Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–11.
ISSN-0037-8712 in press
doi:10.5269/bspm.66848

Existence of Solutions to Elliptic Equations on Compact Riemannian Manifolds

Hind Bouaam, Said Ait Temghart, Chakir Allalou and Said Melliani

Abstract: The aim of this paper is to investigate the existence of weak solutions of a nonlinear elliptic problem with Dirichlet boundary value condition, in the framework of Sobolev spaces on compact Riemannian manifolds.

Key Words: Weak solutions, compact Riemannian manifolds, nonlinear elliptic problem.

Contents

1. Introduction

In this paper, we study the following nonlinear elliptic problem

$$(P)\begin{cases} -div_g\left(a\left(x,u,\nabla u\right)\right)+b\left(x,u,\nabla u\right)+\lambda|u|^{q-2}u+h(x)|u|^{r-2}u=f(x,u) & \text{in } M,\\ u=0 & \text{on } \Gamma, \end{cases}$$

(M,g) is a smooth compact Riemannian manifold of dimension N, $h(x)\in L^s(M)$ with $s=\frac{q}{q-r}$, $\lambda>0$, $1<r<p<q<p^*$, here $p^*=\frac{Np}{N-p}$ if $p<N$ or $p^*=\infty$ if $p\geq N$, and

$$-div_g(a(x,u,\nabla u))=-\sum_{i,j=1}^{N}\frac{\partial a_i}{\partial x_i}+a_j\Gamma^i_{ij},$$

where Γ^i_{ij} represents the symbol of Christoffel and $a=\sum_{i=1}^N a_i\left(\frac{\partial}{\partial x_i}\right)$ is a Carathtéodory's function defined from $M\times\mathbb{R}\times\mathbb{R}^N$ into $\mathbb{R}$, (measurable with respect to x in M for every $(s,\,\eta)$ in $\mathbb{R}\times\mathbb{R}^N$ and continuous with respect to $(s,\,\eta)$ in $\mathbb{R}\times\mathbb{R}^N$ for almost every x in M), and satisfy the assumptions of growth, ellipticity and monotonicity. b: $M\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory's function that satisfy a few conditions that we will review in the section 3, and $f:M\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory's function which is decreasing with respect to the second variable and satisfy the growth assumption.
Boccardo and Gallouët [11] considered the following problem

$$\begin{cases} -\operatorname{div}\left(a(x,u,\nabla u)\right)=f(x,u,\nabla u) & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the right-hand side f is a bounded Radon measure, they proved the existence and some regularity results. In [13], Duc and Vu showed the existence of weak solutions for the problem (1.1), by using a variation of the Mountain Pass theorem which was introduced by Duc in [14]. Moreover, we can even cite the work of S. Liu [23], he established the existence of weak solutions to the particular case for the

2010 *Mathematics Subject Classification:* 35D30 , 35R01 , 65L12.
Submitted January 26, 2023. Published May 27, 2023

problem (1.1) by using the Morse theory. Bensoussan et al. [7] proved the existence of a solution for the problem

$$\begin{cases} -\operatorname{div}\left(a(x,u,\nabla u)\right)+g\left(x,u,\nabla u\right)=f & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $f \in W^{-1,p\prime}(\Omega)$. Drabek and Nicolosi in [12] proved the existence of bounded solution for the degenerated problem (1.2) where $g(x,u,\nabla u)=-c_0|u|^{p-2}u$.

A abbassi et al. [1] established the existence results of weak solutions via the recent Berkovits topological degree for the following nonlinear p-elliptic problems :

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=\lambda|u|^{q-2}u+f(x,u,\nabla u) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where the vector field f is a Carathéodory function which satisfies only the growth condition.

In [6], the authors studied the following problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)+m(x)|u|^{p-2}u=f(x,u) \qquad x\in\mathbb{R}^N, \tag{1.4}$$

by using Browder's theorem, they proved the existence and uniqueness of a weak solution. When $p=2$, the problem (1.4) is a normal Schrodinger equation which has been extensively studied.

The Sobolev space on Riemannian manifolds is another area that is rapidly developing. Thierry Aubin's work in 1976 is credited with providing the first understanding of the Sobolev spaces on the Riemannian manifolds (See [2,3,4,5,8,9,16,17,18,21,22,23]). He applied his findings to the non-linear EDPs on the manifolds. Sobolev spaces on compact manifolds have been used for a long time (Ebin works); in essence, they are the same as Sobolev spaces on a ball of $\mathbb{R}^n$.

This paper is organized as follows. In Section 2, we recall some preliminaries about the framework of Sobolev space on the Riemannian manifolds and some technical lemmas. In Section 3, we introduce some assumptions on the Carathéodory functions $a_i\left(x,s,\xi\right)$, $b(x,s,\xi)$ and $f(x,s)$ for which our problem has at least one solution and we prove our main result.

