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Multiplicity of Solutions for Anisotropic Dirichlet Problem With Variable Exponent

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ABSTRACT: We establish some results on the existence of multiple nontrivial solutions for a general anisotropic elliptic equations. Our approach relies on the variable exponent theory of generalized Lebesgue-Sobolev spaces, combined with adequate variational methods and a variant of the Mountain Pass lemma.

Key Words: Anisotropic equations, generalized Lebesgue-Sobolev spaces, variational method, multiple solutions.

Contents

1. Introduction

In the last few decades, one of the topics from the field of partial differential equations that has continuously attracted interest is that concerning the Sobolev space with variable exponents, $W^{1,p(.)}$ (where $p(.)$ is a function depending on x), see for example the monograph [4] and the references therein. Naturally, problems involving the $p(x)$-Laplacian operator were intensively studied.

On the other hand, it has been experimentally shown that the above-mentioned fluids may have their viscosity undergoing a significant change, see [1]. Consequently, the mathematical modelling of such fluids requires the introduction of the so-called anisotropic variable spaces. Indeed, there is by now a large number of papers and increasing interest about anisotropic problems. With no hope of being complete, let us mention some pioneering works on anisotropic Sobolev spaces [12,15]. Therefore, in the recent years, the study of various mathematical problems modeled by quasilinear elliptic and parabolic equations with both anisotropic and variable exponent has received considerable attention.

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with smooth boundary. In this paper we study the following nonlinear anisotropic elliptic equations

$$(\mathcal{P}) \begin{cases} -\triangle_{\overrightarrow{p}(x)}(u) = f(x,u) & in \quad \Omega \\ u = 0 & on \quad \partial\Omega, \end{cases}$$

where $\triangle_{\overrightarrow{p}(.)}$ represents the $\overrightarrow{p}(.)$-Laplace operator, that is,

$$\triangle_{\overrightarrow{p}(x)}(u) = \sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u),$$

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function, $\overrightarrow{p}(x) = (p_1(x), p_2(x), ..., p_N(x))$,

$$p_M(x) = \max_{i\in\{1,2,...,N\}} p_i(x), \quad p_m(x) = \min_{i\in\{1,2,...,N\}} p_i(x)$$

and for $i = 1, ..., N$, we assume that p_i is a continuous function on $\overline{\Omega}$ such that $\inf_{\Omega} p_i(x) > 1$.
We set,

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) | \min_{x\in\overline{\Omega}} h(x) > 1\}.$$

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For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \quad and \quad h^- = \inf_{x\in\overline{\Omega}} h(x).$$

Moreover, let's put the positive real numbers P_M^+, P_m^+, P_m^- which defined as the following

$$P_M^+ = \max\{p_1^+, ..., p_N^+\},\ P_m^+ = \max\{p_1^-, ..., p_N^-\},\ P_m^- = \min\{p_1^-, ..., p_N^-\}.$$

Throughout this paper, we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{1.1}$$

Define $P_-^*, P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^N \frac{1}{p_i^-} - 1},\ P_{-,\infty} = \max\{P_m^+, P_-^*\}.$$

The $\triangle_{\overrightarrow{p}(.)}$-Laplacian problems on a bounded domain have been investigated and some interesting results have been obtained (see [6,8,13,14] and references therein).

Inspired by the above references and the work in [7], we prove that there exist two nontrivial solutions for $(\mathcal{P})$.

Let $F(x,t) = \int_0^t f(x,s)\,ds$, and we assume that f satisfies the following conditions:
(F_0) $|f(x,t)| \le c(1 + |t|^{q(x)-1})$, $\forall t \in \mathbb{R}$, a.e. $x \in \Omega$,
for some $c > 0$, $q \in C(\bar{\Omega})$ and $1 < q(x) < P_{-,\infty}$, $\forall x \in \bar{\Omega}$.
(F_1) There exist $\theta > P_M^+$ and $M > 0$ such that

$$|\ t\ | \ge M \Rightarrow 0 < \theta F(x,t) \le tf(x,t)$$

for a.e. $x \in \Omega$ and each $t \in \mathbb{R}$.
(F_2) $f(x,t) = o(|\ t\ |^{P_M^+ - 1})$ as $t \to 0$ and uniformly for $x \in \Omega$, with $q^- > P_M^+$.

