ and $\phi_k(t) = \sum_{n=0}^{+\infty} \dfrac{t^{2n+1}}{(2n+1)!}v_{n-k} = \sum_{n=0}^{+\infty} \dfrac{t^{2(n+k)+1}}{(2(n+k)+1)!}v_n$ take the following form

$$\psi_k(t) = \sum_{j=1}^{r} \frac{1}{\lambda_j^{k+1}R'(\lambda_j)} \left[\sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!}\lambda_j^n\right] \text{ and } \phi_k(t) = \sum_{j=1}^{r} \frac{1}{\lambda_j^{k+1}R'(\lambda_j)} \left[\sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2(n+1))!}\lambda_j^n\right].$$

For every λ_j $(1 \leq j \leq r)$, let μ_j $(1 \leq j \leq r)$ be its principal square root. Then, we have

$$\psi_k(t) = \sum_{j=1}^{r} \frac{1}{\lambda_j^{k+1} R'(\lambda_j)} \cosh(t\mu_j) \text{ and } \phi_k(t) = \sum_{j=1}^{r} \frac{1}{\lambda_j^{k+1} R'(\lambda_j)} \sinh(t\mu_j).$$

In summary, we have the following result.

Theorem 5.8. *Let consider Equation (1.1), where the solution is submitted to initial data $X(0)$ and $X'(0)$. Let $R(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-2} z - a_{r-1}$, be the polynomial such that $R(A) = O_d$. Assume that, $R(z)$ owns simple roots λ_j $(1 \leq j \leq r)$. Then, the unique solution of Equation (1.1), is as follows*

$$X(t) = \left[\sum_{k=0}^{r-1} \psi_k(t) U_k\right] X(0) + \left[\sum_{k=0}^{r-1} \phi_k(t) U_k\right] X'(0),$$

where $U_0 = I_r$, $U_i = A^i - b_0 A^{i-1} - \cdots - b_{i-1} I_r$, for $i = 1, \ldots, r-1$ and $\psi_k(t)$ and $\phi_k(t)$ are given. Moreover, we have

$$\psi_k(t) = \sum_{j=1}^{r} \frac{1}{\lambda_j^{k+1} R'(\lambda_j)} \left[\sum_{n=0}^{+\infty} \frac{t^{2n}}{(2n)!} \lambda_j^n\right], \ \phi_k(t) = \sum_{j=1}^{r} \frac{1}{\lambda_j^{k+1} R'(\lambda_j)} \left[\sum_{n=0}^{+\infty} \frac{t^{2n+1}}{(2(n+1))!} \lambda_j^n\right].$$

If μ_j $(1 \leq j \leq r)$ are the principal square root of λ_j $(1 \leq j \leq r)$, respectively, we get

$$X(t) = \left[\sum_{k=0}^{r-1} \left(\sum_{j=1}^{r} \frac{1}{\lambda_j^{k+1} R'(\lambda_j)} \cosh(t\mu_j)\right) U_k\right] X(0) + \left[\sum_{k=0}^{r-1} \left(\sum_{j=1}^{r} \frac{1}{\lambda_j^{k+1} R'(\lambda_j)} \sinh(t\mu_j)\right) U_k\right] X'(0).$$

We can observe that, for Equation (1.1), Theorem 5.8 is analogous to Theorem 5.2. Moreover, the principal square root of the matrix A considered in Proposition 2.2, is analogous to the principal square root of the simple eigenvalues λ_j $(1 \leq j \leq r)$ of A.