2. Preliminaries

In this section, we recall the most important and relevant properties and notations about Sobolev spaces on the Riemannian manifolds, and we give some properties and lemmas, that we will need in our analysis of the problem (P), by that, referring to [15,16,17] for more details.

2.1. Definitions and properties

Let (M,g) a Riemannian manifold, and $d\sigma_g$ the Riemannian measure associated with it. Given $u:M\to\mathbb{R}$ a function of class $C^\infty(M)$, and k an integer, we denote by $\nabla^k u$ the k^{th} covariant derivative of u (with the Convention $\nabla^0 u=u$) and $|\nabla^k(u)|$ the norm of $\nabla^k u$ defined by

$$\left|\nabla^k u\right| = g^{i_1j_1}\ldots\ldots g^{i_kj_k}\left(\nabla^k u\right)_{i_1\ldots i_k}\left(\nabla^k u\right)_{j_1\ldots j_k}$$

where the Einstein summons is adopted.
Let $p\geq 1$ be a real number, and k is a positive integer. We define the following spaces:

$$L^p(M)=\left\{u:M\to\mathbb{R} \text{ measurable : } \int_M |u|^p d\sigma_g<\infty\right\}$$

and

$$C_k^p(M)=\left\{u\in C^\infty:\ \forall j=0,\ldots,k\ \int_M \left|\nabla^j u\right|^p d\sigma_g<\infty\right\}.$$

Definition 2.1. *The Sobolev space $W^{k,p}(M)$ is the complete space $C_k^p(M)$ for the norm*

$$\|u\|_{k,p}=\sum_{j=0}^k \left\|\nabla^j u\right\|_p,$$

where

$$\|u\|_{1,p} = \|\nabla u\|_p + \|u\|_p.$$

Definition 2.2. *We must recall the notion of the geodesic distance for every curve:*

$$\Upsilon : [a,b] \to M.$$

We define the length of Υ *by:*

$$l(\Upsilon) = \int_a^b \sqrt{g(\Upsilon(t))\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)}dt.$$

Remark 2.3. *For* $x, y \in M$ *defining a distance* d_g *by:*

$$d_g(x,y) = \inf\{l(\Upsilon) : \ \Upsilon : [0,1] \to M, \ \Upsilon(0) = x, \ \Upsilon(1) = y\}.$$

By the theorem of Hopf-Rinow, we obtain that if M *a Riemannian manifold then compact for all* x, y *in* M *can be joined by a minimizing curve* Υ *i.e* $l(\Upsilon) = d_g(x,y)$.

Proposition 2.4. *If* $p = 2$*, the space* $W^{k,2}(M)$ *is a Hilbert space for the following scalar product*

$$(u,v)_{W^{k,2}} = \sum_{j=0}^{k} \left(\nabla^j u, \nabla^i v\right)_{L^2}.$$

Proposition 2.5. *If* $p > 1$ *then* $W^{k,p}(M)$ *is Banach reflexive space.*

Definition 2.6. *The Sobolev space* $W_0^{k,p}(M)$ *is the closure of* $\mathcal{D}(M)$ *in* $W^{k,p}(M)$.

Theorem 2.7. *If* (M,g) *is complete, then for all* $p \geq 1$ $W_0^{1,p}(M) = W^{1,p}(M)$.

2.2. Embeddings of Sobolev :

Theorem 2.8. *[17] Let* (M,g) *be a complete Riemannian* n*-manifold. Then, if the embedding* $W^{1,1}(M) \hookrightarrow L^{\frac{n}{n-1}}(M)$ *holds, then whenever the real numbers* q *and* p *satisfy*

$$1 \leq q < n$$

and

$$q \leq p \leq q* = \frac{nq}{n-q}$$

the embedding $W^{1,q}(M) \hookrightarrow L^p(M)$ *also holds.*

Theorem 2.9. *[17] Let* (M,g) *a compact Riemannian manifold of dimension* n*. Given* $1 \leq q < p$ *two real numbers, and* $0 \leq m < k$ *two integers such that* $\frac{1}{p} = \frac{1}{q} - \frac{(k-m)}{n}$*, then*

$$W^{k,q}(M) \hookrightarrow W^{m,p}(M).$$

In particular, for all $1 \leq q < n$

$$W^{1,q}(M) \hookrightarrow L^p(M),$$

where $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$.

Theorem 2.10. ***(Rellich-Kondrakov's Theorem)*** *[17] Let* (M,g) *a compact Riemannian manifold of* n *dimension,* $j \geq 0$ *and* $m \geq 1$ *two integers,* $q \geq 1$ *and* p *two real numbers that verify*

$$1 \leq p < nq/(n-mq).$$

Then

$$W^{j+m,q}(M) \hookrightarrow W^{j,p}(M)$$

is compact.

