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# Lagrangian and Clairaut anti-invariant semi-Riemannian submersions in para-Kaehler geometry

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ABSTRACT: Purpose of this article is to examine some geometric features of Clairaut anti-invariant semi-Riemannian submersions from para-Kaehler manifold to a Riemannian manifold. We give Lagrangian semi-Riemannian submersion in para-Kaehler space froms. Then, we investigate under what conditions Clairaut submersions can become anti-invariant semi-Riemannian submersions. After, we obtain conditions for totally geodesic on vertical and horizontal distributions. We also supply a non-trivial example of Clairaut submersion.

Key Words: Para-Kaehler manifold, Para-Kaehler space form, semi Riemannian submersion, Lagrangian semi-Riemannian submersion, anti-invariant semi-Riemannian submersion, Clairaut submersion.

## Contents



## 1. Introduction

Clairaut's theorem specifies that for any timelike geodesic $\zeta$ on a surface of revolution $\bar{M}_1$, for $\rho$ is the distance from a point on the surface to the rotation axis, $\rho\cosh\phi$ is constant along timelike geodesic $\zeta$, where $\phi$ is the angle between $\zeta$ and the meridian through $\zeta$ in the theory of surface. Allison in ([3]) was applied this opinion to the pseudo-Riemannian submersions ([27]). He also presented a necessary and sufficient condition for Clairaut submersions can become anti-invariant semi-Riemannian submersions and the submersions have quirky implementations in static space-times.

Some researchers studies of $C^\infty-$submersion $\Phi$ from a (semi)Riemannian manifold $(\bar{M}_1, g_{\bar{M}_1})$ onto a (semi)-Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$, according to the circumstances on the map $\Phi : (\bar{M}_1, g_{\bar{M}_1}) \to (\bar{M}_2, g_{\bar{M}_2})$ such as:
a (semi) Riemannian submersion ([4], [12],[18],[26],[30]), an almost Hermitian submersion ([32]), a Clairaut submersion ([6],[31],[24]), an anti-invariant submersion ([13],[16,17],[25],[29],[23]), a conformal anti-invariant submersion ([1],[2]), a para-contact para-complex submersion ([15]), a para-contact submersion ([14]), a (para) quaternionic submersion ([21],[9]), a H-anti-invariant submersion ([28]), etc. As we know that O'Neill ([26]) and Gray ([18]) were severally introduced Riemannian submersions in 1960s. Especially, in ([32]), Watson presented several differential geometric properties between total manifolds, fibers and base manifolds by utilize the notion of almost Hermitian submersions. After that, there are many consequences on this issue. As we know that Riemannian submersions are related to physics and have their applications such as in the Kaluza-Klein theory ([7],[19]), supergravity and superstring theories ([20]), Yang-Mills theory ([8]) and many more.

The paper is organized as follows. In part 2, we give brief information about semi-Riemannian manifolds, para-Kaehler manifolds and distributions that are defined by the semi-Riemannian submersion. In part 3, we give Lagrangian semi-Riemannian submersion in para-Kaehler space forms. In part 4, we describe Clairaut anti-invariant semi Riemannian submersions and symbolize them as semi-Riemannian

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submersions which have totally umbilic fibers with a gradient field as mean curvature vector field. Finally, we give a non-trivial example for Clairaut anti-invariant semi-Riemannian submersion.

## 2. Preliminaries

Let $\Phi : (\bar{M}_1, g_{\bar{M}_1}) \to (\bar{M}_2, g_{\bar{M}_2})$ be a semi-Riemannian submersion between two Riemannian manifolds. The map satisfies the following axioms:
1. $\Phi_{*|q}$ is onto for all $q \in \bar{M}_1$;
2. The fibers $\Phi^{-1}(\bar{q})$, $\bar{q} \in \bar{M}_2$, are $k-$ dimensional semi-Riemannian submanifolds of $\bar{M}_1$, where $k = dim(\bar{M}_1) - dim(\bar{M}_2)$.
3. $\Phi_*$ preserves scalar products of vectors normal to fibres.

The tangent bundle $T\bar{M}_1$ of $\bar{M}_1$ has got an orthogonal decomposition

$$T\bar{M}_1 = ker\Phi_* \oplus (ker\Phi_*)^{\perp},$$

where $ker\Phi_*$ is called the vertical distribution and $(ker\Phi_*)^{\perp}$ denotes its orthogonal distribution. A semi-Riemannian submersion $\Phi : (\bar{M}_1, g_{\bar{M}_1}) \to (\bar{M}_2, g_{\bar{M}_2})$ specifies two tensor fields as T and A on $\bar{M}_1$, with O'Neill formulas ( [26]):

$$\mathrm{T}_{E_1}E_2 = h\nabla_{vE_1}vE_2 + v\nabla_{vE_1}hE_2 \tag{2.1}$$

and

$$\mathrm{A}_{E_1}E_2 = v\nabla_{hE_1}hE_2 + h\nabla_{hE_1}vE_2 \tag{2.2}$$

for any $E_1, E_2 \in \chi(\bar{M}_1)$, where $h$ and $v$ are the horizontal and vertical projections respectively. It is simple to see that $\mathrm{A}_{E_1}$ and $\mathrm{T}_{E_1}$ are skew-symmetric operators on the tangent bundle $T\bar{M}_1$ of the total space $\bar{M}_1$ reversing the horizontal and the vertical distributions.
From (2.1) and (2.2), we obtain the followings:

$$\nabla_{E_1}E_2 = \mathrm{T}_{E_1}E_2 + \hat{\nabla}_{E_1}E_2; \tag{2.3}$$

$$\nabla_{E_1}F_1 = \mathrm{T}_{E_1}F_1 + h(\nabla_{E_1}F_1); \tag{2.4}$$

$$\nabla_{F_1}E_1 = \mathrm{A}_{F_1}E_1 + v(\nabla_{F_1}E_1), \tag{2.5}$$

$$\nabla_{F_1}F_2 = \mathrm{A}_{F_1}F_2 + h(\nabla_{F_1}F_2), \tag{2.6}$$

for any $F_1, F_2 \in \Gamma((ker\Phi_*)^{\perp}), E_1, E_2 \in \Gamma(ker\Phi_*)$. Besides, if $F_1$ is fundamental vector field then we have $h(\nabla_{E_1}F_1) = h(\nabla_{F_1}E_1) = \mathrm{A}_{F_1}E_1$.