6. Equation (1.2): another approach for the commutative case

When the commutativity condition, $A_0 A_1 = A_1 A_0$, is fulfilled, it was established in Proposition 3.4 that Equation (1.2) owns a unique solution $X(t)$, with the prescribed initial data $X'(0)$ and $X(0)$ given by

$$X(t) = [I_d + A_1 g_0(t)] X(0) + [I_d t + A_0 g_0(t) + A_1 g_1(t)] X'(0),$$

where $g_0(t)$ and $g_1(t)$, are the exponential generating matrix functions of $\{\rho(n,2)\}_{n\geq 0}$ and $\{\rho(n-1,2)\}_{n\geq 0}$ (respectively), with $\rho(n,2)$ is as (3.6). And we can verify that $\frac{dg_1}{dt}(t) = g_0(t)$ or equivalently $g_1(t) = \int_0^t g_0(x)dx$.

Suppose that $P_1(A_0) = P_2(A_1) = O_d$, for some polynomials $P_1(z)$ and $P_2(z)$. For clarity, we consider $P_1(z) = \det(A_0 - zI_d)$ and $P_2(z) = \det(A_1 - zI_d)$ are the characteristic polynomial of A_0 and A_1, respectively. For reason of simplicity, we suppose that the eigenvalues $\{\lambda_1, \ldots, \lambda_d\}$ (respectively $\{\mu_1, \ldots, \mu_d\}$) of A_0 (respectively A_1) are simple. Because of the commutativity condition $A_0 A_1 = A_1 A_0$, there exists an invertible matrix $P \in \mathbb{C}^{d\times d}$ such that

$$P^{-1} A_0 P = \text{diag}(\lambda_1, \ldots, \lambda_d) \text{ and } P^{-1} A_1 P = \text{diag}(\mu_1, \ldots, \mu_d).$$

Therefore, we have $A_0^n = P\text{diag}(\lambda_1^n, \ldots, \lambda_d^n)P^{-1}$ and $A_1^n = P\text{diag}(\mu_1^n, \ldots, \mu_d^n)S^{-1}$. Hence, we acquire

$$\rho(n,2) = P\text{diag}(\rho(n,2)[\lambda_1, \mu_1], \ldots, \rho(n,2)[\lambda_j, \mu_j], \ldots, \rho(n,2)[\lambda_d, \mu_d])P^{-1},$$

where $u_{n,j} = \rho(n,2)[\lambda_j, \mu_j] = \displaystyle\sum_{k_0+2k_1=n-2} \frac{(k_0+k_1)!}{k_0!k_1!}\lambda_j^{k_0}\mu_j^{k_1}$. Hence, for each fixed j ($1 \leq j \leq d$), the sequence $\{u_{n,j}\}_{n\geq 1}$ is a recursive sequence of order 2 of type (4.4), whose coefficients are $a_{0,j} = \lambda_j$, $a_{1,j} = \mu_j$, and initial values $u_{1,j} = 0$, $u_{2,j} = 1$, namely, $u_{n+1,j} = \lambda_j u_{n,j} + \mu_j u_{n-1,j} = a_{0,j}u_{n,j} + a_{1,j}u_{n-1,j}$ for every $n \geq 2$. Suppose that $\lambda_j^2 + 4\mu_j \neq 0$, then, we have $u_{n,j} = \alpha_{1,j}z_{1,j}^n + \alpha_{2,j}z_{2,j}^n$, for every $n \geq 0$, with $z_{1,j} = \frac{\lambda_j+\sqrt{\lambda_j^2+4\mu_j}}{2}$, $z_{2,j} = \frac{\lambda_j-\sqrt{\lambda_j^2+4\mu_j}}{2}$, where the scalar $\alpha_{1,j}$, $\alpha_{2,j}$ are obtained by solving the Vandermonde linear system $\alpha_{1,j}z_{1,j}^n + \alpha_{2,j}z_{2,j}^n = u_{n,j}$, for $n = 1, 2$. Therefore, we have $g_0(t) = \displaystyle\sum_{n=0}^{+\infty} \rho(n,2)\frac{t^n}{n!} = P\mathrm{diag}(\Lambda_1(t), \ldots, \Lambda_j(t), \ldots, \Lambda_d(t))P^{-1}$, where $\Lambda_j(t) = \displaystyle\sum_{n=0}^{+\infty} u_{n,j}\frac{t^n}{n!}$. Using the analytic expression of the recursive sequences $\{u_{n,j}\}_{n\geq 1}$, we get

$$\Lambda_j(t) = \sum_{n=0}^{+\infty}[\alpha_{1,j}z_{1,j}^n + \alpha_{2,j}z_{2,j}^n]\frac{t^n}{n!} = \alpha_{1,j}e^{z_{1,j}t} + \alpha_{2,j}e^{z_{2,j}t}.$$

In summary, we get the following result.

