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About the Image of Strongly Generalized Derivations of Order n

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ABSTRACT: Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. A linear mapping $\Delta : \mathcal{A} \to \mathcal{B}$ is called a *strongly generalized derivation of order* n, if there exist the families of linear mappings $\{E_k : \mathcal{A} \to \mathcal{B}\}_{k=1}^n$, $\{F_k : \mathcal{A} \to \mathcal{B}\}_{k=1}^n$, $\{G_k : \mathcal{A} \to \mathcal{B}\}_{k=1}^n$ and $\{H_k : \mathcal{A} \to \mathcal{B}\}_{k=1}^n$ which satisfy $\Delta(ab) = \sum_{k=1}^n [E_k(a)F_k(b) + G_k(a)H_k(b)]$ for all $a, b \in \mathcal{A}$. The main purpose of this paper is to study the image of such derivations. Our main result on the image of strongly generalized derivations of order one reads as follows:
Let $\mathcal{A}$ be a unital, commutative Banach algebra and let $\Delta : \mathcal{A} \to \mathcal{A}$ be a continuous strongly generalized derivation of order one; that is, there exist the linear mappings $E, F, G, H : \mathcal{A} \to \mathcal{A}$ satisfying $\Delta(ab) = E(a)F(b)+G(a)H(b)$ for all $a, b \in \mathcal{A}$. Let E, F, G and H be continuous linear mappings. We prove that, under certain conditions, $H(\mathcal{A})$, $E(\mathcal{A})$ and $\Delta(\mathcal{A})$ are contained in the Jacobson radical of $\mathcal{A}$. This result generalizes Singer-Wermer theorem about the image of continuous derivations on commutative Banach algebras.

Key Words: Derivation, strongly generalized derivation of order n, Jacobson radical, Singer-Wermer theorem.

Contents

1. Introduction and preliminaries

As a pioneering work, Singer and Wermer [18] achieved a fundamental result which started investigation into the image of derivations on Banach algebras. The so-called Singer-Wermer theorem, which is a classical theorem of complex Banach algebra theory, states that every continuous derivation on a commutative Banach algebra maps the algebra into its Jacobson radical, and Thomas [19] proved that the Singer-Wermer theorem remains true without assuming the continuity of the derivation. So far, many authors have studied the image of derivations, see, e.g. [1,7,8,11,14,15,16,19,20,21] and references therein. In this article, we are going to introduce a new class of derivations called strongly generalized derivation of order n and investigate its image. First, let us recall some basic definitions and set the notations which are used in what follows. Let $\mathcal{A}$ be an algebra. The set of all primitive ideals of $\mathcal{A}$ is denoted by $\Pi(\mathcal{A})$. The Jacobson radical of an algebra $\mathcal{A}$ is the intersection of all primitive ideals of $\mathcal{A}$ which is denoted by $rad(\mathcal{A})$. Indeed, $rad(\mathcal{A}) = \bigcap_{\mathcal{P}\in\Pi(\mathcal{A})} \mathcal{P}$. The algebra $\mathcal{A}$ is called semisimple if $rad(\mathcal{A}) = \{0\}$. A nonzero linear functional φ on an algebra $\mathcal{A}$ is called a *character* if $\varphi(ab) = \varphi(a)\varphi(b)$ for every $a, b \in \mathcal{A}$. The set of all characters on $\mathcal{A}$ is denoted by $\Phi_{\mathcal{A}}$. According to [4, Proposition 1.3.37], the kernel of φ, $\ker\varphi$, is a maximal ideal of $\mathcal{A}$ for every $\varphi \in \Phi_{\mathcal{A}}$. Recall that an algebra (or ring) $\mathcal{A}$ is called prime if for $a, b \in \mathcal{A}$, $a\mathcal{A}b = \{0\}$ implies that $a = 0$ or $b = 0$, and is semiprime if for $a \in \mathcal{A}$, $a\mathcal{A}a = \{0\}$ implies that $a = 0$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras and let n be a positive integer. A linear mapping $\Delta : \mathcal{A} \to \mathcal{B}$ is called a *strongly generalized derivation of order* n, if there exist the families of linear mappings $\{E_k : \mathcal{A} \to \mathcal{B}\}_{k=1}^n$, $\{F_k : \mathcal{A} \to \mathcal{B}\}_{k=1}^n$, $\{G_k : \mathcal{A} \to \mathcal{B}\}_{k=1}^n$ and $\{H_k : \mathcal{A} \to \mathcal{B}\}_{k=1}^n$ which satisfy

$$\Delta(ab) = \sum_{k=1}^{n} [E_k(a)F_k(b) + G_k(a)H_k(b)]$$

for all $a, b \in \mathcal{A}$. Clearly, for $n = 1$, we have $\Delta(ab) = E(a)F(b) + G(a)H(b)$ for all $a, b \in \mathcal{A}$, where $E, F, G, H : \mathcal{A} \to \mathcal{B}$ are linear mappings. As can be seen, if Δ is a strongly generalized derivation of order one, then it covers the notion of a derivation (if $\Delta = E = H$ and $F = G = I$), the notion of a generalized

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(σ,τ)-derivation associated with a linear mapping d (if $\Delta = E$, $F=\sigma$, $G=\tau$ and $H=d$), the notion of a left σ-centralizer (if $\Delta = E$, $F=\sigma$ and G or H is zero), the notion of a right τ-centralizer (if E or F is zero, $G=\tau$ and $H=\Delta$), the notion of a generalized derivation associated with a mapping d (if $\Delta = E$, $F=G=I$ and $H=d$), the notion of a homomorphism (if $\Delta = E = F$ and $G=0$ or $H=0$), and the notion of a ternary derivation (if $F=G=I$). Let $\mathcal{A}$ be an algebra. Recall that a triplet of linear maps (D,E,H) of $\mathcal{A}$ is a ternary derivation of $\mathcal{A}$ if $D(ab)=E(a)b+aH(b)$ for all $a,b\in\mathcal{A}$.

