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Essential Ideal of a Matrix Nearring and Ideal Related Properties of Graphs

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ABSTRACT: In this paper, we consider matrix maps over a zero-symmetric right nearring N with 1. We define the notions of f-essential ideal, f-superfluous ideal, generalized f-essential ideal of a matrix nearring and prove results which exhibit the interplay between these ideals and the corresponding ideals of the base nearring N. We discuss the combinatorial properties such as connectivity, diameter, completeness of a graph (denoted by $\mathcal{L}_g(H)$) defined on generalized essential ideals of a finitely generated module H over N. We prove a characterization for $\mathcal{L}_g(H)$ to be complete. We also prove $\mathcal{L}_g(H)$ has diameter at-most 2 and obtain related properties with suitable illustrations.

Key Words: Essential ideal, matrix nearring, N-group, graph.

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1. Introduction

Nearring is a classical generalization of a ring. Rings can be considered as algebraic systems of linear maps on groups, while nearrings describe a general non-linear case [23]. In this paper, we consider a zero-symmetric right nearring N, and matrix maps over N [19]. Meldrum and Van der Walt [19] defined the notion of a matrix nearring, denoted by $M_n(N)$, which is the subnearring of the nearring $M(N^n)$, the set of all maps from N^n to N^n. Van der Walt [4] explored the relationship between primitive modules over a nearring N and those of the matrix nearring $M_n(N)$. For recent developments in matrix nearrings, we refer to [8,9,13,14,25,27]. The notion of an essential submodule of module over a ring is a discretized analogue to the notion of dense subspace in a topological space [3]. The idea of the graph constructed from a ring was initiated from the concept of a zero-divisor graph (see [2]). Later, based on a ring structure, several types of graphs like annihilator essential graph (see [5]), essential graph (see [22]), total graph (see [2]), prime graph (see [11]), and in case of nearrings zero-divisor graph of a nearring [15] and graph with respect to ideal of a nearring [6,12] were studied. In commutative rings, the author (see [1]) studied the properties of an essential ideal graph and characterized rings based on the different types of graphs. They considered the set of all non-trivial ideals in a commutative ring as the vertex set and an edge is defined if the sum of two ideals is essential in the underlying ring. This concept was generalized to modules over rings by [18]. In [28], small essential ideals and Morita duality of rings were discussed, and in [16], the authors characterized classes of commutative and non-commutative rings for which maximal small and minimal essential ideals coincide. In [20,21,24], the authors discussed the dual aspects like generalized supplements, superfluous ideals etc.
In Section 3 of this paper, we introduce the notion of g-essential ideal in a matrix nearring. We establish a one-one correspondence between the set of all generalised essential (resp. essential) ideals of the nearring N and the set of all generalised f-essential (resp. f-essential) full ideals of $M_n(N)$. In section 4, we define generalized essential ideal graph of a module over a nearring N (denoted as, $\mathcal{L}_g(H)$). We derive properties

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such as diameter, completeness based on a given N-group. We prove that if H is a finitely generated over a zero-symmetric nearring N, then any maximal ideal of H is a universal vertex. Furthermore, the graph is connected with the diameter less than 3. The notion of g-complement of an ideal is introduced as a generalization of a complement and show that every non-zero, non-g-essential ideal is adjacent to its g-complement. We consider a subgraph of g-essential ideal graph, induced by the set of all non-g-essential ideals of H. Finally, it is observed that there exists a path between every two superfluous ideals.

2. Preliminaries

Let N be a zero-symmetric right nearring and H be an N-group [23]. A normal subgroup K of H is an ideal if $n(g+k)-ng \in K$ for all $n \in N$, $g \in H$ and $k \in K$, (we denote $A \trianglelefteq_N H$ if A is a ideal of H). $A \trianglelefteq_N H$ is essential if $A \cap B \neq (0)$ for any $(0) \neq B \trianglelefteq_N H$, and we write it as $A \leq_e H$. A uniform N-group is the one in which every non-zero ideal is essential. $A \trianglelefteq_N H$ is superfluous if $A + B = H$ where $B \trianglelefteq_N H$, implies $B = H$, and we denote it by $A \ll H$. $A \trianglelefteq_N H$ is g-essential if $A \cap B = (0)$ and $B \ll H$, implies $B = (0)$ (denoted as $A \leq_{ge} H$). Moreover, if every non-zero ideal of H is g-essential, then we call H is g-uniform. An N-subgroup H of N is said to be finitely generated (as an ideal) if there exists a subset S of H such that $\langle S \rangle = H$, where $\langle S \rangle$ represents the ideal of H generated by S. We refer to [7,8,9,17,19] for the notions of essential ideals, superfluous ideals of N-groups and modules, and we refer to [23] for the notions of maximal ideal, minimal ideal and completely reducible N-group etc. Throughout, H denotes a finitely generated N-group where N is a zero-symmetric right nearring.
For a zero-symmetric nearring N with 1, let N^n will be the direct sum of n copies of $(N,+)$. The elements of N^n are column vectors and written as $(r_1, \cdots, r_n)$. The symbols i_i and π_j respectively, denote the i^{th} coordinate injective and j^{th} coordinate projective maps. For an element $a \in N$, $i_i(a) = (0, \cdots, \underbrace{a}_{i^{th}}, \cdots, 0)$, and $\pi_j(a_1, \cdots, a_n) = a_j$, for any $(a_1, \cdots, a_n) \in N^n$. The nearring of $n \times n$ matrices over N, denoted by $M_n(N)$, is defined to be the subnearring of $M(N^n)$, generated by the set of maps $\{f_{ij}^a : N^n \to N^n : a \in N, 1 \leq i,j \leq n\}$ where $f_{ij}^a(k_1, \cdots, k_n) := (l_1, l_2, \cdots, l_n)$ with $l_i = ak_j$ and $l_p = 0$ if $p \neq i$. Clearly, $f_{ij}^a = i_i f^a \pi_j$, where $f^a(x) = ax$, for all $a, x \in N$. If N happens to be a ring, then f_{ij}^a corresponds to the $n \times n$-matrix with a in position (i,j) and zeros elsewhere. We refer to [9,13,19] for further definitions and notations in matrix nearrings.
The graphs considered are simple graphs. We denote the vertex set as V, we use $d(u,v)$ to represent the shortest $u \sim v$ path, while the eccentricity of a vertex u, is denoted as $e(u)$ which is $max\{d(u,v) : v \in V\}$; radius is the minimum eccentricity, and the diameter is the maximum eccentricity. A vertex is universal if it is adjacent to every other vertex. For all other notions and definitions in graph theory, we refer to [10,9], and for nearrings, we refer to [9,23]. We use $\sim$ to denote an edge, and $\iff$ for "if and only if".