Let $F_1, F_2$ be horizontal and $E_1, E_2$ be vertical vector fields on $\bar{M}_1$. Then, we have the fundamental tensor fields $\mathrm{T}, \mathrm{A}$ as follows:

$$\mathrm{T}_{E_1}E_2 = \mathrm{T}_{E_2}E_1, \quad E_1, E_2 \in \Gamma(ker\Phi_*), \tag{2.7}$$

$$\mathrm{A}_{F_1}F_2 = -\mathrm{A}_{F_2}F_1 = \frac{1}{2}v[F_1, F_2], \quad F_1, F_2 \in \Gamma((ker\Phi_*)^{\perp}). \tag{2.8}$$

**Lemma 2.1.** *(see [12], [27])Let $\Phi : (\bar{M}_1, g_{\bar{M}_1}) \to (\bar{M}_2, g_{\bar{M}_2})$ be a semi-Riemannian submersion and $F_1, F_2$ basic vector fields on $\bar{M}_1$. If $\Phi-$related to $F_{1*}$ and $F_{2*}$ on $\bar{M}_2$, then we can say that the following properties are valid:*

*1.$h[F_1, F_2]$ is a fundamental and $\Phi_* h[F_1, F_2] = [F_{1*}, F_{2*}] \circ \Phi$;*

*2.$h(\nabla_{F_1}F_2)$ is a fundamental $\Phi-$related to $(\nabla^*_{F_{1*}}F_{2*})$, where $\nabla^*$ and $\nabla$ are respectively the Riemannian connection on $\bar{M}_2$ and $\bar{M}_1$, ;*

*3.$[E, E_1] \in \Gamma(ker\Phi_*)$, for every $E_1 \in \Gamma(ker\Phi_*)$ and any vector field $E$.*

**Remark 2.1.** *Let $\Phi$ be a semi-Riemannian submersion from a semi-Riemannian manifold $(\bar{M}_1, g_{\bar{M}_1})$ onto a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$. In this case, one can see that the main equations are the same in the semi-Riemannian case( [5]).*

Let $(\bar{M}_1, g_{\bar{M}_1})$ and $(\bar{M}_2, g_{\bar{M}_2})$ be (semi) Riemannian manifolds and $\Phi : (\bar{M}_1, g_{\bar{M}_1}) \to (\bar{M}_2, g_{\bar{M}_2})$ is a differentiable map. At that time, the second fundamental form of $\Phi$ is given by

$$(\nabla\Phi_*)(E_1, E_2) = \nabla^{\Phi}_{E_1}\Phi_* E_2 - \Phi_*(\nabla_{E_1}E_2) \tag{2.9}$$

for $E_1, E_2 \in \Gamma(T\bar{M}_1)$. Remind that $\Phi$ is called *harmonic* if $trace(\nabla\Phi_*) = 0$ and $\Phi$ is said to be a *totally geodesic* map if $(\nabla\Phi_*)(E_1, E_2) = 0$ for $E_1, E_2 \in \Gamma(T\bar{M}_1)$ [22].

Let $\bar{M}_1$ be an almost para-Hermitian manifold equipped with a semi-Riemannian metric $g_{\bar{M}_1}$ and an almost para-complex structure $\mathrm{P} \neq \pm I$, where $I$ is the identity map. Then, we have

$$\mathrm{P}^2 = I, \quad g_{\bar{M}_1}(\mathrm{P}E_1, \mathrm{P}E_2) = -g_{\bar{M}_1}(E_1, E_2) \tag{2.10}$$

for $E_1, E_2$ tangent to $\bar{M}_1$. The signature of $g_{\bar{M}_1}$ is $(n, n)$ and the dimension of $\bar{M}_1$ is even, where $dim\bar{M}_1 = 2n$. Take into account an almost para-Hermitian manifold $(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1})$ and denote the Levi-Civita connection by $\nabla$ on $\bar{M}_1$ with respect to $g_{\bar{M}_1}$. If P is parallel with respect to $\nabla$ then we can say that $\bar{M}_1$ is said to be a para-Kaehler manifold, that means,

$$(\nabla_{E_1}\mathrm{P})E_2 = 0 \tag{2.11}$$

for $E_1, E_2$ tangent to $\bar{M}_1$ [11].

**Definition 2.2.** *([16]) Let $(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1})$ be an almost para-Hermitian manifold and $(\bar{M}_2, g_{\bar{M}_2})$ a (semi) Riemannian manifold. Suppose that there exists a semi-Riemannian submersion $\Phi : \bar{M}_1 \to \bar{M}_2$ such that $ker\Phi_*$ is anti-invariant with respect to P, i.e., $\mathrm{P}(ker\Phi_*) \subseteq (ker\Phi_*)^{\perp}$. Then we say that $\Phi$ is an anti-invariant semi-Riemannian submersion from an almost para-Hermitian manifold to a (semi) Riemannian manifold.*

We can see that $\mathrm{P}(ker\Phi_*)^{\perp} \cap (ker\Phi_*) \neq 0$, from Definition (2.2), If we indicate the complementary orthogonal distribution to $\mathrm{P}(ker\Phi_*)$ in $(ker\Phi_*)^{\perp}$ by $\eta$, then we can write

$$(ker\Phi_*)^{\perp} = \mathrm{P}(ker\Phi_*) \oplus \eta, \quad \mathrm{P}\eta \subset \eta. \tag{2.12}$$

Thus, for space-like vector field $F_1 \in \Gamma((ker\Phi_*)^{\perp})$, we have

$$\mathrm{P}F_1 = \mu F_1 + \nu F_1, \tag{2.13}$$

here $\mu F_1 \in \Gamma(ker\Phi_*)$ and $\nu F_1 \in \Gamma(\eta)$. Moreover, we can write

$$T\bar{M}_2 = \Phi_*(\mathrm{P}(ker\Phi_*)) \oplus \Phi_*(\eta). \tag{2.14}$$

An anti-invariant semi-Riemannian submersion $\Phi$ is called a Lagrangian semi-Riemannian submersion if $\mathrm{P}(ker\Phi_*) = (ker\Phi_*)^{\perp}$. So, if $\Phi$ is a Lagrangian semi-Riemannian submersion, then we have $\mathrm{P}F_1 = \mu F_1$, $\nu F_1 = 0$, for any space-like vector field $F_1 \in \Gamma((ker\Phi_*)^{\perp})$.