Also, if Δ is a strongly generalized derivation of order 2, we have

$$\Delta(ab)=E_1(a)F_1(b)+G_1(a)H_1(b)+E_2(a)F_2(b)+G_2(a)H_2(b)$$

for all $a,b\in\mathcal{A}$, where $E_i,F_i,G_i,H_i:\mathcal{A}\to\mathcal{B}$ are linear mappings for any $i\in\{1,2\}$. For example, every (δ,ε)-double derivation is a strongly generalized derivation of order 2. For more material about (δ,ε)-double derivations, see, e.g. [9,17]. Now, we provide another example in this regard. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. A sequence $\{d_n\}$ of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a higher derivation if $d_n(ab)=\sum_{k=0}^{n} d_{n-k}(a)d_k(b)$ for all $a,b\in\mathcal{A}$ and all nonnegative integer n. Let n be a positive integer and let $\{d_n\}$ be a higher derivation. Then every d_n is a strongly generalized derivation of order m in which

$$m=\begin{cases} \frac{n+2}{2} & n \textit{ is even},\\ \frac{n+1}{2} & n \textit{ is odd}.\end{cases}$$

We know that derivations are used in quantum mechanics (see [2,3]), and it is interesting to note that the applications of generalized types of derivations, such as generalized derivations and (σ,τ)-derivations, to important physical topics have recently been studied. For example, see [13] for the application of generalized derivations in general relativity, and see [6,12] for the application of (σ,τ)-derivations in theoretical physics. Therefore, it is possible that *strongly generalized derivations of order n* will be considered by physicists in the future and used in the study of physical topics. Hence, it is interesting to investigate details of this general notion of derivations. If $\Delta:\mathcal{A}\to\mathcal{B}$ is a strongly generalized derivation of order one associated with the mappings $E,F,G,H:\mathcal{A}\to\mathcal{B}$, then we say Δ is an (E,F,G,H)-derivation. Also, if Δ is a strongly generalized derivation of order n associated with the families $\{E_k:\mathcal{A}\to\mathcal{B}\}_{k=1}^n$, $\{F_k:\mathcal{A}\to\mathcal{B}\}_{k=1}^n$, $\{G_k:\mathcal{A}\to\mathcal{B}\}_{k=1}^n$ and $\{H_k:\mathcal{A}\to\mathcal{B}\}_{k=1}^n$ of mappings, we say Δ is an $\left(\{E_k\}_{k=1}^n,\{F_k\}_{k=1}^n,\{G_k\}_{k=1}^n,\{H_k\}_{k=1}^n\right)$-derivation.

Now, we state some of the results in this paper. Before stating the main result of this article about the image of strongly generalized derivations of order one, the symbol $\mathcal{E}_{(z,T,S)}$ is introduced. Let $\mathcal{A}$ be a Banach algebra and let $T,S:\mathcal{A}\to\mathcal{A}$ be two continuous linear mappings. For any complex number z, we define the linear mapping $\mathcal{E}_{z,T,S}:\mathcal{A}\to\mathcal{A}$ by $\mathcal{E}_{(z,T,S)}(a)=\sum_{n=0}^{\infty}\frac{z^nT^n(a)}{n!}$ in which $T^0=S$. It is clear that $\mathcal{E}_{(z,T,S)}$ is a continuous linear mapping on $\mathcal{A}$. A triplet of linear mappings (Ω,G,F) of $\mathcal{A}$ is called a ternary homomorphism if $\Omega(ab)=G(a)F(b)$ holds for all $a,b\in\mathcal{A}$. For more material regarding this notion, see [10]. Now, we state our main result concerning the image of (E,F,G,H)-derivations.
Let $\mathcal{A}$ be a unital Banach algebra with identity element $\mathbf{e}$, and let $\Delta:\mathcal{A}\to\mathcal{A}$ be a continuous (E,F,G,H)-derivation such that E,F,G and H are continuous, $G^2=G$, $F^2=F$, $GE=EG=E$ and $FH=HF=H$. Let $(\Omega,G,F):\mathcal{A}\to\mathcal{A}$ be a ternary homomorphism. Suppose that $\varphi\left(\mathcal{E}_{(z,\Delta,\delta)}(\mathbf{e})\right)\neq 0$ for all $z\in\mathbb{C}$ and all $\varphi\in\Phi_{\mathcal{A}}$.
(i) If $\varphi(\mathcal{E}_{(z,H,F)}(\mathbf{e}))\neq 0$ for all $z\in\mathbb{C}$ and all $\varphi\in\Phi_{\mathcal{A}}$, then $H(\mathcal{A})\subseteq\bigcap_{\varphi\in\Phi_{\mathcal{A}}}\ker(\varphi)$. In particular, if $\mathcal{A}$ is a commutative Banach algebra, then $H(\mathcal{A})\subseteq rad(\mathcal{A})$.
(ii) If $\varphi(\mathcal{E}_{(z,E,G)}(\mathbf{e}))\neq 0$ for all $z\in\mathbb{C}$ and all $\varphi\in\Phi_{\mathcal{A}}$, then $E(\mathcal{A})\subseteq\bigcap_{\varphi\in\Phi_{\mathcal{A}}}\ker(\varphi)$. In particular, if $\mathcal{A}$ is a commutative Banach algebra, then $E(\mathcal{A})\subseteq rad(\mathcal{A})$.
(iii) If both $\varphi(\mathcal{E}_{(z,H,F)}(\mathbf{e}))$ and $\varphi(\mathcal{E}_{(z,E,G)}(\mathbf{e}))$ are nonzero for all $z\in\mathbb{C}$ and all $\varphi\in\Phi_{\mathcal{A}}$, then $\Delta(\mathcal{A})\subseteq\bigcap_{\varphi\in\Phi_{\mathcal{A}}}\ker(\varphi)$. In particular, if $\mathcal{A}$ is a commutative Banach algebra, then $\Delta(\mathcal{A})\subseteq rad(\mathcal{A})$.
Obviously, the above-mentioned result generalizes Singer-Wermer theorem to (E,F,G,H)-derivations. Using this theorem, we obtain some results on the image of generalized (σ,τ)-derivations and (δ,ε)-double derivations. Also, some other related results are presented.