Now, we can state the following result.

Theorem 1.1. *Suppose (F_0), (F_1), (F_2) and $f(x,0) = 0$ for a.e. $x \in \Omega$. Then the problem $(\mathcal{P})$ has at least two nontrivial solutions, in which one is non-negative and one is non-positive.*

The $\triangle_{\overrightarrow{p}(.)}$- laplacian operator possesses more complicated nonlinearities than the p -laplacian operator, mainly due to the fact that it is not homogeneous.

This paper contains three sections. We will first introduce some basic preliminary results and lemmas in section 2. In section 3, we will give the proof of our main result.

2. Preliminary results

We recall in this section some definitions and basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$.

Throughout this paper, we assume that $p(x) > 1$, $p(x) \in C^{0,\alpha}(\bar{\Omega})$ with $\alpha \in (0,1)$.

For any $p(x) \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ mesurable and } \int_\Omega |\ u(x)\ |^{p(x)}\ dx < \infty\}.$$

We define a norm, the so-called Luxemburg norm, on this space by the formula

$$|\ u\ |_{p(x)} = \inf\{\mu > 0;\ \int_\Omega |\ \frac{u(x)}{\mu}\ |^{p(x)}\ dx \le 1\}.$$

Variable exponent Lebesgue space resemble classical Lebesgue space in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions are dense, if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 <| \Omega |< \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, the Hölder type inequality

$$| \int_\Omega uv\, dx |\leq (\frac{1}{p^-} + \frac{1}{q^-}) \mid u \mid_{p(x)} \mid v \mid_{q(x)},$$

holds true. For more details, we can refer to [11].

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the Modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $J : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega \mid u \mid^{p(x)}\ dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold true

$$\mid u \mid_{p(x)} > 1 \Rightarrow \mid u \mid_{p(x)}^{p^-} \leq J(u) \leq \mid u \mid_{p(x)}^{p^+}, \tag{2.1}$$

$$\mid u \mid_{p(x)} < 1 \Rightarrow \mid u \mid_{p(x)}^{p^+} \leq J(u) \leq \mid u \mid_{p(x)}^{p^-}, \tag{2.2}$$

$$\mid u_n - u \mid_{p(x)} \to 0 \Leftrightarrow J(u_n - u) \to 0, \tag{2.3}$$

$$\mid u \mid_{p(x)} < 1(\text{resp.} = 1; > 1) \Leftrightarrow J(u) < 1(\text{resp.} = 1; > 1). \tag{2.4}$$

Spaces with $p^+ = \infty$ have been studied by [5].

Next, we define $W_0^{1,\overrightarrow{p}(.)}(\Omega)$, the anisotropic variable exponent Sobolev space, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\| u \| = \| u \|_{\overrightarrow{p}(.)} = \sum_{i=1}^N |\partial_{x_i} u|_{p_i(.)}. \tag{2.5}$$

The space $(W_0^{1,\overrightarrow{p}(.)}(\Omega), \| \ . \ \|)$ is a separable and reflexive Banach space (see [3,6]). We note that if $q \in C_+(\bar{\Omega})$ and $q(x) < P_{-,\infty}$ for all $x \in \bar{\Omega}$, then the embedding $W_0^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous. We refer to [2,14] for further properties of variable exponent Lebesgue- anisotropic sobolev spaces.

Recall that the weak solutions of $(\mathcal{P})$ are the critical points of the associated energy functional Φ, given by

$$\Phi(u) = \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\, dx - \int_\Omega F(x,u)\, dx$$

acting on the generalized Sobolev space $W_0^{1,\overrightarrow{p}(.)}(\Omega)$. It is well known that under (F_0), Φ is well defined and is a C^1 functional with derivative given by

$$\langle \Phi'(u), v\rangle = \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \partial_{x_i} v\, dx - \int_\Omega f(x,u) v\, dx,$$

for all $u, v \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$.

Now, we consider the truncated problem

$$(\mathcal{P}_\pm) \begin{cases} -\sum_{i=1}^N \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u) = f_\pm(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$f_\pm(x,t) = \begin{cases} f(x,t) & \text{if } \pm t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$ the positive and negative parts of u.