Let $R^4_2$ be a semi-Euclidean space given with coordinates $(z_1, z_2, z_3, z_4)$. Naturally, we can get an almost para-complex structure P on $R^4_2$ as given follows:

$$\mathrm{P}(\frac{\partial}{\partial z_1}) = \frac{\partial}{\partial z_2}, \ \mathrm{P}(\frac{\partial}{\partial z_2}) = \frac{\partial}{\partial z_1}, \ \mathrm{P}(\frac{\partial}{\partial z_3}) = \frac{\partial}{\partial z_4}, \ \mathrm{P}(\frac{\partial}{\partial z_4}) = \frac{\partial}{\partial z_3}.$$

Let $R^4_2$ be a semi-Euclidean space with respect to the canonical basis $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4})$ and have signature $(-, +, -, +)$.

Thus, we can give an example of anti-invariant semi-Riemannian submersions.

**Example 2.3.** *Let $\Phi$ be a map from semi-Euclidean space $\bar{R}^4_2$ to a Riemannian manifold $\bar{R}^2_0$ given by*

$$\Phi(z_1, ..., z_4) = (\sinh az_1 + \cosh az_2, \sinh bz_3 + \cosh bz_4).$$

*By direct calculations, we have*

$$ker\Phi_* = Span\{E_1 = -\cosh a\frac{\partial}{\partial z_1} + \sinh a\frac{\partial}{\partial z_2}, E_2 = -\cosh b\frac{\partial}{\partial z_3} + \sinh b\frac{\partial}{\partial z_4}\}$$

*and*

$$(ker\Phi_*)^\perp = Span\{F_1 = -\sinh a\frac{\partial}{\partial z_1} + \cosh a\frac{\partial}{\partial z_2}, F_2 = -\sinh b\frac{\partial}{\partial z_3} + \cosh b\frac{\partial}{\partial z_4}\}.$$

*We can easily see that $\Phi$ satisfies the conditions for being a semi-Riemannian submersion. Moreover, $\mathrm{P}E_1 = -F_1$, $\mathrm{P}E_2 = -F_2$ imply that $\mathrm{P}(ker\Phi_*) \subseteq (ker\Phi_*)^\perp$. Consequently, $\Phi$ is an anti-invariant semi-Riemannian submersion.*

We note that $ker\Phi_*$ is a time-like subspace and $(ker\Phi_*)^\perp$ is a space-like subspace of $T_q\bar{R}^4_2$ for every $q \in \bar{R}^4_2$.

**Remark 2.2.** *In ([5]), the authors stated the semi-Riemannian submersions from a semi-Riemannian manifold $(\bar{M}_1, g_{\bar{M}_1})$ onto a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$.*
*That is why we have defined the Definition (2.2) anti-invariant semi-Riemannian submersions from an almost para-Hermitian manifold onto a Riemannian manifold.*

## 3. Lagrangian semi-Riemannian submersions in para-Kaehler space forms

Let $(\bar{M}^{2n}_1, \mathrm{P}, g_{\bar{M}_1})$ be a para-Hermitian manifold and $(\bar{M}_2, g_{\bar{M}_2})$ be a Riemannian manifold and let $\Phi : \bar{M}^{2n}_1 \to \bar{M}_2$ be an anti-invariant Riemannian submersion. Then we call $\Phi$ a Lagrangian Riemannian submersion, if $dim(ker\Phi_*) = dim(ker\Phi_*)^\perp$. In this instance, the para-complex structure P of $\bar{M}_1$ reverses the horizontal and the vertical distributions, i.e., $\mathrm{P}(ker\Phi_*)^\perp = ker\Phi_*$ and $\mathrm{P}(ker\Phi_*) = (ker\Phi_*)^\perp$.

The Riemannian curvature tensor of para-Kaehler space forms $(\bar{M}^{2n}_1(\nu), \mathrm{P}, g_{\bar{M}_1})$ of constant para-sectional curvature $\nu$ satisfies [10]

$$\begin{aligned} R_1(E_1, E_2)E_3 \;\; = \;\; & \frac{\nu}{4}\{g_{\bar{M}_1}(E_2, E_3)E_1 - g_{\bar{M}_1}(E_1, E_3)E_2 \\ & + g_{\bar{M}_1}(\mathrm{P}E_2, E_3)\mathrm{P}E_1 \\ & - g_{\bar{M}_1}(PE_1, E_3)PE_2 + 2g_{\bar{M}_1}(E_1, PE_2)PE_3\} \end{aligned} \tag{3.1}$$

for all non-null $E_1, E_2, E_3 \in (T\bar{M}_1)$.
Let $\Phi$ be a Lagrangian semi-Riemannian submersion from a para-Kaehler space form $(\bar{M}^{2n}_1(\nu), \mathrm{P}, g_{\bar{M}_1})$ to a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$. Suppose $\{X_1, X_2, ..., X_n\}$ is a timelike orthonormal basis of the vertical space $ker\Phi_{*q}$, for $q \in \bar{M}_1$, and $\{\bar{X}_{n+1}, ..., \bar{X}_{2n}\}$ be a spacelike orthonormal basis of the horizontal space $(ker\Phi_{*q})^\perp$.
We defined the scalar curvature $\tau^{ker\Phi_*}$ on the vertical space $ker\Phi_{*q}$ by

$$\tau^{ker\Phi_*} = \Sigma^n_{k,s=1}\epsilon_k\epsilon_s g_{\bar{M}_1}(\tilde{R}(X_k, X_s)X_s, X_k).$$

We obtain $g_{\bar{M}_1}(X_k, X_s) = \epsilon_k\delta_{ks}$ for every $k, s$ (here $\epsilon_k \in \{-1\}$).
Then, we can write

$$\begin{aligned} \mathrm{T}^\beta_{ks} &= \epsilon_k g_{\bar{M}_1}(T(X_k, X_s), X_\beta),\ k, s = 1, ..., n,\ \beta = n+1, ..., 2n, \\ \|\mathrm{T}\|^2 &= \Sigma^n_{k,s=1}\epsilon_k\epsilon_s g_{\bar{M}_1}(\mathrm{T}(X_k, X_s), \mathrm{T}(X_k, X_s)), \\ trace\mathrm{T} &= \Sigma^n_{k=1}\epsilon_k\mathrm{T}(X_k, X_k),\ \|trace\mathrm{T}\|^2 = g_{\bar{M}_1}(trace\mathrm{T}, trace\mathrm{T}) \end{aligned}$$

and the squared norm of T over the manifold $\bar{M}_1$, denoted by $\mathcal{C}^{ker\Phi_*}$, is called the vertical Casorati curvatures of the vertical space $(ker\Phi_*)_q$. Thus, we get

$$\mathcal{C}^{ker\Phi_*} = \frac{1}{n}\|\mathrm{T}\|^2 = \frac{1}{n}\Sigma_{\beta=n+1}^{2n}\Sigma_{k,s=1}^{n}\epsilon_k\epsilon_s(\mathrm{T}_{ks}^{\beta})^2.$$