3. Generalized essential ideals in $M_n(N)$

In this section, we introduce the notion of generalised f-essential ideal in a matrix nearring. We establish a one-one correspondence between the set of all generalised essential (resp. essential) ideals of the nearring N and the set of all generalised f-essential (resp. f-essential) full ideals of $M_n(N)$.

Definition 3.1. *[9]*

1. *Let $\mathcal{K} \trianglelefteq M_n(N)$. Then $\mathcal{K}_\star = \{x \in N : x \in im(\pi_j A)$ for some $A \in \mathcal{K}$ and $j, 1 \leq j \leq n\}$.*

2. *Let $I \trianglelefteq N$. Then $I^\star = \{A \in M_n(N) : A\rho \subseteq I^n$ for all $\rho \in N^n\}$.*

Lemma 3.2. *Let $I, J \trianglelefteq N$. Then $(I \cap J)^\star = I^\star \cap J^\star$.*

Proof. Let $A \in (I \cap J)^\star \iff A\rho \in (I \cap J)^n$ for every $\rho \in N^n \iff \pi_j A\rho \in I \cap J$ for every $\rho \in N^n$ and $1 \leq j \leq n \iff \pi_j A\rho \in I$ and $\pi_j A\rho \in J$ for every $\rho \in N^n$ and $1 \leq j \leq n \iff A\rho \in I^n$ and $A\rho \in J^n$ for every $\rho \in N^n \iff A \in I^\star \cap J^\star$. ☐

Lemma 3.3. *Let $\mathcal{S}, \mathcal{T} \trianglelefteq M_n(N)$ with $\mathcal{S} \cap \mathcal{T} = (0)$. Then $\mathcal{S}_\star \cap \mathcal{T}_\star = (0)$*

Proof. Let $x \in \mathcal{S}_\star \cap \mathcal{T}_\star$. By Lemma 4.4 of [19], we get $f_{11}^x \in \mathcal{S}$ and $f_{11}^x \in \mathcal{T}$, implies $f_{11}^x \in \mathcal{S} \cap \mathcal{T} = (0)$. Therefore, $f_{11}^x = (0)$, and so $(x, 0, \cdots, 0) = f_{11}^x(1, 1, \cdots, 1) = (0, 0, \cdots, 0)$, we get $x = 0$. ☐

Lemma 3.4. *Let $I,\ J \trianglelefteq N$ be such that $I + J = N$. Then $I^\star + J^\star = M_n(N)$.*

Proof. Clearly $I^\star + J^\star \subseteq M_n(N)$. Let $A \in M_n(N)$. We use the induction on weight of A. Let $w(A) = 1$. Then $A = f_{ij}^a$ for some $a \in N$, $1 \leq i, j \leq n$. Since $N = I + J$ and $a \in N$, there exists $x \in I$ and $y \in J$ such that $a = x+y$. By Corollary 4.5 of [19], we get $f_{ij}^x \in I^\star$ and $f_{ij}^y \in J^\star$. Now $f_{ij}^a = f_{ij}^{x+y} = f_{ij}^x + f_{ij}^y \in I^\star + J^\star$. Assume that whenever $w(A) \lneq m$, then $A \in I^\star + J^\star$. Now let $w(A) = m$. Then $A = B + C$ or $A = BC$ with $w(B) \lneq m$ and $w(C) \lneq m$.
Case 1: $A = B + C$. Since $w(B)$ and $w(C)$ is less than m, by induction hypothesis, we can write $B = B_1 + B_2$ and $C = C_1 + C_2$ for some $B_1,\ B_2$ in $I^\star$ and $C_1,\ C_2$ in $J^\star$. Now $A = B + C = B_1 + B_2 + C_1 + C_2 \in I^\star + J^\star + I^\star + J^\star \in I^\star + J^\star$ since $I^\star + J^\star$ is an ideal of $M_n(N)$.
Case 2: $A = BC$. This implies $A = (B_1 + B_2)(C_1 + C_2) = B_1(C_1 + C_2) + B_2(C_1 + C_2) \in I^\star + J^\star$ since $B_1 \in I^\star$ and $B_2 \in J^\star$ and $I^\star,\ J^\star$ are right ideals of $M_n(N)$.
Therefore, $A \in M_n(N)$ implies $A \in I^\star + J^\star$. Hence $I^\star + J^\star = M_n(N)$. □

Lemma 3.5. *Let $I, J \trianglelefteq N$. Then $I^\star + J^\star \subseteq (I + J)^\star$.*

Proof. Let $A \in I^\star + J^\star$. Then $A = B + C$ for some $B \in I^\star$ and $C \in J^\star$. This implies $A\rho \in I^n$ and $B\rho \in J^n$ for every $\rho \in N^n$. Now $A\rho = (B+C)\rho = B\rho + C\rho \in I^n + J^n = (I+J)^n$ and hence $A \in (I+J)^\star$. □

Definition 3.6. *[19] $\mathcal{K} \trianglelefteq M_n(N)$ is called a full ideal if $\mathcal{K} = J^\star$ for some $J \trianglelefteq N$.*

Proposition 3.7. *[19] There is a bijection between the set of all ideals of N and the set of all full ideals of $M_n(N)$ given by $I \to I^\star$ and $\mathcal{S} \to \mathcal{S}_\star$ such that $(I^\star)_\star = I$ and $(\mathcal{S}_\star)^\star$ for an ideal I of N and $\mathcal{S}$ of $M_n(N)$.*

Definition 3.8. *$\mathcal{S} \trianglelefteq M_n(N)$ is called an f-essential ideal if for any full ideal $\mathcal{T}$ of $M_n(N)$, $\mathcal{S} \cap \mathcal{T} = (0)$ implies $\mathcal{T} = (0)$ and it is denoted by $\mathcal{S} \leq_e^f M_n(N)$.*

Proposition 3.9. *Let $I \trianglelefteq N$ and $\mathcal{S} \trianglelefteq M_n(N)$. Then*

1. *$\mathcal{S} \leq_e^f M_n(N)$ implies $\mathcal{S}_\star \leq_e N$.*
2. *$I \leq_e N$ implies $I^\star \leq_e^f M_n(N)$.*
3. *There is a one-one correspondence between the set of all essential ideals of N and the set of all essential full ideals of $M_n(N)$.*