We need the following lemmas.

Lemma 2.1. *1. If $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ then u^+, $u^- \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ and*
$$\nabla u^+ = \begin{cases} \nabla u, & \text{if } [u > 0], \\ 0, & \text{if } [u \leq 0], \end{cases} \qquad \nabla u^- = \begin{cases} 0, & \text{if } [u \geq 0], \\ \nabla u, & \text{if } [u < 0]. \end{cases}$$

2. The mappings $u \mapsto u^{\pm}$ are continuous on $W_0^{1,\overrightarrow{p}(.)}(\Omega)$.

Proof. 1. Let $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ be fixed. Then there exists a sequence $(\phi_n) \in C_0^{\infty}(\Omega)$ such that

$$\mid \partial_{x_i}(\phi_n - u) \mid_{p_i(.)} \to 0, \text{ for } i \in \{1, ..., N\}.$$

Since $1 < P_m^- < p_i(x)$ for all $x \in \Omega$, it follows that $L^{p_i(x)}$ is continuously embedded in $L^{P_m^-}(\Omega)$ and thus

$$\mid \partial_{x_i}(\phi_n - u) \mid_{P_m^-} \to 0.$$

Hence $u \in W_0^{1,P_m^-}(\Omega)$. We obtain

$$u^+, u^- \in W_0^{1,P_m^-}(\Omega) \subset W_0^{1,1}(\Omega). \tag{2.6}$$

On the other hand, Theorem 7.6 in [10] implies
$$\nabla u^+ = \begin{cases} \nabla u, & \text{if } [u > 0], \\ 0, & \text{if } [u \leq 0], \end{cases} \qquad \nabla u^- = \begin{cases} 0, & \text{if } [u \geq 0], \\ \nabla u, & \text{if } [u < 0]. \end{cases}$$
By the above equalities we deduce that

$$\mid u^+(x) \mid^{p_M(x)} \leq \mid u(x) \mid^{p_M(x)}, \ \mid \partial_{x_i} u^+(x) \mid^{p_i(x)} \leq \mid \partial_{x_i} u(x) \mid^{p_i(x)} \tag{2.7}$$

and
$$\mid u^-(x) \mid^{p_M(x)} \leq \mid u(x) \mid^{p_M(x)}, \ \mid \partial_{x_i} u^-(x) \mid^{p_i(x)} \leq \mid \partial_{x_i} u(x) \mid^{p_i(x)}, \tag{2.8}$$

for a.e. $x \in \Omega$.
Since $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ we have

$$\mid u(x) \mid^{p_M(x)}, \mid \partial_{x_i} u(x) \mid^{p_i(x)} \in L^1(\Omega). \tag{2.9}$$

By (2.7), (2.8) and (2.9) and Lebesgue theorem we obtain that

$$u^+, u^- \in W^{1,\overrightarrow{p}(.)}(\Omega), \tag{2.10}$$

where $W^{1,\overrightarrow{p}(.)}(\Omega) = \{u \in L^{p_M(x)}(\Omega); \partial_{x_i} u \in L^{p_i(x)}(\Omega), \forall i \in \{1, ..., N\}\}$.
By (2.6) and (2.10) we conclude that

$$u^+, u^- \in W^{1,\overrightarrow{p}(.)}(\Omega) \cap W_0^{1,1}(\Omega).$$

Since $p_i \in C^{0,\alpha}(\bar{\Omega})$, Theorem 2.6 and Remark 2.9 in [9] show that
$W_0^{1,\overrightarrow{p}(.)}(\Omega) = W^{1,\overrightarrow{p}(.)}(\Omega) \cap W_0^{1,1}(\Omega)$. Thus $u^+, u^- \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$.