Using (3.1) and Proposition 1.3 (i) of ( [4]) we have

$$2\tau^{ker\Phi_*} = \frac{\nu}{4}n(n-1) + n\mathcal{C}^{ker\Phi_*} - \|trace\mathrm{T}\|^2. \tag{3.2}$$

From here, we obtain:

**Theorem 3.1.** *Let $\Phi$ be a Lagrangian semi-Riemannian submersion from a para-Kaehler space form $(\bar{M}_1^{2n}(\nu), \mathrm{P}, g_{\bar{M}_1})$ to a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$with $3 \leq n$. Then the vertical Casorati curvature on the vertical space satisfies*

$$\mathcal{C}^{ker\Phi_*} = -\frac{\nu}{4}(n-1) + \frac{1}{n}\|trace\mathrm{T}\|^2 + \frac{2}{n}\tau^{ker\Phi_*}.$$

**Theorem 3.2.** *Let $\Phi$ be a Lagrangian semi-Riemannian submersion from a para-Kaehler space form $(\bar{M}_1^{2n}(\nu), \mathrm{P}, g_{\bar{M}_1})$ to a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$with $3 \leq n$. We deduce that if the fibres are totally geodesic, $ker\Phi_*$ is Einstein.*

*Proof.* Let's remember that the trace of scalar curvature is Ricci curvature. Then, we get

$$\tilde{S}(E_1, E_2) = \sum_{k=1}^{n}\epsilon_k\tilde{R}(E_1, X_k, X_k, E_2)$$

for all timelike vector fields $E_1, E_2 \in (ker\Phi_*)$ and $\{X_1, ...X_n\}$ is timelike orthonormal basis on $(ker\Phi_*)$. Then, if the fibres are totally geodesic, from (3.1) and Proposition 1.3 (i) of ( [4]), we obtain

$$\tilde{S}(E_1, E_2) = \frac{\nu}{4}(n-1)g_{\bar{M}_1}(E_1, E_2),$$

where $\tilde{S}$ is the scalar curvature of fibres. □

**Lemma 3.3.** *Let $\Phi$ be a Lagrangian semi-Riemannian submersion from a para-Kaehler manifold $(\bar{M}_1^{2n}, \mathrm{P}, g_{\bar{M}_1})$ to a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$. Then we have*

$$\mathrm{A}_{F_1}\mathrm{P}F_2 = -\mathrm{A}_{F_2}\mathrm{P}F_1,$$

*for any spacelike vector fields $F_1, F_2 \in \Gamma((ker\Phi_*)^{\perp})$.*

*Proof.* For any spacelike vector fields $F_1, F_2 \in \Gamma((ker\Phi_{*q})^{\perp})$, Since $\bar{M}_1$ is a para-Kaehler manifold and using (2.5),(2.6) and (2.11) we have $\mathrm{A}_{F_1}\mathrm{P}F_2 = \mathrm{P}\mathrm{A}_{F_1}F_2$. By (2.8), we obtain $\mathrm{P}\mathrm{A}_{F_1}F_2 = -\mathrm{P}\mathrm{A}_{F_2}F_1 = -\mathrm{A}_{F_2}\mathrm{P}F_1$. □

**Proposition 3.4.** *Let $\Phi$ be a Lagrangian semi-Riemannian submersion from a para-Kaehler manifold $(\bar{M}_1^{2n}, \mathrm{P}, g_{\bar{M}_1})$ to a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$. Then, the horizontal distribution $(ker\Phi_*)^{\perp}$ is integrable.*

*Proof.* For any spacelike vector fields $F_1, F_2 \in \Gamma((ker\Phi_{*q})^{\perp})$, since $\mathrm{A}_{F_1}F_2 = \frac{1}{2}[F_2, F_1]$, it is sufficient to show that$\mathrm{A}_{F_1} = 0$. For spacelike vector fields $F_3 \in \Gamma((ker\Phi_*)^{\perp})$, then using (2.6), (2.8),(2.10) and (2.11) and Lemma (3.3), we get

$$\begin{aligned} g_{\bar{M}_1}(\mathrm{A}_{F_1}\mathrm{P}F_2, F_3) &= g_{\bar{M}_1}(\mathrm{A}_{F_2}\mathrm{P}F_1, F_3) = -g_{\bar{M}_1}(\nabla_{F_2}\mathrm{P}F_1, F_3) \\ = -g_{\bar{M}_1}(\mathrm{P}\nabla_{F_2}F_1, F_3) &= g_{\bar{M}_1}(\nabla_{F_2}F_1, \mathrm{P}F_3) = -g_{\bar{M}_1}(\mathrm{A}_{F_2}F_1, \mathrm{P}F_3) \\ = g_{\bar{M}_1}(\mathrm{A}_{F_1}\mathrm{P}F_3, F_1) &= -g_{\bar{M}_1}(\mathrm{A}_{F_3}\mathrm{P}F_1, F_2) = g_{\bar{M}_1}(\mathrm{A}_{F_3}F_2, \mathrm{P}F_1) \\ = -g_{\bar{M}_1}(\mathrm{A}_{F_2}F_3, \mathrm{P}F_1) &= g_{\bar{M}_1}(\mathrm{A}_{F_2}\mathrm{P}F_1, F_3) = -g_{\bar{M}_1}(\mathrm{A}_{F_1}\mathrm{P}F_2, F_3). \end{aligned}$$