Proof.
1. Let $B \trianglelefteq N$ such that $\mathcal{S}_\star \cap B = (0)$. Then $(\mathcal{S}_\star \cap B)^\star = (0)^\star$. By Lemma 3.2, we get $(\mathcal{S}_\star)^\star \cap B^\star = (0)$. Now $\mathcal{S} \cap B^\star \subseteq (\mathcal{S}_\star)^\star \cap B^\star = (0)$. Since $\mathcal{S} \leq_e^f M_n(N)$, we get $B^\star = (0)$ which implies $B = (0)$. Therefore, $\mathcal{S}_\star \leq_e N$.
2. Let $\mathcal{T}$ be a full ideal of $M_n(N)$ such that $I^\star \cap \mathcal{T} = (0)$. Since $\mathcal{T}$ is a full ideal, we have $\mathcal{T} = K^\star$ for some ideal K of N. Therefore, by Lemma 3.2 we get $(I \cap K)^\star = I^\star \cap K^\star = (0)$ and so $I \cap K = (0)$. Since $I \leq_e N$, we get $K = (0)$ which implies $\mathcal{T} = K^\star = (0)^\star = (0)$. Therefore, $I^\star \leq_e^f M_n(N)$.
3. Let $\mathcal{P} = \{A \trianglelefteq N : A \leq_e N\}$ and $\mathcal{Q} = \{\mathcal{A} \trianglelefteq M_n(N) : \mathcal{A} \leq_e^f M_n(N)\}$. Define $\phi : \mathcal{P} \to \mathcal{Q}$ by $\phi(A) = A^\star$ and $\psi : \mathcal{Q} \to \mathcal{P}$ by $\psi(\mathcal{A}) = \mathcal{A}_\star$. Take $A \in \mathcal{P}$. Then $\psi \circ \phi(A) = \psi(\phi(A)) = \psi(A^\star) = (A^\star)_\star = A$. Take $\mathcal{A} \in \mathcal{Q}$. Then since $\mathcal{A}$ is a full ideal, we get $\phi \circ \psi(\mathcal{A}) = \phi(\psi(\mathcal{A})) = \phi(\mathcal{A}_\star) = (\mathcal{A}_\star)^\star = \mathcal{A}$. □

Lemma 3.10. *Let $a,\ u \in N$. If $a \in \langle u \rangle$, then $f_{ij}^a \in \langle f_{ij}^u \rangle$.*

Proof. From the Notation 3.4.6, given in [9], $\langle u \rangle = \cup A_m$, where m ranges from 0 to infinity, $A_0 = \{u\}$ and $A_{m+1} = A_m^{++} \cup A_m^0 \cup A_m^+ \cup A_m^-$ where $A_m^{++} = \{n + x - n : n \in N \text{ and } x \in A_m\}$, $A_m^0 = \{a - b : a, b \in A_m\}$, $A_m^+ = \{n(n' + x) - nn' : n, n' \in N \text{ and } x \in A_m\}$ and $A_m^- = \{xn : n \in N \text{ and } x \in A_m\}$.
We prove by induction on m. Let $m = 0$. Then $A_1 = A_0^{++} \cup A_0^0 \cup A_0^+ \cup A_0^-$. Let $a \in A_1$.
Case 1: If $a \in A_0^{++}$. Then $a = n + u - n$ for some $n \in N$.
Now $f_{ij}^a = f_{ij}^{n+u-n} = f_{ij}^n + f_{ij}^u - f_{ij}^n \in \langle f_{ij}^u \rangle$ as $f_{ij}^u \in \langle f_{ij}^u \rangle$ and $\langle f_{ij}^u \rangle$ is a normal subgroup of $M_n(N)$.

Case 2: Let $a \in A_0^0$. Then $a = 0$. Clearly $f_{ij}^0 \in \langle f_{ij}^u \rangle$.

Case 3: Let $a \in A_0^+$. Then $a = n(n' + u) - nn'$ for some $n, n' \in N$. Now $f_{ij}^a = f_{ij}^{n(n'+u)-nn'} = f_{ij}^n(f_{ij}^{n'} + f_{ij}^u) - f_{ij}^n f_{ij}^{n'} =\in \langle f_{ij}^u \rangle$ as $f_{ij}^u \in \langle f_{ij}^u \rangle$ and $\langle f_{ij}^u \rangle$ is a left ideal of $M_n(N)$.
Case 4: Let $a \in A_0^-$. Then $a = un$ for some $n \in N$. Now $f_{ij}^a = f_{ij}^{un} = f_{ij}^u + f_{ij}^n \in \langle f_{ij}^u \rangle$ as $f_{ij}^u \in \langle f_{ij}^u \rangle$ and $\langle f_{ij}^u \rangle$ is a right ideal of $M_n(N)$.
Therefore, $a \in A_1$ implies $f_{ij}^a \in \langle f_{ij}^u \rangle$. Assume that if $a \in A_m$, then $f_{ij}^a \in \langle f_{ij}^u \rangle$. Now let $a \in A_{m+1} = A_m^{++} \cup A_m^0 \cup A_m^+ \cup A_m^-$.
If $a \in A_m^{++}$, then $a = n + x - n$ for some $n \in N$ and $x \in A_m$. Now $f_{ij}^a = f_{ij}^{n+x-n} = f_{ij}^n + f_{ij}^x - f_{ij}^n \in \langle f_{ij}^u \rangle$ as $\langle f_{ij}^u \rangle$ is a normal subgroup of $M_n(N)$ and by induction hypothesis, $f_{ij}^x \in \langle f_{ij}^u \rangle$. If $a \in A_m^0$, then $a = x - y$ where $x, y \in A_m$. Clearly $f_{ij}^a = f_{ij}^{x-y} = f_{ij}^x - f_{ij}^y \in \langle f_{ij}^u \rangle$ since $f_{ij}^a, f_{ij}^a \in \langle f_{ij}^u \rangle$. Now if $a \in A_m^+$, then $a = n(n' + x) - nn'$ for some $n, n' \in N$ and $x \in A_m$. So $f_{ij}^a = f_{ij}^{n(n'+x)-nn'} = f_{ij}^n(f_{ij}^{n'} + f_{ij}^x) - f_{ij}^n f_{ij}^{n'} \in \langle f_{ij}^u \rangle$ as $f_{ij}^x \in \langle f_{ij}^u \rangle$ by induction hypothesis and $\langle f_{ij}^u \rangle$ is a left ideal of $M_n(N)$. Also, if $a \in A_m^-$, then $a = xn$ for some $n \in N$ and $x \in A_m$. Now $f_{ij}^a = f_{ij}^{xn} = f_{ij}^x + f_{ij}^n \in \langle f_{ij}^u \rangle$ as $f_{ij}^x \in \langle f_{ij}^u \rangle$ by induction hypothesis and $\langle f_{ij}^u \rangle$ is a right ideal of $M_n(N)$. Therefore, $f_{ij}^a \in \langle f_{ij}^u \rangle$ whenever $a \in \langle u \rangle$. □