Proposition 6.1. *Let consider Equation (1.2), where A_0, $A_1 \in \mathbb{C}^{d\times d}$ satisfy the commutativity condition $A_0A_1 = A_1A_0$. Suppose that the eigenvalues $\{\lambda_1, \ldots, \lambda_d\}$ (respectively $\{\mu_1, \ldots, \mu_d\}$) of A_0 (respectively A_1) are simple, with $\lambda_j^2 + 4\mu_j \neq 0$. Then, the unique solution $X(t)$ of Equation (1.2), with the prescribed initial data $X(0)$ and $X'(0)$, is formulated as follows*

$$X(t) = [I_d + A_1g_0(t)]\, X(0) + [I_dt + A_0g_0(t) + A_1g_1(t)]\, X'(0),$$

where

$$g_0(t) = P\,diag(\Lambda_1(t), \ldots, \Lambda_j(t), \ldots, \Lambda_d(t))P^{-1}$$

and

$$g_1(t) = P\,diag(\Phi_1(t), \ldots, \Phi_j(t), \ldots, \Phi_d(t))P^{-1},$$

such that $\Lambda_j(t) = \alpha_{1,j}e^{z_{1,j}t} + \alpha_{2,j}e^{z_{2,j}t}$, with $z_{1,j} = \frac{\lambda_j+\sqrt{\lambda_j^2+4\mu_j}}{2}$, $z_{2,j} = \frac{\lambda_j-\sqrt{\lambda_j^2+4\mu_j}}{2}$, $\Phi_j(t) = \int_0^t \Lambda_j(x)dx$ and $P \in \mathbb{C}^{d\times d}$ is an invertible matrix satisfying

$$P^{-1}A_0P = diag(\lambda_1, \ldots, \lambda_d) \text{ and } P^{-1}A_1P = diag(\mu_1, \ldots, \mu_d).$$

Proposition 6.1 can be extended to the case of multiple eigenvalues of the matrices A_0 and A_1.

7. Concluding Remarks

In this work, we studied the generalized Apostolo-Kolodner second-order matrix differential. We have opted for a method based on the properties of recursive sequences in the algebra of square matrices, the Fibonacci-Hörner decomposition of the powers of a matrix, and the fundamental solution of a differential equation, describing the associated system fundamental of solutions. It seems to us that our approach and results are not current in the literature on this topic.

Finally, it should be mentioned that the results of the previous sections seem to be also important for the resolution of non-homogeneous second-order matrix differential equations of the type, $X''(t) = AX(t) + F(t)$, where $A \in \mathbb{C}^{d\times d}$ and $F(t)$ is a continuous function near $t = t_0$. We can mainly get explicit formulas of the solutions of this type of matrix differential equation. Some preliminary results have been established on this subject.

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Mustapha Rachidi,
Institute of Mathematics - INMA,
Federal University of Mato Grosso do Sul - UFMS,
Campo Grande, MS, 79070-900, Brazil
E-mail address: mu.rachidi@gmail.com; mustapha.rachidi@ufms.br

and

Mohammed Mouniane,
Laboratory of Analysis, Geometry and Applications - LAGA,
Department of Mathematics,
Faculty of Sciences,
Ibn Tofail University,
Kenitra, B.P. 133, Morocco
E-mail address: mohammed.mouniane@uit.ac.ma; mohammed.mouniane@gmail.com