Corollary 2.11. *Let M be a compact Riemannian manifold of n dimension. Assume that*

$$q < N, \quad p < \frac{Nq}{N-q} \quad \text{for } x \in M.$$

Then,

$$W^{1,q}(M) \hookrightarrow L^p(M),$$

is a continuous and compact embedding.

Lemma 2.12. *(**Inequality of Poincaré**)([17].) Let D a regular domain is bounded in a Riemannian manifold M and $1 \leq p < \infty$. Then there is a constant A such as:*

$$\left(\int_D |u - u_D|^p \, d\sigma_g\right)^{\frac{1}{p}} \leq A \left(\int_D |\nabla u|^p d\sigma_g\right)^{\frac{1}{p}}$$

for all $u \in W^{1,p}_{loc}(M)$, where $u_D = \dfrac{1}{vol(D)} \displaystyle\int_D u d\sigma_g$ is the mean value of u on D.

By combining this lemma with the Hölder inequality, we obtain:

Corollary 2.13. *There exists a constant $c = c_D$ such that*

$$\int_D |u - u_D| \, d\sigma_g \leq c_D \left(\int_M |\nabla u|^p d\sigma_g\right)^{\frac{1}{p}} \quad \forall u \in W^{1,p}_{loc}(M).$$

Proof. We apply the inequality of Hölder, we will have

$$\begin{aligned} \int_D 1. \, |u - u_{\mathrm{D}}| \, d\sigma_g &\leq \left(\int_{\mathrm{D}} 1^{p'} d\sigma_g\right)^{\frac{1}{p}} \left(\int_{\mathrm{D}} |u - u_{\mathrm{D}}|^p \, d\sigma_g\right)^{\frac{1}{p}} \\ &\leq A \, vol(D)^{\frac{1}{p}} \left(\int_M |\nabla u|^p d\sigma_g\right)^{\frac{1}{p}}. \end{aligned}$$

2.3. Technical lemmas

Theorem 2.14. *([10].) Let X be a Banach Reflexive space and let $A : X \to X'$ an operator having the following properties:*

(1) *A is bounded hemicontinuous .*

(2) *A is monotone.*

(3) *A is coercive ,e.i, $\dfrac{\langle A(v), v >}{\|v\|} \to \infty$ if $\|v\| \to \infty$.*

Then A is surjective of $X \to X'$, i.e, for every $f \in X'$, there exists $u \in X$ such as:

$$A(u) = f.$$

The next inequalities will be systematically used in this work.

Lemma 2.15. *([22].) Let ξ_1, $\xi_2 \in M$, we have*
1) *If $p < 2$,*

$$|\xi_1 + \xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2}\xi_1\xi_2 \leq C(p)|\xi_2|^p,$$

$$|\xi_2|^p| - |\xi_1|^p - p|\xi_1|^{p-2}\xi_1(\xi_2 - \xi_1) \geq C(p)\frac{|\xi_2 - \xi_1|^2}{(|\xi_2| + |\xi_1|)^{2-p}}.$$

2) *If $p \geq 2$,*

$$|\xi_1 + \xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2}\xi_1\xi_2 \leq |\frac{p(p-1)}{2}(|\xi_1| + |\xi_2|)^{p-2}|\xi_2|^2,$$

$$|\xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2}\xi_1(\xi_2 - \xi_1) \geq \frac{C(p)}{2^p - 1}|\xi_2 - \xi_1|^p.$$

3. Existence Result

In this section, we will state and prove the existence of solutions for problem (P). We start by stating the following assumptions :

(H_1) $h(x) \in L^s(M)$ with $s = \frac{q}{q-r}$,

$$1 < r < p < q < p* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ \infty & \text{if } p \geq N, \end{cases}$$

and

$$h(x) \geq h_0 \geq 0. \tag{3.1}$$

(H_2) The function $a_i(x, \eta, \xi)$ is a Carathéodory function satisfies the following assumptions:

- For a.e $x \in M$, for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $C_1 > 0$ and $k \in L^{p'}(M)$;

$$|a_i\left(x, \eta, \xi\right)| \leq C_1 \left(k(x) + |\eta|^{\frac{q}{p'}} + |\xi|^{p-1}\right), \tag{3.2}$$

- For a.e $x \in M$ and all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$;

$$\sum_{i=1}^{N} \left(a_i(x, u, \xi) - a_i\left(x, u, \xi^*\right)\right)\left(\xi_i - \xi_i^*\right) \geq 0, \tag{3.3}$$

- For a.e $x \in M$, for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $C_0 > 0$ and $K_0 \in L^1(M)$;

$$\sum_{i=1}^{N} a_i\left(x, u, \xi^*\right) \xi_i^* \geq C_0 |\xi|^p - K_0(x). \tag{3.4}$$

(H_3) b is a Carathéodory function satisfies the following assumptions:

- For a.e $x \in M$, for all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $\beta_0 > 0$ and $\gamma \in L^{p'}(M)$;

$$|b\left(x, s, \xi\right)| \leq \gamma(x) + \beta_0 |\xi|^{p-2}\xi, \tag{3.5}$$

- For a.e $x \in M$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$;

$$b\left(x, s, \xi\right).s \geq 0, \tag{3.6}$$

- For a.e $x \in M$, for all $(\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}$ and $C > 0$;

$$b\left(x, \xi_1, \nabla\xi_1\right)\left(\xi_1 - \xi_2\right) \geq C|\xi_1 - \xi_2|^p - |\xi_1|^p + |\xi_2|^p. \tag{3.7}$$

(H_4) f Carathéodory function which is decreasing with respect to the second variable, i.e

$$f(x, \xi_2) \leq f(x, \xi_1) \quad \text{for a.e } x \in M \quad \text{if} \qquad \xi_2 \geq \xi_1, \tag{3.8}$$

- For a.e $x \in M$, for all $\xi \in \mathbb{R}$, $C > 0$ and $f_0 \in L^{q'}(M)$;

$$|f(x, \xi)| \leq f_0(x) + C|\xi|^{q-1}. \tag{3.9}$$

Definition 3.1. *We say that $u \in W_0^{1,p}(M)$ is a weak solution of Dirichlet problem (P) if*

$$\sum_{i=1}^{N} \int_M (a_i(x,u,\nabla u))\, D_i v(x) d\sigma_g + \int_M b\,(x,u,\nabla u)\, v d\sigma_g$$
$$+\lambda \int_{\mathbb{M}} |u|^{q-2} u v d\sigma_g + \int_M h(x)|u|^{r-2} u v d\sigma_g = \int_M f(x,u) v(x) d\sigma_g$$

$\forall u, v \in W_0^{1,p}(M)$.

Theorem 3.2. *Let (M,g) be a compact Riemannian manifold of N dimension, suppose that the hypotheses $(H_1)-(H_4)$ are satisfied. Then the problem (P) admits at least a weak solution $u \in W_0^{1,p}(M)$.*

Proof. We define the operator $T : W_0^{1,p}(M) \to W^{-1,p'}(M)$ by

$$T := A + B + \lambda G + R - F$$

where the operators A,B, G, R and F are defined from $W_0^{1,p}(M)$ into $W^{-1,p'}(M)$ as

$$\langle A(u), v\rangle = \sum_{i=1}^{N} \int_M (a_i(x,u,\nabla u))\, D_i v(x) d\sigma_g,$$

$$\langle B(u), v\rangle = \int_{\mathbb{M}} b\,(x,u,\nabla u)\, v d\sigma_g,$$

$$\langle G(u), v\rangle = \int_{\mathbb{M}} |u|^{q-2} u v d\sigma_g,$$

$$\langle R(u), v\rangle = \int_{\mathbb{M}} h(x)|u|^{r-2} u v d\sigma_g$$

and

$$\langle F(u), v\rangle = \int_{\mathbb{M}} f(x,u) v d\sigma_g$$

for all $u, v \in W_0^{1,p}(M)$.
By Definition 3.1, the main tool in searching the weak solutions of (P) is to finding $u \in W_0^{1,p}(M)$ which satisfies the operator equation $Tu = 0$.
Step 1 : We establish that T is bounded operator.
Let $u \in W_0^{1,p}(M)$, such that $\|u\|_{1,p} \leq M$. Using Hölder's inequality, we obtain

$$|\langle A(u), v\rangle| \leq \sum_{i=1}^{N} \left(\int_M |\,(a_i(x,u,\nabla u))\,|^{p'} d\sigma_g \right)^{\frac{1}{p'}} \left(\int_M |D_i v(x)|^p d\sigma_g \right)^{\frac{1}{p}},$$

by the growth condition (3.2) and the continuous embedding $W_0^{1,p}(M) \hookrightarrow L^q(M)$, we have

$$\begin{aligned}\int_M |\,(a_i(x,u,\nabla u))\,|^{p'} d\sigma_g &\leq C \left(\int_M |k(x)|^{p'} + |u|^q + |\nabla u|^p d\sigma_g \right) \\ &\leq C \left(\|k\|_{p'}^{p'} + C_1 \|u\|_{1,p}^q + \|u\|_{1,p}^p \right).\end{aligned}$$

Therefore,

$$\begin{aligned}\|A(u)\|_{W^{-1,p'}(M)} &= \sup_{\|v\|_{1,p}\leq 1} |\langle A(u), v\rangle| \\ &\leq \sup_{\|v\|_{1,p}\leq 1} C \left(\|k\|_{p'}^{p'} + C_1 \|u\|_{1,p}^q + \|u\|_{1,p}^p \right)^{\frac{1}{p'}} \|v\|_{1,p} \\ &\leq C \left(\|k\|_{p'}^{p'} + C_1 \|u\|_{1,p}^q + \|u\|_{1,p}^p \right)^{\frac{1}{p'}}.\end{aligned}$$