2. Main Results

Throughout the paper, without further mention, I denotes the identity mapping on an algebra and $\mathbf{e}$ stands for the identity element of any unital algebra. We begin with the following propositions expressing some properties of strongly generalized derivations of order one. Recall that the commutator of the mappings S and T is $[T,S]=TS-ST$.

Proposition 2.1. *Let $\mathcal{A}$ be an algebra, let $E,F,G,H:\mathcal{A}\to\mathcal{A}$ be mappings such that $[G,E]=[F,H]=0$. Let $\Delta:\mathcal{A}\to\mathcal{A}$ be an (E,F,G,H)-derivation. Then for each $n\in\mathbb{N}$ and $a,b\in\mathcal{A}$, we have*

$$\Delta^n(ab)=\sum_{k=0}^{n}\binom{n}{k}E^{n-k}\big(G^k(a)\big)H^k\big(F^{n-k}(b)\big), \tag{2.1}$$

where $F^0=G^0=E^0=H^0=I$.

Proof. We use induction on n. For $n=1$ there is nothing to do. Assume that (2.1) holds for n. Indeed, we suppose that

$$\begin{aligned}\Delta^n(ab)&=\sum_{k=0}^{n}\binom{n}{k}E^{n-k}\big(G^k(a)\big)H^k\big(F^{n-k}(b)\big)\\&=\sum_{k=0}^{n}\binom{n}{k}G^k\big(E^{n-k}(a)\big)F^{n-k}\big(H^k(b)\big),\end{aligned}$$

where $F^0=G^0=E^0=H^0=I$. Let a and b be two arbitrary elements of $\mathcal{A}$. We have the following expressions:

$$\begin{aligned}\Delta^{n+1}(ab)&=\Delta\big(\Delta^n(ab)\big)=\Delta\Big(\sum_{k=0}^{n}\binom{n}{k}G^k\big(E^{n-k}(a)\big)F^{n-k}\big(H^k(b)\big)\Big)\\
&=\sum_{k=0}^{n}\binom{n}{k}\Big[E\left(G^k\left(E^{n-k}(a)\right)\right)F\left(F^{n-k}\left(H^k(b)\right)\right)+G\left(G^k\left(E^{n-k}(a)\right)\right)H\left(F^{n-k}\left(H^k(b)\right)\right)\Big]\\
&=\sum_{k=0}^{n}\binom{n}{k}G^k\left(E^{n+1-k}(a)\right)F^{n+1-k}\left(H^k(b)\right)+\sum_{k=0}^{n}\binom{n}{k}G^{k+1}\left(E^{n-k}(a)\right)F^{n-k}\left(H^{k+1}(b)\right)\\
&=\sum_{k=0}^{n}\binom{n}{k}G^k\left(E^{n+1-k}(a)\right)F^{n+1-k}\left(H^k(b)\right)+\sum_{k=1}^{n+1}\binom{n}{k-1}G^k\left(E^{n+1-k}(a)\right)F^{n+1-k}\left(H^k(b)\right)\\
&=\sum_{k=1}^{n}\binom{n}{k}G^k\left(E^{n+1-k}(a)\right)F^{n+1-k}\left(H^k(b)\right)+\sum_{k=1}^{n}\binom{n}{k-1}G^k\left(E^{n+1-k}(a)\right)F^{n+1-k}\left(H^k(b)\right)\\
&\quad+E^{n+1}(a)F^{n+1}(b)+G^{n+1}(a)H^{n+1}(b)\\
&=\sum_{k=1}^{n}\left[\binom{n}{k}+\binom{n}{k-1}\right]G^k\left(E^{n+1-k}(a)\right)F^{n+1-k}\left(H^k(b)\right)+E^{n+1}(a)F^{n+1}(b)+G^{n+1}(a)H^{n+1}(b)\\
&=\sum_{k=1}^{n}\binom{n+1}{k}G^k\left(E^{n+1-k}(a)\right)F^{n+1-k}\left(H^k(b)\right)+E^{n+1}(a)F^{n+1}(b)+G^{n+1}(a)H^{n+1}(b)\\
&=\sum_{k=0}^{n+1}\binom{n+1}{k}G^k\left(E^{n+1-k}(a)\right)F^{n+1-k}\left(H^k(b)\right).\end{aligned}$$

Thereby, our proof is complete. □

An immediate corollary of the previous proposition reads as follows:

Corollary 2.2. *Let $\mathcal{A}$ be an algebra, let $E,F,G,H:\mathcal{A}\to\mathcal{A}$ be mappings such that $GE=EG=E$ and let $FH=HF=H$. Let $\Delta:\mathcal{A}\to\mathcal{A}$ be an (E,F,G,H)-derivation. Then for each $n\in\mathbb{N}$ and $a,b\in\mathcal{A}$,* *we have*

$$\Delta^n(ab) = \sum_{k=1}^{n-1} \binom{n}{k} E^{n-k}(a)H^k(b) + G^n(a)H^n(b) + E^n(a)F^n(b),$$

where $F^0 = G^0 = E^0 = H^0 = I$.