Proposition 3.11. *If u is an essential element in N, then f_{ij}^u is an essential element in $M_n(N)$.*

Proof. To prove f_{ij}^u is an essential element in $M_n(N)$, we need to prove $\langle f_{ij}^u \rangle \leq_e M_n(N)$. Let $\mathcal{S} \trianglelefteq M_n(N)$ be a ideal such that $\langle f_{ij}^u \rangle \cap \mathcal{S} = (0)$. By Lemma 3.3, we get $\langle f_{ij}^u \rangle_\star \cap \mathcal{S}_\star = (0)$. Now we show $\mathcal{S}_\star = (0)$. Let $x \in \mathcal{S}_\star$. Since $\mathcal{S}_\star \trianglelefteq N$, we get $\langle x \rangle \subseteq \mathcal{S}_\star$. Let $0 \neq a \in \langle x \rangle$. Since $\langle x \rangle \subseteq \mathcal{S}_\star$, we get $a \in \mathcal{S}_\star$. Since $a \neq 0$ and $\langle f_{ij}^u \rangle_\star \cap \mathcal{S}_\star = (0)$, we get $a \notin \langle f_{ij}^u \rangle_\star$ and Corollary 4.5 of [19], we get $f_{ij}^a \notin \langle f_{ij}^u \rangle$. Now by Lemma 3.10, we get $a \notin \langle u \rangle$. Therefore, $\langle x \rangle \cap \langle u \rangle = (0)$. Since $\langle u \rangle \leq_e N$, we get $\langle x \rangle = (0)$ which implies $x = 0$. Therefore, $\mathcal{S}_\star = (0)$. Now $(\mathcal{S}_\star)^\star = (0)^\star = (0)$. Since $\mathcal{S} \subseteq (\mathcal{S}_\star)^\star$, we get $\mathcal{S} = (0)$, which implies $\langle f_{ij}^u \rangle \leq_e M_n(N)$. Hence f_{ij}^u is an essential element of $M_n(N)$. □

Definition 3.12. *An ideal $\mathcal{S}$ of $M_n(N)$ is said to be f-superfluous in $M_n(N)$ if for any full ideal $\mathcal{K}$ of $M_n(N)$ with $\mathcal{S} + \mathcal{K} = M_n(N)$ implies $\mathcal{K} = M_n(N)$ and it is denoted by $\mathcal{S} \ll^f M_n(N)$.*
Lemma 3.13.

1. *Let $B \trianglelefteq N$. If $B \ll N$, then $B^\star \ll^f M_n(N)$.*
2. *Let $\mathcal{B}$ be a full ideal of $M_n(N)$. If $\mathcal{B} \ll^f M_n(N)$, then $\mathcal{B}_\star \ll N$.*

Proof. 1. Let $\mathcal{K}$ be a full ideal of $M_n(N)$ such that $B^\star + \mathcal{K} = M_n(N)$. Since $\mathcal{K}$ is a full ideal, there exists an ideal K of N such that $K^\star = \mathcal{K}$, which implies $B^\star + K^\star = M_n(N)$. By Lemma 3.5, we get $B^\star + K^\star \subseteq (B+K)^\star$ and so $(B+K)^\star = M_n(N)$. This implies $((B+K)^\star)_\star = M_n(N)_\star = N$. By Proposition 4.7(2) of [19], we get $B + K = N$. Since $B \ll N$, we get $K = N$. Now, $\mathcal{K} = K^\star = N^\star = M_n(N)$. Hence $B^\star \ll^f M_n(N)$.

2. Let K be an ideal of N such that $\mathcal{B}_\star + K = N$. By Lemma 3.4, we get $(\mathcal{B}_\star)^\star + K^\star = M_n(N)$. Since $\mathcal{B}$ is a full ideal, we get $\mathcal{B} + K^\star = M_n(N)$. Since $\mathcal{B} \ll^f M_n(N)$, we get $K^\star = M_n(N)$, and so $K = (K^\star)_\star = M_n(N)_\star = N$. Hence $\mathcal{B}_\star \ll N$.

□

Definition 3.14. *An ideal $\mathcal{S}$ of $M_n(N)$ is called a generalized f-essential (abbr. g_f-essential) ideal if for any f-superfluous full ideal $\mathcal{T}$ of $M_n(N)$ $\mathcal{S} \cap \mathcal{T} = (0)$ implies $\mathcal{T} = (0)$.*
Proposition 3.15. *Let $\mathcal{S}$ be an ideal of $M_n(N)$ and I be an ideal of N.*

1. *If $\mathcal{S} \leq_{ge}^f M_n(N)$, then $\mathcal{S}_\star \leq_{ge} N$.*
2. *If $I \leq_{ge} N$, then $I^\star \leq_{ge}^f M_n(N)$.*
3. *There is a one-one correspondence between the set of all g-essential ideals of N and the set of all generalised f-essential full ideals of $M_n(N)$.*

Proof.

1. Let $B \ll N$ such that $\mathcal{S}_\star \cap B = (0)$. Then $(\mathcal{S}_\star \cap B)^\star = (0)^\star$. This implies $(\mathcal{S}_\star)^\star = (0)$. Since $\mathcal{S} \subseteq (\mathcal{S}_\star)^\star$, we get $\mathcal{S} \cap B^\star = (0)$. Since $\mathcal{S} \leq_{ge}^{f} M_n(N)$, we have $B^\star = (0)$. Now $(B^\star)_\star = (0)^\star = (0)$. Therefore, $\mathcal{S}_\star \leq_{ge} N$.

2. Let $\mathcal{T}$ be a f-superfluous full ideal of $M_n(N)$ such that $I^\star \cap \mathcal{T} = (0)$. Since $\mathcal{T}$ is a full ideal, we have $\mathcal{T} = K^\star$ for some ideal K of N. Therefore, by Lemma 3.2 we get $(I \cap K)^\star = I^\star \cap K^\star = (0)$ and so $I \cap K = (0)$. Since $\mathcal{T} \ll^f M_n(N)$, by Lemma 3.13 (2), we have $\mathcal{T}_\star = (K^\star)_\star \ll N$. Since $I \leq_{ge} N$, we get $K = (0)$ which implies $\mathcal{T} = K^\star = (0)^\star = (0)$. Therefore, $I^\star \leq_{ge}^{f} M_n(N)$.