Thus $\mathrm{A}_{F_1}\mathrm{P}F_2 = 0$. Hence, we obtain $\mathrm{A}_{F_1} = 0$.
Since A vanishes, for any spacelike vector fields $F_1, F_2, F_3, F_4 \in \Gamma((ker\Phi_{*q})^{\perp})$, using Proposition 1.3 (vi) of ( [4]) we have

$$R_1(F_1, F_2, F_3, F_4) = R^*(F_{1*}, F_{2*}, F_{3*}, F_{4*}) \tag{3.3}$$

here we denote by $R_1, R^*$ the Riemannian tensors of the metrics $g_{\bar{M}_1}, g_{\bar{M}_2}$, respectively. □

Using (3.1) and (3.3) we have:

**Theorem 3.5.** *Let $\Phi$ be a Lagrangian semi-Riemannian submersion from a para-Kaehler space form $(\bar{M}_1^{2n}(\nu), \mathrm{P}, g_{\bar{M}_1})$ to a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$with $3 \leq n$. Then the scalar curvature $\tau^*$ on $\bar{M}_2$ satisfies*

$$\tau^* = \frac{\nu}{4}n(n-1).$$

From (3.1) and (3.3) we obtain:

**Theorem 3.6.** *Let $\Phi$ be a Lagrangian semi-Riemannian submersion from a para-Kaehler space form $(\bar{M}_1^{2n}(\nu), \mathrm{P}, g_{\bar{M}_1})$ to a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$with $3 \leq n$. Then the base manifold $\bar{M}_2$ is Einstein.*

## 4. Clairaut anti-invariant semi Riemannian submersions

Let $(\bar{M}_1, g_{\bar{M}_1})$ be a semi Riemannian manifold and $(\bar{M}_2, g_{\bar{M}_2})$ a Riemannian manifold. Let $\Phi : (\bar{M}_1, g_{\bar{M}_1}) \to (\bar{M}_2, g_{\bar{M}_2})$ be a semi Riemannian submersion. Every horizontal vector field in $T_q\bar{M}_1$ is space-like. So, $(ker\Phi_*)^{\perp}$ is a space-like subspace and $ker\Phi_*$ is a time-like subspace of $T_q\bar{M}_1$ for every $q \in \bar{M}_1$( [27]).

Suppose that $\zeta$ is a time-like geodesic in $(\bar{M}_1, g_{\bar{M}_1})$. Using the notation above, $\dot{\zeta} = Z = F_1 + E_1$, here $F_1$ is horizontal and $E_1$ is vertical. $F_1$ is space-like and the time-like character of $\zeta$ implies $E_1$ is timelike. At each point $\zeta(r)$, we define $\phi(r)$ to be the hyperbolic angle between $Z$ and $E_1$, i.e., $\phi \geq 0$ is the number satisfying

$$g_{\bar{M}_1}(Z, E_1) = -|Z||E_1|\cosh\phi \tag{4.1}$$

where $|Z|^2 = -g_{\bar{M}_1}(Z, Z)$ and $|E_1|^2 = -g_{\bar{M}_1}(E_1, E_1)$.

Let $\phi$ be the angle among a meridian and the velocity vector of a timelike geodesic. Clairaut's relation implies that $\rho\cosh\phi$ is constant, where $\rho$ is the distance to the axis of a surface of revolution. As we know that this concept was defined by Allison in ( [3]), in the submersions theory. According to this expression, a submersion $\Phi : \bar{M}_1 \to \bar{M}_2$ to be a Clairaut submersion if there is a function $\rho : \bar{M}_1 \to R^+$ such that for every timelike geodesic, making angles $\phi$ with the spacelike subspaces $(ker\Phi_*)^{\perp}$, $\rho\cosh\phi$ is constant.

**Theorem 4.1.** *( [3]) Let $\Phi : (\bar{M}_1, g_{\bar{M}_1}) \to (\bar{M}_2, g_{\bar{M}_2})$ be a semi Riemannian submersion with connected fibers. Then $\Phi$ is a Clairaut submersion with $\rho = e^{\delta}$ if and only if each fibre is totally umbilical and has the mean curvature vector field $\mathrm{H} = -\nabla\delta$, here $\nabla\delta$ is the gradient of the function $\delta$ with respect to $g_{\bar{M}_1}$.*

From the above it follows that timelike geodesic on a surface creates the origin of the concept of Clairaut submersion. Then, for a curve on the total space to be timelike geodesic we will try to find necessary conditions.

**Lemma 4.2.** *Let $\Phi$ be an anti-invariant semi-Riemannian submersion from a para-Kaehler manifold $(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1})$ onto a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$. If $\zeta : J \subset R \to \bar{M}_1$ is a regular curve and $E_1(s)$ and $F_1(s)$ are respectively the vertical and horizontal parts of the tangent vector field $\dot{\zeta}(r) = Z$ of $\zeta(r)$, then we can say that $\zeta$ is a timelike geodesic if and only if along $\zeta$ the followings hold:*

$$v\nabla_{\dot{\zeta}}\mu F_1 + \mathrm{A}_{F_1}\mathrm{P}E_1 + \mathrm{T}_{E_1}\mathrm{P}E_1 + (\mathrm{A}_{F_1} + \mathrm{T}_{E_1})ZF_1 = 0, \tag{4.2}$$

*and*

$$h\nabla_{\dot{\zeta}}ZF_1 + h\nabla_{\dot{\zeta}}\mathrm{P}E_1 + (\mathrm{A}_{F_1} + \mathrm{T}_{E_1})\mu F_1 = 0. \tag{4.3}$$

*Proof.* From (2.11), we get

$$\mathrm{P}\nabla_{\dot{\zeta}}\dot{\zeta} = (\nabla_{\dot{\zeta}}\mathrm{P}\dot{\zeta}). \tag{4.4}$$

Since $\dot{\zeta} = E_1 + F_1$, we can write

$$\mathrm{P}\nabla_{\dot{\zeta}}\dot{\zeta} = (\nabla_{E_1+F_1}\mathrm{P}(E_1 + F_1)). \tag{4.5}$$

By direct computations, we get

$$\mathrm{P}\nabla_{\dot{\zeta}}\dot{\zeta} = (\nabla_{E_1}\mathrm{P}E_1 + \nabla_{E_1}\mathrm{P}F_1 + \nabla_{F_1}\mathrm{P}E_1 + \nabla_{F_1}\mathrm{P}F_1).$$