Hence, A is bounded.
For each $u \in W_0^{1,p}(M)$, we have by the growth condition (3.5), inequality of Hölder and the continuous embedding $W_0^{1,p}(M) \hookrightarrow L^q(M)$ that

$$\begin{aligned}
\|B(u)\|_{W^{-1,p'}(M)} &= \sup_{\|v\|_{1,p}\leq 1} |\langle B(u), v\rangle| \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \int_M |b(x,u,\nabla u)||v| d\sigma_g \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \left[\int_M |\gamma||v| d\sigma_g + \beta_0 \int_M |\nabla u|^{p-1}|v| d\sigma_g\right] \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \left[\left(\int_M |\gamma(x)|^{p'} d\sigma_g\right)^{\frac{1}{p'}} \left(\int_M |v|^p d\sigma_g\right)^{\frac{1}{p}} + \beta_0 \left(\int_M |\nabla u|^p\right)^{\frac{1}{p'}} \left(\int_M |v|^p d\sigma g\right)^{\frac{1}{p}}\right] \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \|\gamma\|_{p'}\|v\|_p + \beta_0 \|u\|_{1,p}^{\frac{p}{p'}} \|v\|_{1,p}, \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \left(\|\gamma\|_{p'} + \beta_0\|u\|_{1,p}^{\frac{p}{p'}}\right)\|v\|_{1,p} \\
&\leq \|\gamma\|_{p'} + \beta_0\|u\|_{1,p}^{\frac{p}{p'}}.
\end{aligned}$$

Hence, the operator B is bounded.
For each $u \in W_0^{1,p}(M)$, by using the inequality of Hölder and the continuous embedding $W_0^{1,p}(M) \hookrightarrow L^q(M)$, we get

$$\begin{aligned}
\|G(u)\|_{W^{-1,p'}(M)} &= \sup_{\|v\|_{1,p}\leq 1} |\langle G(u), v\rangle| \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \int_M |u|^{q-1}|v| d\sigma_g \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \left(\int_M |u|^q d\sigma_g\right)^{\frac{1}{q'}} \left(\int_M |v|^q d\sigma_g\right)^{\frac{1}{q}} \\
&\leq \sup_{\|v\|_{1,p}\leq 1} C\|u\|_{1,p}^{\frac{q}{q'}}\|v\|_{1,p} \\
&\leq C\|u\|_{1,p}^{\frac{q}{q'}},
\end{aligned}$$

and

$$\begin{aligned}
\|R(u)\|_{W^{-1,p'}(M)} &= \sup_{\|v\|_{1,p}\leq 1} |\langle R(u), v\rangle| \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \int_M h(x)|u|^{r-1}|v| d\sigma_g \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \left(\int_M |h(x)|^s\right)^{\frac{1}{s}} \left(\int_M |u|^q d\sigma_g\right)^{\frac{1}{m}} \left(\int_M |v|^q d\sigma_g\right)^{\frac{1}{q}} \\
&\leq \sup_{\|v\|_{1,p}\leq 1} C\|h\|_s\|u\|_{1,p}^{\frac{q}{m}}\|v\|_{1,p} \\
&\leq C\|h\|_s\|u\|_{1,p}^{\frac{q}{m}},
\end{aligned}$$

where $m = \frac{q}{r-1}$ and $\frac{1}{s} + \frac{1}{m} + \frac{1}{q} = 1$.
Hence, G and R are bounded.
For each $u \in W_0^{1,p}(M)$, we have by the growth condition (3.9), inequality of Hölder and the continuous

embedding $W_0^{1,p}(M) \hookrightarrow L^q(M)$ that

$$\begin{aligned}
\|F(u)\|_{W^{-1,p'}(M)} &= \sup_{\|v\|_{1,p}\leq 1} |\langle F(u), v\rangle| \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \int_M |f(x,u)||v| d\sigma_g \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \int_M \left(|f_0| + C|u|^{q-1}\right)|v| d\sigma_g \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \left[\left(\int_M |f_0|^{q'} d\sigma_g\right)^{\frac{1}{q'}} \left(\int_M |v|^q d\sigma_g\right)^{\frac{1}{q}} + C\left(\int_M |u|^q d\sigma_g\right)^{\frac{1}{q'}} \left(\int_M |v|^q d\sigma_g\right)^{\frac{1}{q}}\right] \\
&\leq C \sup_{\|v\|_{1,p}\leq 1} \left(\|f_0\|_{q'} + \|u\|_{1,p}^{\frac{q}{q'}}\right)\|v\|_{1,p} \\
&\leq C\left(\|f_0\|_{q'} + \|u\|_{1,p}^{\frac{q}{q'}}\right).
\end{aligned}$$