3. Let $\mathcal{P} = \{A \trianglelefteq N : A \leq_{ge} N\}$ and $\mathcal{Q} = \{\mathcal{A} \trianglelefteq M_n(N) : \mathcal{A} \leq_{ge}^{f} M_n(N)\mathcal{A}$ is a full ideal $\}$. Define the mappings $\phi : \mathcal{P} \to \mathcal{Q}$ by $\phi(A) = A^\star$ and $\psi : \mathcal{Q} \to \mathcal{P}$ by $\psi(\mathcal{A}) = \mathcal{A}_\star$. Then the correspondence follows, similar to the Lemma 3.9(3).

□

4. Generalised essential ideal graph

In this section, we introduce the notion of generalized essential ideal graph (in short, g-essential ideal graph) of a module H over a nearring N (denoted as, $\mathcal{L}_g(H)$). We derive properties such as diameter, completeness based on a given N-group.

Definition 4.1. *The g-essential ideal graph, $\mathcal{L}_g(H)$ of H, is a graph whose vertex set is the set of all non-trivial ideals of H, and two distinct vertices A, B are adjacent if $A + B \leq_{ge} H$.*

Example 4.2.

1. *If H is simple, then $\mathcal{L}_g(H)$ is a null graph.*

2. *Let $N = (\frac{\mathbb{Z}_2(t)}{\langle t^3+t\rangle}, +, \cdot) = \{at^2 + bt + c : a, b, c \in \mathbb{Z}_2\}$, and $H = N$. The non-trivial ideals of H are $\langle t\rangle$,$\langle t+1\rangle$,$\langle t^2+t\rangle$ and $\langle t^2+1\rangle$. The ideal $\langle t^2+t\rangle$ is superfluous. Since $\langle t^2+1\rangle \cap \langle t^2+t\rangle = (0)$, we have $\langle t^2+1\rangle$ is non-g-essential. The corresponding g-essential ideal graph is given in the Figure 1.*

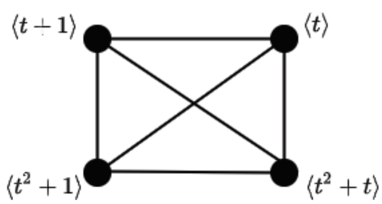

Figure 1: $\mathcal{L}_g(\frac{\mathbb{Z}_2(t)}{\langle t^3+t\rangle})$

Example 4.3. *Consider the nearring $N = (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, +, \cdot)$ and $N = H$, where the addition is carried out component-wise modulo 2 and the multiplication table is given in Table 1. For convenience, the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are denoted as $(0,0,0) = 0$, $(0,0,1) = 1$, $(0,1,0) = 2$, $(0,1,1) = 3$, $(1,0,0) = 4$, $(1,0,1) = 5$, $(1,1,0) = 6$, $(1,1,1) = 7$.*

The non-trivial ideals are $S_1 = \{0,1,4,5\}$, $S_2 = \{0,1,6,7\}$, $S_3 = \{0,1,2,3\}$ and $S_4 = \{0,1\}$. The only non-zero superfluous ideal of H is S_4 and $S_i \cap S_4 = S_4 \neq (0)$ for all $1 \leq i \leq 4$. Therefore, all non-zero ideals of H are g-essential. The g-essential ideal graph is given in the Figure 2.

Example 4.4. *Let $N = (\frac{\mathbb{Z}_4(t)}{\langle t^2+t\rangle}, +, \cdot) = \{at + b : a, b \in \mathbb{Z}_4\}$, and $H = N$. The non-trivial ideals of H are $\langle 2\rangle$,$\langle t\rangle$,$\langle t+1\rangle$,$\langle t+2\rangle$,$\langle t+3\rangle$,$\langle 2t\rangle$ and $\langle 2t+2\rangle$, and out of which the ideals $\langle 2\rangle$, $\langle 2t\rangle$ and $\langle 2t+2\rangle$ are superfluous. Since $\langle t+1\rangle \cap \langle 2t\rangle = (0)$, $\langle t\rangle \cap \langle 2t+2\rangle = (0)$ and $\langle 2t+2\rangle \cap \langle 2t\rangle = (0)$, we have $\langle t+1\rangle$, $\langle t\rangle$, $\langle 2t+2\rangle$ and $\langle 2t\rangle$ are non-g-essential ideals. The corresponding g-essential ideal graph is given in the Figure 3.*

Table 1: Multiplication table of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$\star$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	1	0	1	0	0
3	0	0	0	1	0	1	0	0
4	0	0	0	0	1	1	1	1
5	0	0	0	0	1	1	1	1
6	0	0	0	1	1	0	1	1
7	0	0	0	1	1	0	1	1

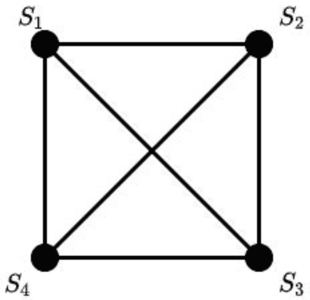

Figure 2: $\mathcal{L}_g(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Example 4.5. *Let $N = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with the multiplication table given in Table 2. Let $N = H$. Consider the notations given in the Example 4.3.*

The ideals are $S_1 = \{0\}$, $S_2 = \{0, 1, 4, 5\}$, $S_3 = \{0, 1, 2, 3\}$, $S_4 = \{0, 2, 4, 6\}$, $S_5 = \{0, 4\}$, $S_6 = \{0, 2\}$, $S_7 = \{0, 1\}$ and $S_8 = H$. We have $S_2 + S_6 = H$, $S_3 + S_5 = H$ and $S_4 + S_7 = H$. Therefore, all the non-zero ideals are non-superfluous, and hence all ideals other than (0) are g-essential. The corresponding g-essential ideal graph is given in the Figure 4.

Proposition 4.6.

1. *If H has DCCI and contains only one minimal ideal, then $\mathcal{L}_g(H)$ is a complete graph.*
2. *Let $A \in V(\mathcal{L}_g(H))$ be a universal vertex. If $A \nleq_{ge} H$, then A is a minimal ideal of H.*
3. *If H has a unique non-zero superfluous ideal, say B, then B is a universal vertex in $\mathcal{L}_g(H)$.*

We denote $Max(H) = \{M : M \text{ is a maximal ideal of } H\}$.