Using (2.13), we have

$$\mathrm{P}\nabla_{\dot{\zeta}}\dot{\zeta} = (\nabla_{E_1}\mathrm{P}E_1 + \nabla_{E_1}(\mu F_1 + ZF_1) + \nabla_{F_1}\mathrm{P}E_1 + \nabla_{F_1}(\mu F_1 + ZF_1)).$$

Using (2.3)−(2.6), we obtain

$$\begin{aligned}\mathrm{P}\nabla_{\dot{\zeta}}\dot{\zeta} &= (h(\nabla_{\dot{\zeta}}\mathrm{P}E_1 + \nabla_{\dot{\zeta}}ZF_1) + (\mathrm{A}_{F_1} + \mathrm{T}_{E_1})(\mu F_1 + ZF_1)\\ &\quad + v\nabla_{\dot{\zeta}}\mu F_1 + \mathrm{A}_{F_1}\mathrm{P}E_1 + \mathrm{T}_{E_1}\mathrm{P}E_1).\end{aligned}$$

if we take the vertical and horizontal parts of this equation, then we obtain

$$v\nabla_{\dot{\zeta}}\mu F_1 + \mathrm{A}_{F_1}\mathrm{P}E_1 + \mathrm{T}_{E_1}\mathrm{P}E_1 + (\mathrm{A}_{F_1} + \mathrm{T}_{E_1})ZF_1 = v\mathrm{P}\nabla_{\dot{\zeta}}\dot{\zeta} \tag{4.6}$$

and

$$h\nabla_{\dot{\zeta}}ZF_1 + h\nabla_{\dot{\zeta}}\mathrm{P}E_1 + (\mathrm{A}_{F_1} + \mathrm{T}_{E_1})\mu F_1 = h\mathrm{P}\nabla_{\dot{\zeta}}\dot{\zeta}. \tag{4.7}$$

From (4.6) and (4.7), we can easily see that $\zeta$ is a timelike geodesic if and only if (4.2) and (4.3) *hold.* □

**Theorem 4.3.** *Let $\Phi$ be an anti-invariant semi-Riemannian submersion from a para-Kaehler manifold $(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1})$ onto a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$. By then, $\Phi$ is a Clairaut submersion with $\rho = e^{\delta}$ if and only if along $\Phi$ the following equation is provided*

$$g_{\bar{M}_1}(h\nabla_{\dot{\zeta}}ZF_1 + (\mathrm{A}_{F_1} + \mathrm{T}_{E_1})\mu F_1, \mathrm{P}E_1) = g_{\bar{M}_1}(\nabla\delta, F_1)\|E_1\|^2, \tag{4.8}$$

*where $E_1(r)$ and $F_1(r)$ are severally the vertical and horizontal parts of the tangent vector field $\dot{\zeta}(r)$ of the timelike geodesic $\zeta(r)$ on $\bar{M}_1$..*

*Proof.* Let $\zeta(r)$ be a timelike geodesic on $\bar{M}_1$, at that time, we get

$$|E_1|^2 = -g_{\bar{M}_1}(E_1, E_1) = -g_{\bar{M}_1}(Z, E_1) = |Z||E_1|\cosh\phi$$

and

$$|E_1| = |Z|\cosh\phi. \tag{4.9}$$

Squaring both sides of (4.9), we have

$$-g_{\bar{M}_1}(E_1, E_1) = |E_1|^2 = |Z|^2\cosh^2\phi = -g_{\bar{M}_1}(Z, Z)\cosh^2\phi.$$

So,

$$g_{\bar{M}_1}(E_1, E_1) = x\cosh^2\phi. \tag{4.10}$$

Since $\zeta$ is a timelike geodesic, $x = g_{\bar{M}_1}(Z, Z)$ is a negative constant. So, we have

$$g_{\bar{M}_1}(F_1, F_1) = -x\sinh^2\phi. \tag{4.11}$$

Differentiating (4.10), we get

$$\frac{d}{dr}g_{\bar{M}_1}(E_1(r), E_1(r)) = 2g_{\bar{M}_1}(\nabla_{\dot{\zeta}(r)}E_1(r), E_1(r)) = 2x\cosh\phi\sinh\phi\frac{d\phi}{dr}. \tag{4.12}$$

Thus, using (2.10) and (2.11), we obtain

$$-g_{\bar{M}_1}(h\nabla_{\dot{\zeta}(r)}\mathrm{P}E_1(r),\mathrm{P}E_1(r)) = x\cosh\phi\sinh\phi\frac{d\phi}{dr}. \tag{4.13}$$

By (4.3), we arrive at along timelike geodesic $\zeta$,

$$g_{\bar{M}_1}(h\nabla_{\dot{\zeta}}ZF_1 + (\mathrm{A}_{F_1} + \mathrm{T}_{E_1})\mu F_1, \mathrm{P}E_1) = x\cosh\phi\sinh\phi\frac{d\phi}{dr}. \tag{4.14}$$

Moreover, $\Phi$ is a Clairaut submersion with $\rho = e^{\delta}$ if and only if

$$\frac{d}{dr}(e^{\delta}\cosh\phi) = 0 \Leftrightarrow e^{\delta}(\frac{d\delta}{dr}\cosh\phi + \sinh\phi\frac{d\phi}{dr}) = 0.$$

If we multiply the last equation by the non-zero factor $x\cosh\phi$, then we have

$$\frac{d\delta}{dr}x\cosh^2\phi + x\cosh\phi\sinh\phi\frac{d\phi}{dr} = 0. \tag{4.15}$$

Using (4.14) and (4.15), we have

$$g_{\bar{M}_1}(h\nabla_{\dot{\zeta}}ZF_1 + (\mathrm{A}_{F_1} + \mathrm{T}_{E_1})\mu F_1, \mathrm{P}E_1) = \frac{d\delta}{dr}(\zeta(r))\|E_1\|^2. \tag{4.16}$$

Since $\frac{d\delta}{dr}(\zeta(r)) = \dot{\zeta}[\delta] = g_{\bar{M}_1}(\nabla\delta,\dot{\zeta}) = g_{\bar{M}_1}(\nabla\delta, F_1)$, the claim (4.8) follows from(4.16). □

The fibers of $\Phi$ is called totally umbilical if

$$\mathrm{T}_{E_1}E_2 = g_{\bar{M}_1}(E_1,E_2)\,H \tag{4.17}$$

for any $E_1, E_2 \in \Gamma(\ker\Phi_*)$, here $H$ is the mean curvature vector field of the fiber of $\Phi$ [5].