Proposition 2.3. *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *are two algebras such that* $\mathcal{B}$ *is semiprime,* $\Delta, E, F, G, H : \mathcal{A} \to \mathcal{B}$ *are mappings such that* Δ *is linear and* $\Delta(ab) = E(a)F(b) + G(a)H(b)$ *holds for all* $a, b \in \mathcal{A}$.
(i) If H *is linear and* E *is surjective, then* F *is linear.*
(ii) If F *is linear and* G *is surjective, then* H *is linear.*
(iii) If G *is linear and* F *is surjective, then* E *is linear.*
(iv) If E *is linear and* H *is surjective, then* G *is linear.*

Proof. (i) For each $a, b, c \in \mathcal{A}$ and for each $\lambda \in \mathbb{C}$, we have

$$\Delta(a\lambda(b+c)) = E(a)F(\lambda b + \lambda c) + G(a)H(\lambda(b+c)).$$

On the other hand, since Δ is a linear mapping, we have the following expressions:

$$\begin{aligned}\Delta(a\lambda(b+c)) &= \lambda\Delta(ab) + \lambda\Delta(ac) \\ &= \lambda E(a)F(b) + \lambda G(a)H(b) + \lambda E(a)F(c) + \lambda G(a)H(c),\end{aligned}$$

for all $a, b, c \in \mathcal{A}$. Comparing the last two equations regarding $\Delta(a\lambda(b+c))$, we get that

$$E(a)\big[F(\lambda b + \lambda c) - \lambda F(b) - \lambda F(c)\big] = G(a)\big[H(b) + \lambda H(c) - H(\lambda b + \lambda c)\big],$$

If H is linear and E is surjective, it follows from the above-mentioned equation that

$$\mathcal{B}\big[F(\lambda b + \lambda c) - \lambda F(b) - \lambda F(c)\big] = 0.$$

Previous equation with the semiprimness of $\mathcal{B}$ imply that F is a linear mapping.
Similarly, one can prove (ii), (iii) and (iv). □

Proposition 2.4. *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *are two unital algebras,* $\Delta : \mathcal{A} \to \mathcal{B}$ *is a* (Δ, F, G, H)*-derivation such that* $G(\boldsymbol{e}) = \boldsymbol{e}$. *Then* $\Delta(\boldsymbol{e})[F(bc) - F(b)F(c)] = 0$ *if and only if* H *is an* (H, F, G, H)*-derivation. In particular, if* F *is a homomorphism, then* H *is an* (H, F, G, H)*-derivation.*

Proof. For each $a, b, c \in \mathcal{A}$, we have

$$\Delta(abc) = \Delta(a)F(bc) + G(a)H(bc).$$

On the other hand, we have

$$\begin{aligned}\Delta(abc) &= \Delta(ab)F(c) + G(ab)H(c) \\ &= \Delta(a)F(b)F(c) + G(a)H(b)F(c) + G(ab)H(c).\end{aligned}$$

Comparing the last two equations regarding $\Delta(abc)$, we get that

$$\Delta(a)\big[F(bc) - F(b)F(c)\big] = G(a)\big[H(b)F(c) - H(bc)\big] + G(ab)H(c),$$

Putting $a = \mathbf{e}$ in the previous equation and using the assumption that $G(\mathbf{e}) = \mathbf{e}$, we obtain that

$$\Delta(\mathbf{e})\big[F(bc) - F(b)F(c)\big] = H(b)F(c) - H(bc) + G(b)H(c).$$

It follows from the above-mentioned equation that $\Delta(\mathbf{e})\big[F(bc) - F(b)F(c)\big] = 0$ if and only if H is an (H, F, G, H)-derivation. It is clear that if F is a homomorphism, then H is an (H, F, G, H)-derivation. □

Let $\mathcal{A}$ be a complex algebra. Recall that an involution over $\mathcal{A}$ is a map $* : \mathcal{A} \to \mathcal{A}$ satisfying the following conditions for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$:
(i) $(a^*)^* = a$,
(ii) $(ab)^* = b^*a^*$,
(iii) $(a+b)^* = a^* + b^*$,
(iv) $(\lambda a)^* = \overline{\lambda} a^*$.
An algebra $\mathcal{A}$ equipped with an involution $*$ is called an involutive algebra or $*$-algebra and is denoted, as an ordered pair, by $(\mathcal{A}, *)$. Let $(\mathcal{A}, *)$ be an involutive algebra. We define a mapping $T^* : \mathcal{A} \to \mathcal{A}$ by $T^*(a) = (T(a^*))^*$ for all $a \in \mathcal{A}$. A mapping $T : \mathcal{A} \to \mathcal{A}$ is called an involution-preserving map or a $*$-map if $T = T^*$.

Proposition 2.5. *Let $\mathcal{A}$ be an $*$-algebra and let Δ be an (E, F, F, H)-derivation such that F is an involution-preserving map. Then there exist the $*$-mappings Δ_1 and Δ_2 such that $\Delta = \Delta_1 + i\Delta_2$ and further Δ_1 is an (α, F, F, α^*)-derivation and Δ_2 is a (β, F, F, β^*)-derivation.*