Theorem 4.7. [23] *Let $N = N_0$ and H be finitely generated. Then every proper ideal of H is contained in a maximal ideal. In particular, H has a maximal ideal.*

Proposition 4.8. *Let H be a completely reducible N-group. Then $\bigcap\limits_{M \in Max(H)} M = (0)$.*

Proof. Let $B = \bigcap\limits_{M \in Max(H)} M$. On the contrary, suppose that B is non-zero. Since H is completely reducible, we have B is a direct summand. Therefore, there exists a proper ideal S of H such that $B + S = H$. As $S \neq H$ and H is finitely generated, we get $S \subseteq X$ for some $X \in Max(H)$. Now since $B = \bigcap\limits_{M \in Max(H)} M \subseteq X$, we have $H = B + S \subseteq B + X = X$, a contradiction. Therefore, $B = (0)$. $\square$

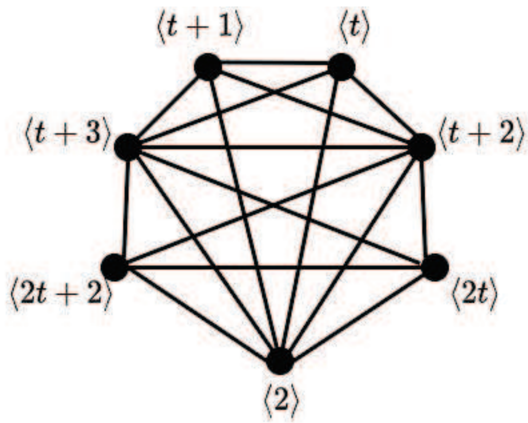

Figure 3: $\mathcal{L}_g(\frac{\mathbb{Z}_4(t)}{\langle t^2+t\rangle})$

Table 2: The multiplication table of $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$

$\star$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	0	1
2	0	0	2	2	0	0	2	2
3	0	1	2	3	0	1	2	3
4	4	4	4	4	4	4	4	4
5	4	5	4	5	4	5	4	5
6	4	4	6	6	4	4	6	6
7	4	5	6	7	4	5	6	7

Proposition 4.9. *If $S\leq_{ge} H$, then $S+P\leq_{ge} H$ for any ideal P of H.*

Proof. Let $S\leq_{ge} H$ and $P\trianglelefteq_N H$. To prove $S+P$ is g-essential, let $K\ll H$ such that $(S+P)\cap K=(0)$. Now $S\cap K\subseteq(S+P)\cap K=(0)$, implies $S\cap K=(0)$. Since $S\leq_{ge} H$, we get $K=(0)$. Therefore, $S+P\leq_{ge} H$. □

Lemma 4.10. *Any proper g-essential ideal of H is a universal vertex in $\mathcal{L}_g(H)$.*

Proof. Let $A\trianglelefteq_N H$ which is proper and g-essential. Let B be a non-trivial ideal of H. We prove $A\sim B\in E(\mathcal{L}_g(H))$. Since $A\leq_{ge} H$, by Proposition 4.9, we have $A+B\leq_{ge} H$ and so $A\sim B\in E(\mathcal{L}_g(H))$. Since B is arbitrary, $A\sim B\in E(\mathcal{L}_g(H))$ for all $B\in V(\mathcal{L}_g(H))$, we conclude that A is a universal vertex in $\mathcal{L}_g(H)$. □

Proposition 4.11. *Suppose that H is not simple. If $\bigcap_{L\in Max(H)} L=(0)$, then $\mathcal{L}_g(H)$ is a complete graph. In particular, if H is completely reducible, then $\mathcal{L}_g(H)$ is a complete graph.*

Proof. Let $B=\bigcap_{L\in Max(H)} L=(0)$. We claim that H has no non-zero superfluous ideals. Let K be a non-zero ideal of H. Since $K\nsubseteq B=(0)$, we have $K\nsubseteq L$ for some $L\in Max(H)$. Now $L\subsetneq B+L\subseteq H$ and $L\in Max(H)$ imply $B+L=H$, which means B is not superfluous. Hence H has no non-zero superfluous ideals and every proper ideal of H is g-essential. By Lemma 4.10, we have every proper g-essential ideal is a universal vertex and hence $\mathcal{L}_g(H)$ is a complete graph. □

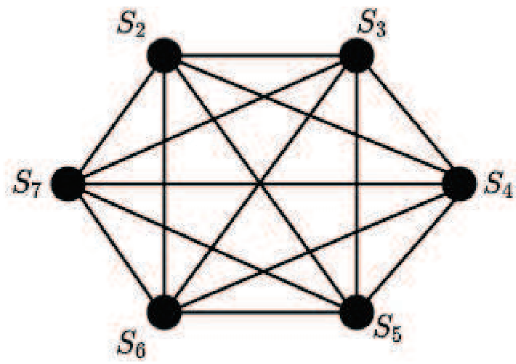

Figure 4: $\mathcal{L}_g(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$

Proposition 4.12. *Let* $B = \bigcap\limits_{L \in Max(H)} L$*. If* $B \neq (0)$*, then* B *is a universal vertex in* $\mathcal{L}_g(H)$*.*

Proof. Let $K \ll H$ such that $K \cap B = (0)$. Let $P \in Max(H)$. Then $P \subseteq P + K \subsetneq H$ implies $P + K = P$ or $P + K = H$. Since $K \ll H$, we get $P + K \neq H$ and so we have $P + K = P$, implies $K \subseteq P$. Since $P \in Max(H)$ is arbitrary, we have that $K \subseteq L$ for every $L \in Max(H)$. Hence $K \subseteq B$. Now $K = B \cap K = (0)$. Therefore, B is a g-essential ideal of H. By Lemma 4.10, we get B is a universal vertex. $\square$

Proposition 4.13.

1. *Every maximal ideal of* H *is a universal vertex in* $\mathcal{L}_g(H)$*.*

2. $Max(H)$ *induces a clique* $\mathcal{L}_g(H)$*.*

Proof.
1. Let $K \in Max(H)$. Let $(0) \neq L \trianglelefteq_N H$ be such that $K \cap L = (0)$. We prove that L is not superfluous. Since $L \neq (0)$ and $K \cap L = (0)$, we get $L \nsubseteq K$. Therefore, $K \subsetneq K + L \subseteq H$ and $K \in Max(H)$, implies that $L + K = H$. So, L is not superfluous in H. Hence, $K \leq_{ge} H$ and by Lemma 4.10, K is a universal vertex in $\mathcal{L}_g(H)$.