2. Since $u^{\pm} = \frac{1}{2}(\mid u \mid \pm u)$, it suffices to prove that the mapping $u \mapsto \mid u \mid$ is continuous on $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ i.e. $u_n \to u$ implies $\mid u_n \mid \to \mid u \mid$. We have $\mid u_n \mid \to \mid u \mid$ in $L^{p_M(x)}(\Omega)$ and $\mid u_n \mid$ is bounded in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$. Thus, from reflexivity of $W_0^{1,\overrightarrow{p}(.)}(\Omega)$, $\mid u_n \mid \rightharpoonup z$ in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ for a subsequence. Hence $z = \mid u \mid$ and $\mid u_n \mid \rightharpoonup \mid u \mid$ for the whole sequence. On the other hand, we have $\partial_{x_i} \mid u \mid = \operatorname{sgn}(u) \partial_{x_i} u$, for $i = 1, 2, ..., N$ and $\| \mid u \mid \| = \| u \|$. By uniform convexity of $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ it follows that $\mid u_n \mid \to \mid u \mid$ in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$. Thus, the lemma follows.

□

Lemma 2.2. *All solutions of $(\mathcal{P}_+)$ (resp. $(\mathcal{P}_-)$) are non-negative (resp. non-positive) solutions of $(\mathcal{P})$.*

Proof. Define $\Phi_{\pm} : W_0^{1,\overrightarrow{p}(.)}(\Omega) \to \mathbb{R}$,

$$\begin{aligned}\Phi_{\pm}(u) &= \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\,dx - \int_\Omega F_{\pm}(x,u)\,dx \\ &= \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\,dx - \int_\Omega F(x,u^{\pm})\,dx,\end{aligned}$$

where $F_{\pm}(x,s) = \int_0^s f_{\pm}(x,t)\,dt$. It is well known that from lemma 2.1 and the condition (F_0), $\Phi_{\pm}$ is well defined on $W_0^{1,\overrightarrow{p}(.)}(\Omega)$, weakly lower semi-continuous and C^1-functionals.

Let u be a solution of $(\mathcal{P}_+)$, or equivalently, u be a critical point of Φ_+. Taking $v = u^-$ in

$$\langle \Phi_+'(u), v\rangle = \int_\Omega (\sum_{i=1}^N (|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u)\partial_{x_i} v - f_+(x,u)v)\,dx = 0,$$

shows that $\int_\Omega \sum_{i=1}^N |\partial_{x_i} u^-|^{p_i(x)}\,dx = 0$. In view of (2.1) and (2.2) we have $\| u^- \| = 0$, so $u^- = 0$ and $u = u^+$ is also a critical point of Φ with critical value $\Phi(u) = \Phi_+(u)$.

Similarly, nontrivial critical points of Φ_- are non-positive solutions of $(\mathcal{P})$. This ends the proof. $\square$

3. Proof of main result

To apply the mountain pass theorem, we will do separate studies of the compactness of $\Phi_{\pm}$ and its geometry.

Lemma 3.1. *Under (F_0) and (F_1), the functional Φ_+ satisfies the (PS) condition.*

Proof. Let $(u_n)_n$ be a (PS) sequence for the functional Φ_+: $\Phi_+(u_n)$ bounded and $\Phi_+'(u_n) \to 0$. Let us show that $(u_n)_n$ is bounded in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$. Using the hypothesis (F_1), since $\Phi_+(u_n)$ is bounded, we have

$$\begin{aligned}C_1 &\geq \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)}\,dx - \int_\Omega F(x,u_n^+)\,dx \\ &\geq \frac{1}{P_M^+}\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\,dx - \int_\Omega \frac{u_n^+}{\theta} f(x,u_n^+)\,dx + C_2,\end{aligned}$$

where C_1 and C_2 are two constants. Note that

$$\begin{aligned}\langle \Phi_+'(u_n), u_n\rangle &= \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\,dx - \int_\Omega f(x,u_n^+)u_n\,dx \\ &= \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\,dx - \int_\Omega f(x,u_n^+)u_n^+\,dx,\end{aligned}$$

which implies

$$C_1 \geq (\frac{1}{P_M^+} - \frac{1}{\theta})\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}\,dx + \frac{1}{\theta}\langle \Phi_+'(u_n), u_n\rangle + C_2. \tag{3.1}$$