Hence, F is bounded.
Step 2 : We establish that T is continuous.
Let $u_n \to u$ in $W_0^{1,p}(M)$. Then

$$u_n \to u \quad \text{in} \quad L^q(M),$$

and

$$\nabla u_n \to \nabla u \quad \text{in} \quad L^p(M).$$

Hence there exist a subsequence (u_n) and measurable functions h in $L^q(M)$ and g in $(L^p(M))^N$ such that

$$u_n(x) \to u(x) \text{ and } \nabla u_n(x) \to \nabla u(x),$$

$$|u_n(x)| \leq h(x) \text{ and } |\nabla_n(x)| \leq |g(x)|,$$

for a.e. $x \in M$ and all $k \in N$. Since a_i satisfies the Carathéodory condition, we obtain that

$$a_i(x, u_n(x), \nabla u_n(x)) \to a_i(x, u(x), \nabla u(x)) \text{ a.e. } x \in M.$$

It follows from (3.2) that

$$|a_i(x, u_n(x), \nabla u_n(x))| \leq C\left(k(x) + |h(x)|^{\frac{q}{p'}} + |g(x)|^{p-1}\right)$$

for a.e. $x \in M$ and for all $k \in \mathbb{N}$. Since

$$k + |h(x)|^{\frac{q}{p'}} + |g(x)|^{p-1} \in L^{p'}(M)$$

and taking into account the inequality

$$\begin{aligned}
&\|A(u_n) - A(u)\|_{W^{-1,p'}(M)} \\
&\leq \sup_{\|v\|_{1,p}\leq 1} \sum_{i=1}^N \left(\int_M |a_i(x, u_n, \nabla u_n) - a_i(x, u, \nabla u)|^{p'} d\sigma_g\right)^{\frac{1}{p'}} \left(\int_M |D_i v(x)|^p d\sigma_g\right)^{\frac{1}{p}} \\
&\leq \sum_{i=1}^N \left(\int_M |a_i(x, u_n, \nabla u_n) - a_i(x, u, \nabla u)|^{p'} d\sigma_g\right)^{\frac{1}{p'}},
\end{aligned}$$

the dominated convergence theorem imply that $Au_n \to Au$ in $W^{-1,p'}(M)$. Then A is continuous.

Similarly,

$$\begin{aligned}
\|B(u_n) - B(u)\|_{W^{-1,p'}(M)} &\leq \sup_{\|v\|_{1,p}\leq 1} \left(\int_M |b(x, u_n, \nabla u_n) - b(x, u, \nabla u)|^{p'} d\sigma_g\right)^{\frac{1}{p'}} \left(\int_M |v(x)|^p d\sigma_g\right)^{\frac{1}{p}} \\
&\leq C\left(\int_M |b(x, u_n, \nabla u_n) - b(x, u, \nabla u)|^{p'} d\sigma_g\right)^{\frac{1}{p'}} \longrightarrow 0,
\end{aligned}$$

$$\|G(u_n)-G(u)\|_{W^{-1,p'}(M)} \leq \sup_{\|v\|_{1,p}\leq 1}\left(\int_M \left||u_n|^{q-2}u_n-|u|^{q-2}u\right|^{q'}d\sigma_g\right)^{\frac{1}{q'}}\left(\int_M |v(x)|^q d\sigma_g\right)^{\frac{1}{q}}$$
$$\leq C\left(\int_M \left||u_n|^{q-2}u_n-|u|^{q-2}u\right|^{q'}d\sigma_g\right)^{\frac{1}{q'}} \longrightarrow 0,$$

$$\|R(u_n)-R(u)\|_{W^{-1,p'}(M)}$$
$$\leq \sup_{\|v\|_{1,p}\leq 1}\left(\int_M |h|^s\right)^{\frac{1}{s}}\left(\int_M \left||u_n|^{r-2}u_n-|u|^{r-2}u\right)^{\frac{q}{r-1}}d\sigma_g\right|^{\frac{r-1}{q}}\left(\int_M |v(x)|^q d\sigma_g\right)^{\frac{1}{q}}$$
$$\leq C\left(\int_M \left||u_n|^{r-2}u_n-|u|^{r-2}u\right|^{\frac{q}{r-1}}d\sigma_g\right)^{\frac{r-1}{q}} \longrightarrow 0,$$

and

$$\|F(u_n)-F(u)\|_{W^{-1,p'}(M)} \leq \sup_{\|v\|_{1,p}\leq 1}\left(\int_M |f(x,u_n)-f(x,u)|^{q'}d\sigma_g\right)^{\frac{1}{q'}}\left(\int_M |v(x)|^q d\sigma_g\right)^{\frac{1}{q}}$$
$$\leq C\left(\int_M |f(x,u_n)-f(x,u)|^{q'}d\sigma_g\right)^{\frac{1}{q'}} \longrightarrow 0.$$