**Theorem 4.4.** *Let $\Phi$ be a Clairaut anti-invariant semi-Riemannian submersion from a para-Kaehler manifold $(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1})$ onto a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$ with $\rho = e^{\delta}$. Then*

$$\mathrm{A}_{PE_3}PF_1 = -F_1(\delta)E_3,$$

*for timelike vector field $E_3 \in ker\Phi_*$ and spacelike vector field $F_1 \in \eta$ such that $PE_3$ is fundamental.*

*Proof.* Let $\Phi$ *be a Clairaut anti-invariant semi-Riemannian submersion from a para-Kaehler manifold* $(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1})$ *onto a Riemannian manifold* $(\bar{M}_2, g_{\bar{M}_2})$ *with* $\rho = e^{\delta}$*. If we crash equation* (4.17) by $PE_3$, timelike vector field $E_3 \in ker\Phi_*$ such that $PE_3$ is fundamental and from (2.3), then we have

$$g_{\bar{M}_1}(\nabla_{E_1}E_2, PE_3) = -g_{\bar{M}_1}(E_1,E_2)\,g_{\bar{M}_1}(\nabla\delta, PE_3).$$

From (2.10), we get

$$g_{\bar{M}_1}(\nabla_{E_1}PE_3, E_2) = g_{\bar{M}_1}(E_1,E_2)\,g_{\bar{M}_1}(\nabla\delta, PE_3).$$

Using (2.9) and (2.11), we have

$$-g_{\bar{M}_1}(\nabla_{E_1}E_3, PE_2) = g_{\bar{M}_1}(E_1,E_2)\,g_{\bar{M}_1}(\nabla\delta, PE_3).$$

Using (2.3), we obtain

$$-g_{\bar{M}_1}(\mathrm{T}_{E_1}E_3, PE_2) = g_{\bar{M}_1}(E_1,E_2)\,g_{\bar{M}_1}(\nabla\delta, PE_3).$$

Using (4.17) in above equation, we get

$$g_{\bar{M}_1}(E_1,E_3)g_{\bar{M}_1}(\nabla\delta, PE_2) = g_{\bar{M}_1}(E_1,E_2)\,g_{\bar{M}_1}(\nabla\delta, PE_3). \tag{4.18}$$

If we take $E_1 = E_3$ and interchanging the role of $E_1$ and $E_2$, then we have

$$g_{\bar{M}_1}(E_2, E_2)g_{\bar{M}_1}(\nabla\delta, PE_1) = g_{\bar{M}_1}(E_1, E_2)\, g_{\bar{M}_1}(\nabla\delta, PE_2). \tag{4.19}$$

Now, just taking $E_1 = E_3$ in (4.18), we get

$$g_{\bar{M}_1}(E_1, E_1)g_{\bar{M}_1}(\nabla\delta, PE_2) = g_{\bar{M}_1}(E_1, E_2)\, g_{\bar{M}_1}(\nabla\delta, PE_1). \tag{4.20}$$

If we multiply (4.19) and (4.20), we get

$$g_{\bar{M}_1}(\nabla\delta, PE_1) = \frac{g^2_{\bar{M}_1}(E_1, E_2)}{\|E_1\|^2\, \|E_2\|^2} g_{\bar{M}_1}(\nabla\delta, PE_1). \tag{4.21}$$

On the other hand, using (2.10) and (2.11), we have

$$g_{\bar{M}_1}(\nabla_{E_2} PE_3, PF_1) = g_{\bar{M}_1}(P\nabla_{E_2} E_3, PF_1) = -g_{\bar{M}_1}(\nabla_{E_2} E_3, F_1),$$

for any spacelike vector field $F_1 \in \eta$. Using (2.3) and (4.17), we get

$$g_{\bar{M}_1}(\nabla_{E_2} PE_3, PF_1) = g_{\bar{M}_1}(E_2, E_3)g_{\bar{M}_1}(\nabla\delta, F_1). \tag{4.22}$$

Since $PE_3$ is fundamental and from $h(\nabla_{E_2} PE_3) = \mathrm{A}_{PE_3} E_2$, we get

$$g_{\bar{M}_1}(h\nabla_{E_2} PE_3, PF_1) = g_{\bar{M}_1}(\mathrm{A}_{PE_3} E_2, PF_1). \tag{4.23}$$

From (4.22), (4.23) and the anti symmetry of A, we obtain

$$g_{\bar{M}_1}(\mathrm{A}_{PE_3} PF_1, E_2) = -g_{\bar{M}_1}(\nabla\delta, F_1)g_{\bar{M}_1}(E_2, E_3). \tag{4.24}$$

Since $\mathrm{A}_{PE_3} PF_1, E_2$ and $E_3$ are vertical and $\nabla\delta$ is horizontal, then we have

$$g_{\bar{M}_1}(\mathrm{A}_{PE_3} PF_1, E_2) = g_{\bar{M}_1}(E_2, -g_{\bar{M}_1}(\nabla\delta, F_1)E_3).$$

Hence, we obtain

$$\mathrm{A}_{PE_3} PF_1 = -g_{\bar{M}_1}(\nabla\delta, F_1)E_3.$$

Therefore by using $g_{\bar{M}_1}(\nabla\delta, F_1) = F_1(\nabla\delta)$, we get result. □

From Theorem (4.4), we can give the following results.

**Corollary 4.5.** *Let $\Phi$ be a Clairaut anti-invariant semi-Riemannian submersion from a para-Kaehler manifold $(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1})$ onto a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$ with $\rho = e^{\delta}$. If $\nabla\delta \in Pker\Phi_*$, then either the fibres of $\Phi$ are 1-dimensional or $\delta$ is constant on $Pker\Phi_*$.*

**Corollary 4.6.** *Let $\Phi$ be a Clairaut anti-invariant semi-Riemannian submersion from a para-Kaehler manifold $(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1})$ onto a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$ with $\rho = e^{\delta}$ and $\nabla\delta \in Pker\Phi_*$. If $dim(ker\Phi_*) > 1$, then the fibres of $\Phi$ are totally geodesic if and only if $\mathrm{A}_{PE_3} PF_1 = 0$ for timelike vector field $E_3 \in ker\Phi_*$ such that spacelike vector field $F_1 \in \eta$ and $PE_3$ is fundamental.*