2. Follows from (1).

$\square$

Proposition 4.14. *Let* $P(H)$ *be the set of all ideals* J *of* H *satisfying the the property that if* J *is non-zero and maximal with respect to* $J \subsetneq K$ *where* $K \in Max(H)$*. Then* $P(H)$ *induces a clique in* $\mathcal{L}_g(H)$*.*

Proof. Let $L, T \in P(H)$.
Case (i) : Suppose that L and T are contained in K where $K \in Max(H)$.
Then $L + T \subseteq K$. Clearly, $L \nsubseteq T$ and $T \nsubseteq L$. Therefore, $L + T \neq L$ and $L + T \neq T$. Now $L \subsetneq L + T \subseteq K$ and $L \in P(H)$, imply $L + T = K$. By Proposition 4.13, we have K is g-essential, and hence $L \sim T \in E(\mathcal{L}_g(H))$.
Case (ii) : Suppose that L and T are contained in two distinct maximal ideals K_i and K_j respectively. Then $L \subseteq K_i$ and $T \nsubseteq K_i$ imply $L + T \nsubseteq K_i$. Similarly, $L + T \nsubseteq K_j$. If $L + T \subseteq M$ for some $M \in Max(H)$, then $L, T \subseteq M$. Using case (i), we can show $L \sim T \in E(\mathcal{L}^{(}_g H))$. If $L + T \nsubseteq M$ for some $M \in Max(H)$, then $L + T = H$, and therefore, $L \sim T \in E(\mathcal{L}_g(G))$. $\square$

Proposition 4.15. $\mathcal{L}_g(H)$ *is an empty graph if and only if* H *has exactly one non-trivial ideal.*

Proof. Suppose that $\mathcal{L}_g(H)$ is an empty graph. Then by Proposition 4.13, we get $|Max(H)| = 1$. Let $K \in Max(H)$. Suppose there exists a non-trivial ideal $B \neq K$ of H. Then by the Proposition 4.13, we get $B \sim K \in E(\mathcal{L}_g(H))$, a contradiction, as $\mathcal{L}_g(H)$ is empty. Conversely, if H has exactly one non-zero, non g-essential ideal, then $\mathcal{L}_g(H) = K_1$, an empty graph. □

Proposition 4.16. *$\mathcal{L}_g(H)$ is a complete graph if and only if every non-zero, non-g-essential ideal is a minimal.*

Proof. Assume that every non-zero, non-g-essential ideal of M is a minimal ideal. To prove $\mathcal{L}_g(H)$ is a complete graph, let A and B be two distinct vertices of $\mathcal{L}_g(H)$. Then A and B are two non-zero proper ideals of H.
Case 1: Either A or B is g-essential. In this case, by Proposition 4.10, $A \sim B \in E(\mathcal{L}_g(H))$.
Case 2: Neither A nor B is g-essential. Since A and B are non-zero, from the hypothesis A and B are minimal. Then $(0) \neq A \subsetneqq A + B$, implies that $A + B$ is not minimal. Again from the hypothesis, $A + B \leq_{ge} H$, and hence $A \sim B \in E(\mathcal{L}_g(H))$.
Conversely, suppose that $\mathcal{L}_g(H)$ is a complete graph. We prove every non-zero non-g-essential ideal is a minimal ideal. On the contrary, let $(0) \neq A \nleq_{ge} H$ which is not minimal. Then, there exists $(0) \neq B \trianglelefteq_N H$ such that $B \subsetneqq A$. Since $\mathcal{L}_g(H)$ is complete, $A \sim B \in E(\mathcal{L}_g(H))$, which implies $A + B = A \leq_{ge} H$, a contradiction to the assumption. Thus, every non-zero, non-g-essential ideal is a minimal ideal. □

Corollary 4.17. *If every non-maximal ideal of H is a minimal ideal, then $\mathcal{L}_g(H)$ is a complete graph.*

Proof. Follows from Proposition 4.13 and 4.16. □

Definition 4.18. *Let $A \trianglelefteq_N H$. An ideal B of H is a g-complement of A in H if B is maximal with respect to ($A \cap B = (0)$ and B is superfluous in H).*
Example 4.19. *Consider the nearring $N = (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, +, \cdot)$ where the addition is carried out component-wise modulo 2 and the multiplication table is in the Table ??. Elements are denoted as given in the Example 4.3. The ideals are $I_1 = H$, $I_2 = \{0, 2, 4, 6\}$, $I_3 = \{0, 2, 1, 3\}$, $I_4 = \{0, 2, 5, 7\}$, $I_5 = \{0, 2\}$,*

$\star$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	2	2
5	0	0	0	0	0	0	2	2
6	0	0	0	0	0	0	2	2
7	0	0	0	0	0	0	2	2

Table 3: The multiplication table of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

$I_6 = \{0, 1\}$ and $I_7 = \{0\}$. The g-complement of I_6 is I_5.
Example 4.20. *In $\frac{\mathbb{Z}_4(t)}{\langle t^2+t \rangle}$ given in Example 4.4, the ideal $\langle 2t \rangle$ is a g-complement of both $\langle t + 1 \rangle$ and $\langle 2t + 2 \rangle$.*
Lemma 4.21. *Let $A \trianglelefteq_N H$ and C be its g-complement. Then $A + C \leq_{ge} H$.*

Proof. Let $K \ll H$ such that $(A + C) \cap K = (0)$. Then $A \cap (C + K) = (0)$. We claim that $C + K \ll H$. Let $D \trianglelefteq_N H$ such that $(C + K) + D = H$. Since $C + (K + D) = H$ and $C \ll H$, we have $K + D = H$. Again, since K is superfluous in H, we get $D = H$. Therefore, $C + K \ll H$. Since C is a g-complement of A and $C + K \ll H$ satisfying $A \cap (C + K) = (0)$, we get a contradiction. Therefore, $A + C \leq_{ge} H$. □

Proposition 4.22. *Every non g-essential ideal of H is adjacent to its g-complement in $\mathcal{L}_g(H)$.*

Proof. Let $A \neq_{ge} H$. Then there exists a non-zero superfluous ideal $(0) \neq B \ll H$ such that $A \cap B = (0)$. Now by Zorn's lemma, A has a non-zero g-complement, say $A^{'}$. Then by Lemma 4.21, $A + A^{'} \leq_{ge} H$. Therefore, $A \sim A^{'} \in E(\mathcal{L}_g(H))$. $\square$

Lemma 4.23. *Let $B \in V(\mathcal{L}_g(H))$. If $deg_{\mathcal{L}_g(H)}B = 1$, then either B is minimal or $V(\mathcal{L}_g(H)) = \{A, B\}$ where A is a minimal ideal properly contained in B.*

Proof. Suppose that B is not minimal. Then there exists a non-zero proper ideal, say A of H such that $A \subsetneq B$. If $B \leq_{ge} H$, then B is a universal vertex. Since $deg_{\mathcal{L}_g(H)}B = 1$, we get $V(\mathcal{L}_g(H)) = \{A, B\}$. Suppose $B \nleq_{ge} H$, then it has a non-zero g-complement, say C. Now $B \oplus C \leq_{ge} H$ and also $B + (A \oplus C) \leq_{ge} H$. As B cannot be adjacent to two ideals, we get $A \oplus C$ is equal to either C or H. If $A \oplus C = C$, then $A = (0)$, a contradiction. If $A \oplus C = H$, then $B \oplus C = H$. Since both direct sums yield H, we get $A = B$, a contradiction. $\square$