Proof. Considering $\Delta_1 = \frac{\Delta+\Delta^*}{2}$ and $\Delta_2 = i\left(\frac{\Delta^*-\Delta}{2}\right)$, it is observed that $\Delta_1^* = \Delta_1$, $\Delta_2^* = \Delta_2$ and $\Delta = \Delta_1 + i\Delta_2$. We have the following expressions:

$$\begin{aligned}
\Delta_1(ab) &= \frac{E(a)F(b) + F(a)H(b) + \big(E(b^*)F(a^*) + F(b^*)H(a^*)\big)^*}{2} \\
&= \frac{E(a)F(b) + F(a)H(b) + F(a)E^*(b) + H^*(a)F(b)}{2} \\
&= \alpha(a)F(b) + F(a)\alpha^*(b),
\end{aligned}$$

where $\alpha = \frac{E+H^*}{2}$. It means that Δ_1 is an (α, F, F, α^*)-derivation. Similarly, we obtain that $\Delta_2(ab) = i\Big[\left(\frac{H^*(a)-E(a)}{2}\right)F(b) + F(a)\left(\frac{E^*(b)-H(b)}{2}\right)\Big]$ for all $a, b \in \mathcal{A}$. Considering $\beta = i\left(\frac{H^*-E}{2}\right)$, we see that $\Delta_2(ab) = \beta(a)F(b) + F(a)\beta^*(b)$ for all $a, b \in \mathcal{A}$. This means that Δ_2 is a (β, F, F, β^*)-derivation. Thereby, the proof is complete. ☐

Let $T, S : \mathcal{A} \to \mathcal{A}$ be two continuous linear mappings. For any complex number z, we define a linear mapping $\mathcal{E}_{(z,T,S)} : \mathcal{A} \to \mathcal{A}$ by $\mathcal{E}_{(z,T,S)}(a) = \sum_{n=0}^{\infty} \frac{z^n T^n(a)}{n!}$ in which $T^0 = S$. Recall that a triplet of linear mappings (Ω, G, F) of $\mathcal{A}$ is called a ternary homomorphism if $\Omega(ab) = G(a)F(b)$ holds for all $a, b \in \mathcal{A}$.

We are now in a position to establish our main result about the image of strongly generalized derivations of order one.

Theorem 2.6. *Let $\mathcal{A}$ be a unital Banach algebra and let $\Delta : \mathcal{A} \to \mathcal{A}$ be a continuous (E, F, G, H)-derivation such that E, F, G and H are continuous, $G^2 = G$, $F^2 = F$, $GE = EG = E$ and $FH = HF = H$. Let $(\Omega, G, F) : \mathcal{A} \to \mathcal{A}$ be a ternary homomorphism, i.e. $\Omega(ab) = G(a)F(b)$ for all $a, b \in \mathcal{A}$. Suppose that $\varphi\left(\mathcal{E}_{(z,\Delta,\Omega)}(\boldsymbol{e})\right) \neq 0$ for all $z \in \mathbb{C}$ and all $\varphi \in \Phi_{\mathcal{A}}$.*
(i) If $\varphi(\mathcal{E}_{(z,H,F)}(\boldsymbol{e})) \neq 0$ for all $z \in \mathbb{C}$ and all $\varphi \in \Phi_{\mathcal{A}}$, then $H(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker(\varphi)$. In particular, if $\mathcal{A}$ is a commutative Banach algebra, then $H(\mathcal{A}) \subseteq rad(\mathcal{A})$.
(ii) If $\varphi(\mathcal{E}_{(z,E,G)}(\boldsymbol{e})) \neq 0$ for all $z \in \mathbb{C}$ and all $\varphi \in \Phi_{\mathcal{A}}$, then $E(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker(\varphi)$. In particular, if $\mathcal{A}$ is a commutative Banach algebra, then $E(\mathcal{A}) \subseteq rad(\mathcal{A})$.
(iii) If both $\varphi(\mathcal{E}_{(z,H,F)}(\boldsymbol{e}))$ and $\varphi(\mathcal{E}_{(z,E,G)}(\boldsymbol{e}))$ are nonzero for all $z \in \mathbb{C}$ and all $\varphi \in \Phi_{\mathcal{A}}$, then $\Delta(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker(\varphi)$. In particular, if $\mathcal{A}$ is a commutative Banach algebra, then $\Delta(\mathcal{A}) \subseteq rad(\mathcal{A})$.

Proof. (i) Considering the above-mentioned conditions for the mappings E, F, G and H, we get that

$$\Delta^n(ab) = \sum_{k=0}^{n} \binom{n}{k} E^{n-k}(a) H^k(b),$$

in which $E^0 = G$, $H^0 = F$ and $\Delta^0(ab) = \Omega(ab) = G(a)F(b)$ for all $a, b \in \mathcal{A}$. Then we have the following expressions:

$$\begin{aligned}
\mathcal{E}_{(z,\Delta,\Omega)}(ab) &= \sum_{n=0}^{\infty} \frac{z^n \Delta^n(ab)}{n!} \\
&= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k} E^{n-k}(a) H^k(b) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k} E^{n-k}(a)}{(n-k)!} \frac{z^k H^k(b)}{k!} \\
&= \mathcal{E}_{(z,E,G)}(a) \mathcal{E}_{(z,H,F)}(b),
\end{aligned}$$

which means that

$$\mathcal{E}_{(z,\Delta,\Omega)}(ab) = \mathcal{E}_{(z,E,G)}(a) \mathcal{E}_{(z,H,F)}(b), \qquad (z \in \mathbb{C}, \ a, b, \in \mathcal{A}).$$