Also, for a Lagrangian submersion we can give the following result:

**Corollary 4.7.** *Let $\Phi$ be a Clairaut Lagrangian anti-invariant semi-Riemannian submersion from a para-Kaehler manifold $(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1})$ onto a Riemannian manifold $(\bar{M}_2, g_{\bar{M}_2})$ with $\rho = e^{\delta}$. Then either the fibers of $\Phi$ are totally geodesic or the fibres are 1-dimensional.*

*Proof.* Let $\Phi$ *be a Clairaut Lagrangian anti-invariant semi-Riemannian submersion from a para-Kaehler manifold* $(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1})$ *onto a Riemannian manifold* $(\bar{M}_2, g_{\bar{M}_2})$ *with* $\rho = e^{\delta}$*. Then* $\eta = \{0\}$. So, $\mathrm{A}_{PE_3} PF_1 = 0$ always. □

Lastly, we give a non-trivial example for *Clairaut anti-invariant semi Riemannian submersion from a para Kaehler manifold.*

**Example 4.8.** *Let $(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1})$ be a para-Kaehler manifold equipped with semi-Euclidean metric $g_{\bar{M}_1}$ on $\bar{M}_1$ expressed as*

$$\bar{M}_1 = \left\{(x, y, z, w) \in R^4_2 : (y, z, w) \neq 0, x \neq 0\right\}.$$

*we define the semi-Euclidean metric $g_{\bar{M}_1}$ on $\bar{M}_1$ given by*

$$g_{\bar{M}_1} = -e^{2x}dx^2 + e^{2x}dy^2 - dz^2 + dw^2.$$

*Let $\bar{M}_2 = \left\{(x, t) \in R^2_0\right\}$ be a Riemannian manifold with Riemannian metric $g_{\bar{M}_2}$ on $\bar{M}_2$ given by*

$$g_{\bar{M}_2} = e^{2x}dx^2 + dt^2.$$

*Now, we define a map $\Phi :(\bar{M}_1, \mathrm{P}, g_{\bar{M}_1}) \to (\bar{M}_2, g_{\bar{M}_2})$ by*

$$\Phi(x, y, z, w) = (\sinh ax + \cosh ay, w).$$

*Then we find $ker\Phi_*$ and $(ker\Phi_*)^\perp$ as follow:*

$$ker\Phi_* = Span\{E_1 = -e^{-x}\cosh a\frac{\partial}{\partial x} + e^{-x}\sinh a\frac{\partial}{\partial y}, E_2 = \frac{\partial}{\partial z}\}$$

$$(ker\Phi_*)^\perp = Span\{F_1 = -e^{-x}\sinh a\frac{\partial}{\partial x} + e^{-x}\cosh a\frac{\partial}{\partial y}, F_2 = \frac{\partial}{\partial w}\}.$$

*It is easy to see that*

$$\begin{aligned} g_{\bar{M}_1}(F_i, F_i) &= g_{\bar{M}_2}(\Phi_* F_i, \Phi_* F_i) = 1 \\ g_{\bar{M}_1}(PE_i, PE_i) &= g_{\bar{M}_2}(\Phi_*(PE_i), \Phi_*(PE_i) = 1, \end{aligned}$$

*for $i = 1, 2$. Thus $\Phi$ is a Riemannian submersion. On the other hand, we obtain $PE_1 = -F_1, PE_2 = F_2$. It is implies that $P(ker\Phi_*) \subseteq (ker\Phi_*)^\perp$. Therefore, $\Phi$ is a anti-invariant semi-Riemannian submersion. Now, we will find smooth function $\delta$ on $\bar{M}_1$ satisfying $\mathrm{T}_E E = -g_{\bar{M}_1}(E, E)\,\mathrm{grad}\,\delta$ for $E \in \Gamma\,(ker\Phi_*)$. We can calculate that*

$$\begin{aligned} \nabla_{E_1}E_1 &= e^{-2x}\sinh a(\sinh a\frac{\partial}{\partial x} - \cosh a\frac{\partial}{\partial y}), \\ \nabla_{E_2}E_2 &= 0, \\ \nabla_{E_1}E_2 &= \nabla_{E_2}E_1 = 0. \end{aligned}$$

*If we take $E = k_1E_1 + k_2E_2$ for $k_1, k_2 \in R$, then we have*

$$\mathrm{T}_E E = k_1^2\mathrm{T}_{E_1}E_1 + 2k_1k_2\mathrm{T}_{E_1}E_2 + k_2^2\mathrm{T}_{E_2}E_2.$$

*From (2.7) and (2.3)-(2.6), by direct calculations, we have*

$$\mathrm{T}_E E = k_1^2 e^{-2x}\sinh a(\sinh a\frac{\partial}{\partial x} - \cosh a\frac{\partial}{\partial y}).$$

*Since $E = k_1E_1 + k_2E_2$, then by direct calculations, we have*

$$g_{\bar{M}_1}(E, E) = -(k_1^2 + k_2^2).$$

*On the other hand, for any smooth function $\delta$ on $R^4_2$, the gradient of $\delta$ with respect to the metric $g_{\bar{M}_1}$ is given by*

$$\begin{aligned} \nabla\delta &= \sum_{i,j=1}^{4} g^{ij}_{\bar{M}_1}\frac{\partial\delta}{\partial u_i}\frac{\partial}{\partial u_j} \\ &= -e^{-2x}\frac{\partial\delta}{\partial x}\frac{\partial}{\partial x} + e^{-2x}\frac{\partial\delta}{\partial y}\frac{\partial}{\partial y} - \frac{\partial\delta}{\partial z}\frac{\partial}{\partial z} + \frac{\partial\delta}{\partial w}\frac{\partial}{\partial w}. \end{aligned}$$

*Thus,* $\nabla\delta = \frac{k_1^2}{(k_1^2+k_2^2)} e^{-2x} \sinh a(\sinh a \frac{\partial}{\partial x} - \cosh a \frac{\partial}{\partial y})$ *for the function*

$$\delta = -\frac{k_1^2}{(k_1^2+k_2^2)} \sinh a(\sinh a.x + \sinh a \cosh a.y).$$

Therefore, we can easily see that $\mathrm{T}_E E = -g_{\bar{M}_1}(E,E)\,\mathrm{grad}\,\delta$. Hence $\Phi$ is a Clairaut anti-invariant semi-Riemannian submersion.


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