Then B,R, G and F are continuous.
Step 3 : We establish that T is monotone.
We have

$$\begin{aligned}\langle G(u)-G(v),u-v\rangle &= \int_M \left(|u|^{q-2}u-|v|^{q-2}v\right)(u-v)\\ &= \int_M |u|^{q-2}u(u-v)-\int_M |v|^{q-2}v(u-v)\\ &= \alpha+\beta.\end{aligned}$$

By Lemma 2.15, we have

$$\alpha \geq \frac{1}{q}\int_M \frac{C(q)}{2^q-1}|u-v|^q d\sigma_g-\frac{1}{q}\int_M |v|^q d\sigma_g+\frac{1}{q}\int_M |u|^q d\sigma_g,$$

and

$$\beta \geq \frac{1}{q}\int_M \frac{C(q)}{2^q-1}|u-v|^q d\sigma_g-\frac{1}{q}\int_M |u|^q d\sigma_g+\frac{1}{q}\int_M |v|^q d\sigma_g,$$

then

$$\langle G(u)-G(v),u-v\rangle \geq \frac{2}{q}\frac{C(q)}{2^q-1}\|u-v\|_q^q. \tag{3.10}$$

Similarly, by Lemma 2.15 and condition (3.1), we have

$$\langle R(u)-R(v),u-v\rangle \geq h_0\frac{2}{r}\frac{C(r)}{2^r-1}\|u-v\|_r^r. \tag{3.11}$$

On the other hand,

$$\begin{aligned}\langle B(u)-B(v),u-v\rangle &= \int_M (b(x,u,\nabla u)-b(x,v,\nabla v))(u-v)d\sigma_g\\ &= \int_M b(x,u,\nabla u)(u-v)d\sigma_g-\int_M b(x,v,\nabla v)(u-v)d\sigma_g\\ &= \alpha+\beta,\end{aligned}$$

by using condition (3.7), we have

$$\alpha \geq C|u-v|^p - |u|^p + |v|^p, \quad \text{and} \quad \beta \geq C|u-v|^p - |v|^p + |u|^p,$$

hence

$$\langle B(u) - B(v), u - v\rangle \geq 2C\|u-v\|_p^p. \tag{3.12}$$

Further, since f is decreasing with respect to the second variable,

$$\langle F(u) - F(v), u - v\rangle = \int_M (f(x,u) - f(x,v))(u-v)d\sigma_g \quad \leq 0. \tag{3.13}$$

According to (3.3), (3.10), (3.11), (3.12) and (3.13), we get

$$\langle T(u) - T(v), u - v\rangle \geq 0.$$

Therefore T is monotone.
Step 4 : We establish that T is a coercive operator.
Using the growth condition (3.9), inequality of Hölder and the continuous embedding $W_0^{1,p}(M) \hookrightarrow L^q(M)$, we have

$$\begin{aligned}
|\langle F(u), u\rangle| &\leq \int_M \left(|f_0| + C|u|^{q-1}\right)|u|d\sigma_g \\
&\leq \left(\int_M |f_0|^{q'} d\sigma_g\right)^{\frac{1}{q'}} \left(\int_M |u|^q d\sigma_g\right)^{\frac{1}{q}} + C\int_M |u|^q d\sigma_g \\
&\leq \|f_0\|_{q'}\|u\|_q + C\|u\|_q^q \\
&\leq C'\|u\|_{1,p},
\end{aligned}$$

according to (3.1), (3.6), by inequality of Poincaré and condition (3.4), we get

$$\begin{aligned}
|\langle T(u), u\rangle| = &\sum_{i=1}^N \int_M (a_i(x,u,\nabla u))\, D_i u(x) d\sigma_g + \int_M b\,(x,u,\nabla u)\, u d\sigma_g + \lambda \int_{\mathbb{M}} |u|^{q-2}u u d\sigma_g \\
&+ \int_M h(x)|u|^{r-1}u u d\sigma_g - \int_M f(x,u) u d\sigma_g \\
&\geq C_0 \int_M |\nabla u|^p d\sigma_g - \int_M k(x) d\sigma_g + \lambda \int_M |u|^q d\sigma_g + h_0 \int_M |u|^r - C'\|u\|_{1,p} \\
&\geq C_2\|u\|_{1,p}^p - C_1 + \lambda\|u\|_q^q + h_0\|u\|_r^r - C'\|u\|_{1,p}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{|\langle T(u), u\rangle|}{\|u\|_{1,p}} &\geq C_2\|u\|_{1,p}^{p-1} - \frac{C_1}{\|u\|_{1,p}} + \lambda\frac{\|u\|_q^q}{\|u\|_{1,p}} + h_0\frac{\|u\|_r^r}{\|u\|_{1,p}} - C' \\
&\geq C_2\|u\|_{1,p}^{p-1} - \frac{C_1}{\|u\|_{1,p}} - C',
\end{aligned}$$

which gives directly the coercivity of the operator T.
By applying Theorem 2.14 we deduce that the problem (P) admits at least one solution $u \in W_0^{1,p}(M)$.