Let φ be an arbitrary character on $\mathcal{A}$. For $z \in \mathbb{C}$, we define the mappings $\psi_z, \theta_z, \phi_z : \mathcal{A} \to \mathbb{C}$ by

$$\begin{aligned}
\psi_z(a) &= \varphi \mathcal{E}_{(z,\Delta,\Omega)}(a) = \varphi \left(\Omega(a) + z\Delta(a) + \frac{z^2 \Delta^2(a)}{2!} + \frac{z^3 \Delta^3(a)}{3!} + \ldots \right), \\
\theta_z(a) &= \varphi \mathcal{E}_{(z,E,G)}(a) = \varphi \left(G(a) + zE(a) + \frac{z^2 E^2(a)}{2!} + \frac{z^3 E^3(a)}{3!} + \ldots \right), \\
\phi_z(a) &= \varphi \mathcal{E}_{(z,H,F)}(a) = \varphi \left(F(a) + zH(a) + \frac{z^2 H^2(a)}{2!} + \frac{z^3 H^3(a)}{3!} + \ldots \right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\psi_z(ab) = \varphi \mathcal{E}_{(z,\Delta,\Omega)}(ab) &= \varphi \left(\mathcal{E}_{(z,E,G)}(a) \mathcal{E}_{(z,H,F)}(b) \right) \\
&= \varphi \left(\mathcal{E}_{(z,E,G)}(a) \right) \varphi \left(\mathcal{E}_{(z,H,F)}(b) \right) \\
&= \theta_z(a) \phi_z(b),
\end{aligned}$$

for all $a, b \in \mathcal{A}$. Considering $M = 1 + |\phi_z(\mathbf{e})|$, for any $z \in \mathbb{C}$, we define a linear mapping $\lambda_z : \mathcal{A} \to \mathcal{A}$ by $\lambda_z(a) = \frac{\phi_z(a)}{M}$. Now, we show that $\|\lambda_z\| \leq 1$ for all $z \in \mathbb{C}$. To obtain a contradiction, suppose that there exists $a_0 \in \mathcal{A}$ such that $\|a_0\| < 1$ and $|\lambda_z(a_0)| > 1$. Letting $x = \frac{\phi_z(\mathbf{e}) a_0}{M \lambda_z(a_0)}$, we see that $\lambda_z(x) = \frac{\phi_z(\mathbf{e})}{M}$ and $\|x\| < 1$. By [5, Theorem 1.4.2(i)], there exists $a_1 \in \mathcal{A}$ with $a_1(\mathbf{e} - x) = \mathbf{e}$. We have

$$\begin{aligned}
\frac{\psi_z(\mathbf{e})}{M} &= \theta_z(a_1) \frac{\phi_z(\mathbf{e} - x)}{M} = \theta_z(a_1)\left(\frac{\phi_z(\mathbf{e})}{M} - \frac{\phi_z(x)}{M}\right) \\
&= \theta_z(a_1)\left(\frac{\phi_z(\mathbf{e})}{M} - \lambda_z(x)\right) = \theta_z(a_1)\left(\frac{\phi_z(\mathbf{e})}{M} - \frac{\phi_z(\mathbf{e})}{M}\right) \\
&= 0,
\end{aligned}$$

which means that $\psi_z(\mathbf{e}) = 0$. The required contradiction is obtained, since we are assuming $\psi_z(\mathbf{e}) = \varphi \mathcal{E}_{(z,\Delta,\Omega)}(\mathbf{e}) \neq 0$ for all $z \in \mathbb{C}$ and all $\varphi \in \Phi_{\mathcal{A}}$. Hence, $\|\lambda_z\| \leq 1$ for all $z \in \mathbb{C}$. For an arbitrary element $a \in \mathcal{A}$, we define a mapping $\Lambda_a : \mathbb{C} \to \mathbb{C}$ by $\Lambda_a(z) = \lambda_z(a) = \frac{\phi_z(a)}{M} = \frac{\varphi \mathcal{E}_{(z,H,F)}(a)}{M}$. It is clear that $|\Lambda_a(z)| \leq \|a\|$ and this means that Λ_a is a bounded and analytical function. Using Liouville's theorem, we get that Λ_a is a constant function. So there exists $\lambda_0 \in \mathbb{C}$ such that $\Lambda_a(z) = \lambda_0$ for all $z \in \mathbb{C}$, which means that $\varphi \left(F(a) + zH(a) + \frac{z^2 H^2(a)}{2!} + \frac{z^3 H^3(a)}{3!} + \ldots \right) = \lambda_0 M$ for all $z \in \mathbb{C}$. Therefore, we have

$$0 = \frac{d}{dz} \left(\varphi \left(F(a) + zH(a) + \frac{z^2 H^2(a)}{2!} + \frac{z^3 H^3(a)}{3!} + \ldots \right) \right) \Big|_{z=0} = \varphi \left(H(a) \right),$$

which means that $H(a) \in ker(\varphi)$. Since we are assuming that φ and a are arbitrary, we have $H(\mathcal{A}) \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker(\varphi)$. If $\mathcal{A}$ is a commutative algebra, $\bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker(\varphi) = rad(\mathcal{A})$ (see [4]). Consequently, $H(\mathcal{A}) \subseteq rad(\mathcal{A})$, as desired.
Similarly, we can prove (ii) and (iii) and we leave them to the interested reader. □

In the following, there are some immediate consequences of Theorem 2.6.

Corollary 2.7 (Singer-Wermer theorem)**.** *Let $\mathcal{A}$ be a commutative Banach algebra and let $d : \mathcal{A} \to \mathcal{A}$ be a continuous derivation. Then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

Corollary 2.8. *Let $\mathcal{A}$ be a unital Banach algebra.*
Let $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be two continuous linear mappings such that $\tau(ab) = \tau(a)\sigma(b)$ for all $a, b \in \mathcal{A}$ and let $f : \mathcal{A} \to \mathcal{A}$ be a continuous generalized (σ, τ)-derivation associated with a continuous linear mapping $d : \mathcal{A} \to \mathcal{A}$ such that $\tau^2 = \tau$, $\sigma^2 = \sigma$, $\tau f = f\tau = f$ and $\sigma d = d\sigma = d$. If both $\varphi(\mathcal{E}_{(z,f,\tau)}(\boldsymbol{e}))$ and $\varphi(\mathcal{E}_{(z,d,\sigma)}(\boldsymbol{e}))$ are nonzero for all $z \in \mathbb{C}$ and all $\varphi \in \Phi_{\mathcal{A}}$, then $f(\mathcal{A}), d(\mathcal{A}) \subseteq \bigcap_{\varphi\in\Phi_{\mathcal{A}}} \ker(\varphi)$. In particular, if $\mathcal{A}$ is a commutative Banach algebra, then $f(\mathcal{A}), d(\mathcal{A}) \subseteq rad(\mathcal{A})$.