Proposition 4.24. *$\mathcal{L}_g(H)$ is a connected graph of diameter less than 3.*

Proof. Since H is finitely generated, H has a maximal ideal, say K. By Proposition 4.13, K is a universal vertex. Let A and B be two non-zero proper ideal of H which are not maximal. Then, since K is a universal vertex, $K \sim A$ and $K \sim B$ are edges in $\mathcal{L}_g(H)$, which give a length 2 path from A to B. Therefore, $diam(\mathcal{L}_g(H))$ is less than or equal to 2. $\square$

Proposition 4.25. *If $\mathcal{L}_g(H)$ has exactly one universal vertex, then H has a unique proper essential ideal.*

Proof. Suppose $\mathcal{L}_g(H)$ has a unique universal vertex, say A. By Proposition 4.13 and from the hypothesis, it is clear that $Max(M) = \{A\}$. If A is not minimal, then there exists $(0) \neq B \trianglelefteq_N H$ such that $B \subsetneq A$. Since A is a universal vertex, $A \sim B \in E(\mathcal{L}_g(H))$, which implies $A + B = A \leq_{ge} H$. If A is minimal, then we prove that A is g-essential. On the contrary, suppose $A \nleq_{ge} H$. Then there exists $(0) \neq L \trianglelefteq_N H$ such that $J \cap L = (0)$. Since A is minimal, we have $L \nsubseteq A$. By Theorem 4.7, we have $L \subseteq P$ for some $P \in Max(H)$. Now, since A and P are maximal, we get $A + P = H$. Also, since A is minimal, we get $A \cap P = (0)$. Then it can be easily verified that H is isomorphic to $\frac{H}{A} \times \frac{H}{P}$. Since A and P are maximal, $\frac{H}{A}$ and $\frac{H}{P}$ are simple. Therefore, the only non-trivial ideals of $\frac{H}{A} \times \frac{H}{P}$ are $(0) \times \frac{H}{P}$ and $\frac{H}{A} \times (0)$. Hence, H has only two proper ideals, and $\mathcal{L}_g(H) = K_2$, which has two universal vertices, a contradiction. Therefore, A is g-essential. $\square$

Proposition 4.26. *Let A be a non-zero proper ideal of H and $B \trianglelefteq_N H$. If $A \sim B \in E(\mathcal{L}_g(H))$, then $A \sim K \in E(\mathcal{L}_g(H))$ for any proper ideal K of H which contains B.*

Proof. Let $(0) \neq K$ be a proper ideal of H such that $B \subseteq K$. Let $P \ll H$ such that $(A + K) \cap P = (0)$. Then $(A + B) \cap P \subseteq (A + K) \cap P = (0)$. Since $A \sim B \in E(\mathcal{L}_g(H))$, we have $A + B \leq_{ge} H$, and so $P = (0)$. Therefore, $A + K \leq_{ge} H$. Hence, $A \sim K \in E(\mathcal{L}_g(H))$. $\square$

Definition 4.27. *The proper g-essential ideal graph of H, denoted by $\mathcal{P}_g(H)$ is a subgraph of $\mathcal{L}_g(H)$, induced by the ideals which are non-g-essential in H.*

Definition 4.28. *H is g-uniform if every $(0) \neq I \trianglelefteq_N H$ is g-essential.*

Note 1.

1. *Every uniform N-group is g-uniform but the converse need not be true. For instance, a completely reducible N-group is g-uniform but not uniform.*

2. *$\mathcal{P}_g(H)$ is a null graph if and only if H is g-uniform.*

Proposition 4.29. *Let H be an N-group in which every non-maximal ideal is a minimal ideal. Then $\mathcal{P}_g(H)$ is an empty graph if and only if H has only one non-zero and non-g-essential ideal.*

Proof. Suppose H has exactly one non-trivial and non-g-essential ideal. Then $\mathcal{P}_g(H)$ is an empty graph. Conversely, suppose $\mathcal{P}_g(H)$ is an empty graph. We prove that H has only one non-zero and non-g-essential ideal. On the contrary, suppose that H has two proper non-g-essential ideals, say A and B. Then A,

$B \in V(\mathcal{P}_g(H))$. Since every non-maximal ideal is a minimal ideal, by Corollary 4.17, we get that $\mathcal{L}_g(H)$ is a complete graph, which implies $A \sim B \in E(\mathcal{L}_g(H))$. Since $\mathcal{P}_g(H)$ is an induced subgraph of $\mathcal{L}_g(H)$, we get $A \sim B \in E(\mathcal{P}_g(H))$, a contradiction. Therefore, $\mathcal{P}_g(H)$ is an empty graph. □

Proposition 4.30. *There exists a path between any two superfluous ideals in $\mathcal{P}_g(H)$.*

Proof. Let I, J be two distinct superfluous ideals of H such that $I, J \in V(\mathcal{P}_g(H))$. If $I + J \leq_{ge} H$, then $I \sim J \in E(\mathcal{P}_g(H))$. Assume that $I + J$ is not g-essential. Let S_I, S_J be g-complements of I and J respectively. Since I and J are superfluous ideals, we have that S_I and S_J are not g-essential. Therefore, S_I, $S_J \in V(\mathcal{P}_g(H))$. Also, by Lemma 4.21, we get $I + S_I \leq_{ge} H$ and $J + S_J \leq_{ge} H$. If $I + S_J$ or $J + S_I$ is g-essential, then either $I \sim S_J \sim J$ or $I \sim S_J \sim J$ is an IJ path. If neither $J + S_I$ nor $I + S_J$ is g-essential, then $J + S_I$, $I + S_J \in V(\mathcal{P}_g(H))$ and hence $I \sim J + S_I \sim I + S_J \sim J$ is a path from I to J. □

5. Conclusion

We have defined g-essential ideal graph of an N-group. For a finitely generated N-group H, we have shown that the maximal ideal is always a universal vertex and hence the g-essential ideal graph of such N-groups is always connected with diameter not more than 2. We have obtained several properties of g-essential ideal graphs based on the notions of connectivity, completeness etc. As future scope, we will explore to study the lattice aspects and graph theoretical properties of essential elements and superfluous elements as motivated by the authors in [26]. Furthermore, the notions discussed in this paper can be extended to study the finite dimensional aspects in matrix nearrings.

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