Suppose, by contradiction that $(u_n)_n$ unbounded in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$, so $\| u_n \| \geq 1$ for rather large values of n. For each $i \in \{1, ..., N\}$ and n we define

$$\alpha_{i,n} = \begin{cases} P_M^+ & \text{if } |\partial_{x_i} u_n|_{p_i(.)} < 1, \\ P_m^- & \text{if } |\partial_{x_i} u_n|_{p_i(.)} > 1. \end{cases}$$

Using relations (2.1) and (2.2) we have

$$\begin{aligned}\int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u_n|^{p_i(x)}\, dx &\geq \sum_{i=1}^{N} |\partial_{x_i} u_n|_{p_i(.)}^{P_m^-} - \sum_{\{i:\alpha_{i,n}=P_M^+\}} (|\partial_{x_i} u_n|_{p_i(.)}^{P_m^-} - |\partial_{x_i} u_n|_{p_i(.)}^{P_M^+})\\ &\geq \frac{1}{N^{P_m^-}} \parallel u_n \parallel^{P_m^-} - N.\end{aligned}$$

Furthermore, $\Phi'_+(u_n) \to 0$ assure that there exists $C_3 > 0$ such that

$$-C_3 \parallel u_n \parallel \leq \langle \Phi'_+(u_n), u_n \rangle \leq C_3 \parallel u_n \parallel$$

for rather large values of n. Consequently,

$$C_1 \geq (\frac{1}{P_M^+} - \frac{1}{\theta})\frac{1}{N^{P_m^-}} \parallel u_n \parallel^{P_m^-} - (\frac{1}{P_M^+} - \frac{1}{\theta})N - \frac{C_3}{\theta} \parallel u_n \parallel + C_2.$$

Since $P_m^- > 1$ and $(\frac{1}{P_M^+} - \frac{1}{\theta}) > 0$, we have

$$(\frac{1}{P_M^+} - \frac{1}{\theta})\frac{1}{N^{P_m^-}} \parallel u_n \parallel^{P_m^-} - (\frac{1}{P_M^+} - \frac{1}{\theta})N - \frac{C_3}{\theta} \parallel u_n \parallel + C_2 \to +\infty \text{ as n} \to +\infty,$$

what is a contradiction. So $(u_n)_n$ is a bounded sequence in $W_0^{1,\overrightarrow{p}(.)}(\Omega)$. The proof of lemma 3.1 is complete. □

Lemma 3.2. *There exist $r > 0$ and $\alpha > 0$ such that $\Phi_+(u) \geq \alpha$, for all $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ with $\parallel u \parallel = r$.*

Proof. The conditions (F_0) and (F_2) assure that

$$| F(x,t) | \leq \varepsilon \mid t \mid^{P_M^+} + C(\varepsilon) \mid t \mid^{q(x)} \text{ for all } (x,t) \in \Omega \times \mathbb{R}.$$

For $\parallel u \parallel$ small enough, we have

$$\Phi_+(u) \geq \frac{1}{P_M^+} \int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)}\, dx - \int_\Omega F(x,u^+)\, dx \tag{3.2}$$

For such an element u we have $|\partial_{x_i} u|_{p_i(.)} < 1$ and , by relation (2.2), we obtain

$$\frac{\parallel u \parallel^{P_M^+}}{N^{P_M^+-1}} = N(\frac{\sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(.)}}{N})^{P_M^+} \leq \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(.)}^{P_M^+} \leq \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(.)}^{p_i^+} \leq \int_\Omega \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)}\, dx \tag{3.3}$$

Relations (3.2)-(3.3) imply

$$\begin{aligned}\Phi_+(u) &\geq \frac{1}{P_M^+ N^{P_M^+-1}} \parallel u \parallel^{P_M^+} - \varepsilon \int_\Omega \mid u^+ \mid^{P_M^+}\, dx - C(\varepsilon) \int_\Omega \mid u^+ \mid^{q(x)}\, dx\\ &\geq \frac{1}{P_M^+ N^{P_M^+-1}} \parallel u \parallel^{P_M^+} - \varepsilon \int_\Omega \mid u \mid^{P_M^+}\, dx - C(\varepsilon) \int_\Omega \mid u \mid^{q(x)}\, dx.\end{aligned} \tag{3.4}$$