Here, we give an example of an (E, F, G, H)-derivation such that $G^2 = G$, $F^2 = F$, $GE = EG = E$ and $FH = HF = H$.

Example 2.9. *Let $\mathcal{A}$ be an algebra, and let*

$$\mathfrak{A} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \ : \ a, b, c \in \mathcal{A} \right\}$$

Clearly, $\mathfrak{A}$ is an algebra. Define the linear mappings $\Delta, E, F, G, H : \mathfrak{A} \to \mathfrak{A}$ by

$$\Delta\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 2c \\ 0 & 0 & 0 \end{bmatrix},$$

$$F\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & c \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix},$$

$$G\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix},$$

$$H\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that

$$\Delta(AB) = E(A)F(B) + G(A)H(B), \qquad A, B \in \mathfrak{A},$$

which means that Δ is a (E, F, G, H)-derivation on $\mathfrak{A}$ and also one can easily see that $G^2 = G$, $F^2 = F$, $GE = EG = E$ and $FH = HF = H$.

Now, we are going to present some results about the image of (δ, ε)-double derivations. Mirzavaziri and Omidvar Tehrani [17] introduced the concept of a (δ, ε)-double derivation as follows:

Definition 2.10. *Let $\mathcal{A}$ be an algebra and let $\delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ be linear mappings. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a (δ, ε)-double derivation if*

$$d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b),$$

for all $a, b \in \mathcal{A}$. By a δ-double derivation we mean a (δ, δ)-double derivation.

Considering $\varepsilon = I$, the identity mapping on $\mathcal{A}$, we have $d(ab) = d(a)b + ad(b) + \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. Considering $g = d + \delta$, we have

$$d(ab) = g(a)b + ag(b), \qquad a, b \in \mathcal{A}.$$

Here, we provide a result on the image of a (δ, I)-double derivation as a strongly generalized derivation of order 2 in which $E_1 = H_1 = d$, $F_1 = G_1 = F_2 = G_2 = I$ and $E_2 = H_2 = \delta$.

Corollary 2.11. *Let $\mathcal{A}$ be a unital Banach algebra, let $\delta : \mathcal{A} \to \mathcal{A}$ be a continuous linear mapping, and let $d : \mathcal{A} \to \mathcal{A}$ be a continuous $(\delta, I)-$ double derivation. Suppose that $\varphi(\mathcal{E}_{(z,d,I)}(\boldsymbol{e})) \neq 0$ for all $z \in \mathbb{C}$ and all $\varphi \in \Phi_{\mathcal{A}}$. If $\varphi(\mathcal{E}_{(z,g,I)}(\boldsymbol{e})) \neq 0$, where $g = \delta + d$, then $d(\mathcal{A}), \delta(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker(\varphi)$. In particular, if $\mathcal{A}$ is a commutative Banach algebra, then $d(\mathcal{A}), \delta(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

Proof. According to the above discussion, $d(ab) = g(a)b + ag(b)$ for all $a, b \in \mathcal{A}$, which means that d is a (g, I, I, g)-derivation. It follows from Theorem 2.6 that $d(\mathcal{A}), g(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker(\varphi)$. So we can deduce that $\delta(\mathcal{A}) \subseteq \bigcap_{\varphi \in \Phi_{\mathcal{A}}} \ker(\varphi)$ as well. Clearly, if $\mathcal{A}$ is commutative, then we get that $d(\mathcal{A}), \delta(\mathcal{A}) \subseteq rad(\mathcal{A})$. $\square$

In the following theorem, we obtain a result about the image of a (Δ, F, G, Δ)-derivation without assuming the continuity of Δ. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras and let $T : \mathcal{A} \to \mathcal{B}$ be a mapping. We say that T is symmetric whenever $T(ab) = T(ba)$ for all $a, b \in \mathcal{A}$. Recall that an element x in a normed algebra $\mathfrak{Z}$ is called quasi-nilpotent if $lim_{n \to +\infty} \|x^n\|^{\frac{1}{n}} = 0$. The set of all quasi-nilpotent elements of $\mathfrak{Z}$ is denoted by $Q(\mathfrak{Z})$. To prove the next theorem, we use some ideas from [8].

Theorem 2.12. *Let $\mathcal{A}$ be a Banach $*$-algebra and let $\Delta : \mathcal{A} \to \mathcal{A}$ be an involution-preserving (Δ, F, G, Δ)-derivation, i.e. Δ, F and G are involution-preserving maps. If $\frac{F+G}{2}$ is a nonzero, continuous symmetric homomorphism of $\mathcal{A}$ and $(\frac{F+G}{2})^2 = \frac{F+G}{2}$, then $\left(\frac{F+G}{2}\right) \Delta \left(\frac{F+G}{2}\right)(\mathcal{A}) \subseteq Q(\mathcal{A})$.*

Proof. We have

$$\Delta(ab) = \Delta(a)F(b) + G(a)\Delta(b), \qquad (a, b \in \mathcal{A}). \tag{2.2}$$