References

1. A. Abbassi, C. Allalou and A. Kassidi, Existence of Weak Solutions for Nonlinear p-Elliptic Problem by Topological Degree, Nonlinear Dynamics and Systems Theory, 20 (3) (2020) 229–241.

2. T. Aubin, Problemes isopérimétriques et espaces de Sobolev. Journal of differential geometry, 11(4),(1976), 573-598.

3. T. Aubin, Nonlinear analysis on manifolds. Monge-Ampere equations (Vol. 252) (2012). Springer Science and Business Media.

4. A.Abnoune, E.Azroul and M.T.K.Abbassi: Study of a Second-Order Nonlinear Elliptic Problem Generated by a Divergence Type Operator on a Compact Riemannian Manifold, November 2018. Filomat Vol 32, No 14 (2018)(14):4811-4820.

5. A.Abnoune and A.Azroul, Quasi Linear Elliptic Equations with Data in L^1 on Riemannian Manifold. January 2019.

6. M. V. Abdelkadery and A. Ourraouiz, Existence And Uniqueness Of Weak Solution For p-Laplacian Problem In R^N, Appl. Math. E-Notes, 13 (2013), 228–233.

7. A.Bensoussan, L. Boccardo, and F. Murat, On a non linear partial differential equation having natural growth terms and unbounded solution. 5(4),(1988), 347-364.

8. H. Bouaam, M. El Ouaarabi, C. Allalou and S. Melliani, Variable exponent $q(m)$-Kirchhoff-type problems with nonlocal terms and logarithmic nonlinearity on compact Riemannian manifolds. Bulletin of the Malaysian Mathematical Sciences Society,(2023) 46(3), 97.

9. H. Bouaam, M. El Ouaarabi, C. Allalou, Nonlocal Kirchhoff-type problem involving variable exponent and logarithmic nonlinearity on compact Riemannian manifolds. Anal.Math.Phys. 13, 48 (2023). https://doi.org/10.1007/s13324-023-00810-0.

10. L. Boccardo and B. Daccorogna. A characterization of pseudo-monotone differential operators in divergence form, comm in partial diff equations 9 (11) (1984), 1107-1117.

11. L. Boccardo, T. Gallouët, Nonlinear elliptic equations with right-hand side measures, Comm. Partial Diferential Equations,no. 3-4, 17 (1992), 641-655.

12. P. Drabek and F. Nicolosi, Existence of bounded solutions for some degenerated quasilinear elliptic equations, Ann. Mat. Pura Appl. (4) 165 (1993), 217–238.

13. D. M. Duc and N. T. Vu, Nonuniformly elliptic equations of p-Laplacian type, Nonlinear Analysis 61 (2005).

14. D. M. Duc, Nonlinear singular elliptic equations, J. London Math. Soc. 40 (2) (1989).

15. E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Lecture Notes, vol. 5, American Mathematical Society, 2000.

16. E. Hebey, Introduction l'analyse non linaire sur les variétès. Ed. Diderot (1997).

17. E. Hebey, Sobolev spaces on Riemannian manifolds, Springer lect. Notes in Mth. 1635 .

18. El Ouaarabi, M., Allalou, C., Melliani, S.: On a class of p(x)-Laplacian-like Dirichlet problem depending on three real parameters. Arab. J. Math. 11(2), 227–239 (2022)/

19. El Ouaarabi, M., Allalou, C., Melliani, S.: Weak solution of a Neumann boundary value problem with p(x)-Laplacian-like operator. Analysis. 42(4), 271–280 (2022).

20. El Ouaarabi, M., Allalou, C., Melliani, S.: Existence result for Neumann problems with p(x)-Laplacian-like operators in generalized Sobolev spaces. Rend. Circ. Mat. Palermo, II. Ser 72, 1337–1350 (2023).

21. El Ouaarabi, M., Allalou, C., Melliani, S.: Existence of weak solution for a class of p(x)-Laplacian problems depending on three real parameters with Dirichlet condition. Bol. Soc. Mat. Mex. 28, 31(2022).

22. P. Lindqvist, On the equation $\nabla p(u) + \lambda |u|^{p-2}u = 0$. Proc. Amer. Math. Soc. Vol.109, no. (1990), 157-164.

23. S. Liu, Existence of solutions to a superlinear p-Laplacian equation. (2001) 49J35, 35J65, 35B34.

H. Bouaam (Corresponding author),
Said Ait Temghart,
Chakir Allalou,
Said Melliani ,
Laboratory LMACS, FST of Beni-Mellal,
Sultan Moulay slimane University, Morocco.
E-mail address: hind15292@gmail.com
E-mail address: saidotmghart@gmail.com
E-mail address: chakir.allalou@yahoo.fr
E-mail address: s.melliani@usms.ma