By the condition (F_0), it follows

$$P_M^+ < q^- \leq q(x) < P_{-,\infty}$$

then

$$W_0^{1,\overrightarrow{p}(.)}(\Omega) \subset L^{P_M^+}(\Omega) \text{ and } W_0^{1,\overrightarrow{p}(.)}(\Omega) \subset L^{q(x)}(\Omega),$$

with a continuous and compact embedding, what implies the existence of C_4, $C_5 > 0$ such that

$$\| u \|_{L^{P_M^+}} \leq C_4 \| u \| \ \text{ and } \ | u |_{q(x)} \leq C_5 \| u \|$$

for all $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$. Since $\| u \|$ is small enough, we deduce

$$\int_\Omega | u |^{q(x)} \leq | u |_{q(x)}^{q^-} \leq C_6 \| u \|^{q^-} .$$

Replacing in (3.4), it results that

$$\Phi_+(u) \geq \frac{1}{P_M^+ N^{P_M^+ -1}} \| u \|^{P_M^+} - \varepsilon C_4^{P_M^+} \| u \|^{P_M^+} - C_7 \| u \|^{q^-},$$

with C_i are positives constants. Let us choose $\varepsilon > 0$ such that $\varepsilon C_4^{P_M^+} \leq \frac{1}{2P_M^+ N^{P_M^+ -1}}$, we obtain

$$\begin{aligned} \Phi_+(u) &\geq \frac{1}{2P_M^+ N^{P_M^+ -1}} \| u \|^{P_M^+} - C_7 \| u \|^{q^-} \\ &\geq \| u \|^{P_M^+} \left(\frac{1}{2P_M^+ N^{P_M^+ -1}} - C_7 \| u \|^{q^- - P_M^+}\right). \end{aligned}$$

Since $P_M^+ < q^-$, the function $t \to (\frac{1}{2P_M^+ N^{P_M^+ -1}} - C_7 t^{q^- - P_M^+})$ is strictly positive in a neighborhood of zero. It follows that there exist $r > 0$ and $\alpha > 0$ such that

$$\Phi_+(u) \geq \alpha \ \forall u \in W_0^{1,\overrightarrow{p}(.)}(\Omega) : \| u \| = r.$$

The proof is completed. □

Proof of theorem 1.1. In order to apply the Mountain Pass Theorem, we must prove that

$$\Phi_+(su) \to -\infty \text{ as s} \to +\infty,$$

for a certain $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$. From the condition (F_1), we obtain

$$F(x,t) \geq c \mid t \mid^\theta \ \text{ for all (x, t)} \in \bar{\Omega} \times \mathbb{R}.$$

Let $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ and $s > 1$ we have

$$\begin{aligned} \Phi_+(su) &= \int_\Omega \sum_{i=1}^N \frac{s^{p_i(x)}}{p_i(x)} |\partial_{x_i} u|^{p_i(x)}\, dx - \int_\Omega F(x,(su)^+)\, dx, \\ &\leq s^{P_M^+} \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)}\, dx - cs^\theta \int_\Omega | u^+ |^\theta \ dx. \end{aligned}$$

The fact $\theta > P_M^+$, gives that

$$\Phi_+(su) \to -\infty \text{ as s} \to +\infty.$$

It follows that there exists $e \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ such that $\| e \| > r$ and $\Phi_+(e) < 0$.
According to the Mountain Pass Theorem, Φ_+ admits a critical value $\mu \geq \alpha$ which is characterized by

$$\mu = \inf_{h \in \Lambda} \sup_{t \in [0,1]} \Phi_+(h(t))$$

where

$$\Lambda = \{h \in C([0,1], W_0^{1,\overrightarrow{p}(.)}(\Omega)) : h(0) = 0 \text{ and h(1) = e}\}.$$

Then, the functional Φ_+ has a critical point u^+ with $\Phi_+(u^+) \geq \alpha$. But, $\Phi_+(0) = 0$, that is, $u^+ \neq 0$. Therefore, the problem $(\mathcal{P}_+)$ has a nontrivial solution which, by lemma 2.2, is a non-negative solution of the problem $(\mathcal{P})$.

Similarly, using Φ_-, we show that there exists furthermore a non-positive solution. The proof of theorem 1.1 is now complete. □

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