Also, we have the following statements:

$$\begin{aligned} \Delta(ab) &= (\Delta(ab)^*)^* = \left(\Delta(b^*)F(a^*) + G(b^*)\Delta(a^*)\right)^* \\ &= F(a)\Delta(b) + \Delta(a)G(b), \end{aligned}$$

which means that

$$\Delta(ab) = F(a)\Delta(b) + \Delta(a)G(b), \qquad (a, b \in \mathcal{A}). \tag{2.3}$$

Adding (2.2) and (2.3), we arrive at

$$2\Delta(ab) = \Delta(a)(F(b) + G(b)) + (F(a) + G(a))\Delta(b), \qquad (a, b \in \mathcal{A}). \tag{2.4}$$

Considering $\sigma = \frac{F+G}{2}$ in (2.4), we get that

$$\Delta(ab) = \Delta(a)\sigma(b) + \sigma(a)\Delta(b), \qquad (a, b \in \mathcal{A}). \tag{2.5}$$

This means that Δ is a $(\Delta, \sigma, \sigma, \Delta)$-derivation on $\mathcal{A}$. Since we are assuming that σ is a homomorphism and $\sigma^2 = \sigma$, the linear mapping $\Psi = \sigma\Delta\sigma : \mathcal{A} \to \mathcal{A}$ satisfies $\sigma\Psi = \Psi\sigma = \Psi$ and also we have

$$\begin{aligned}\Psi(ab) &= \sigma\Delta\sigma(ab) = \sigma\Delta(\sigma(a)\sigma(b)) \\ &= \sigma(\Delta\sigma(a)\sigma(b) + \sigma(a)\Delta\sigma(b)) \\ &= \Psi(a)\sigma(b) + \sigma(a)\Psi(b)\end{aligned}$$

for all $a, b \in \mathcal{A}$. Now, we define another multiplication on $\mathcal{A}$ as follows: $a \circledast b = \sigma(ab)$ for all $a, b \in \mathcal{A}$. This new algebra is denoted by $\mathfrak{A}_{\circledast}$. Since σ is nonzero, continuous and $\sigma^2 = \sigma$, it is observed that $\|\sigma\| \geq 1$. We define the following new norm on $\mathfrak{A}_{\circledast}$:

$$\|a\|_\sigma = \|\sigma\|\|a\|.$$

It is clear that $(\mathfrak{A}_{\circledast}, \|.\|_\sigma)$ is a Banach algebra. Ψ is a derivation on the Banach algebra $(\mathfrak{A}_{\circledast}, \|.\|_\sigma)$, since

$$\begin{aligned}\Psi(a \circledast b) &= \Psi(\sigma(a)\sigma(b)) = \Psi(\sigma(a))\sigma(b) + \sigma(a)\Psi(\sigma(b)) \\ &= \sigma(\Psi(a))\sigma(b) + \sigma(a)\sigma(\Psi(b)) \\ &= \sigma(\Psi(a)b) + \sigma(a\Psi(b)) \\ &= \Psi(a) \circledast b + a \circledast \Psi(b).\end{aligned}$$

We are assuming that σ is a symmetric homomorphism on $\mathcal{A}$, that is, $\sigma(ab) = \sigma(ba)$ $(a, b \in \mathcal{A})$, and so, $a \circledast b = b \circledast a$. This means that $\mathfrak{A}_{\circledast}$ is a commutative Banach algebra. Hence, Ψ is a derivation on the commutative Banach algebra $\mathfrak{A}_{\circledast}$ and it follows from [19, Theorem 4.4] that $\Psi(\mathcal{A}) = \Psi(\mathfrak{A}_{\circledast}) \subseteq rad(\mathfrak{A}_{\circledast})$. According to [4, Proposition 1.5.32(ii)], $rad(\mathfrak{A}_{\circledast}) \subseteq Q(\mathfrak{A}_{\circledast})$ and so $\Psi(\mathcal{A}) \subseteq Q(\mathfrak{A}_{\circledast})$. It is easy to see that $\sigma(a) \in Q(\mathcal{A})$ if and only if $a \in Q(\mathfrak{A}_{\circledast})$. So $\Psi(\mathcal{A}) = \sigma\Psi(\mathcal{A}) \subseteq Q(\mathcal{A})$, which means that $\sigma\Delta\sigma(\mathcal{A}) \subseteq Q(\mathcal{A})$, as desired. □

Corollary 2.13. *Let $\mathcal{A}$ be a unital, commutative Banach algebra and let $\Delta : \mathcal{A} \to \mathcal{A}$ be a (Δ, F, G, Δ)-derivation. If $\frac{F+G}{2}$ is a nonzero, continuous homomorphism of $\mathcal{A}$ and $(\frac{F+G}{2})^2 = \frac{F+G}{2}$, then $\left(\frac{F+G}{2}\right)\Delta\left(\frac{F+G}{2}\right)(\mathcal{A}) \subseteq rad(\mathcal{A})$. In particular, if $\Delta(\mathcal{A}) \subseteq \left(\frac{F+G}{2}\right)\Delta\left(\frac{F+G}{2}\right)(\mathcal{A})$, then $\Delta(\mathcal{A}) \subseteq rad(\mathcal{A})$.*

Proof. Since $\mathcal{A}$ is a commutative algebra, one can easily get that $\Delta(ab) = \Delta(a)\sigma(b) + \sigma(a)\Delta(b)$ for all $a, b \in \mathcal{A}$, where $\sigma = \frac{F+G}{2}$. It follows from Theorem 2.12 that $\sigma\Delta\sigma(\mathcal{A}) \subseteq Q(\mathcal{A})$. Since $\mathcal{A}$ is a unital, commutative Banach algebra, $Q(\mathcal{A}) = rad(\mathcal{A})$ (see [5, Proposition 2.2.3(iii)]) and consequently, $\sigma\Delta\sigma(\mathcal{A}) \subseteq rad(\mathcal{A})$. □

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