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Indirect Method for Solving Non-Linear Optimal Control of a Non-Rectilinear Motion of a Rocket With Variable Mass

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ABSTRACT: In this paper, an optimal trajectory of the rocket angle with a variable mass will be calculated by considering the aerodynamic forces, the acceleration of gravity and moves with a non-rectilinear motion from a initial state to a final state with a known altitude. The aim is to optimize the lateral offset of the rocket. For this, we formulate an optimal control problem where the rocket angle is the control. In order to solve the problem, let applied Shooting method's based on the Pontryagin's maximum principle, and study the precision and a duration time. Finally, we validate the results by using MATLAB software.

Key Words: Optimal control, shooting method, aerodynamic forces, rocket.

Contents

1. Introduction

In this paper, we have presented a nonlinear optimal control model with free final time to maximize the lateral offset of a rocket moving from an initial position to a final position, where the control represents the flight bath angle of the rocket. In order to solve the considered problem, first the Pontryagin's maximum principle [15] is applied, then a numerical solution is found using the shooting method.

The problem of maximizing the lateral offset of a rocket which a non-rectilinear motion is formulated in Section 2. The resolution is detailed in Section 3. In particular, numerical results obtained with the help of the software MATLAB are given in Section 3.2. Finally, Section 4 concludes the article.

2. Problem Statement

Let consider $m(t),\ t \in [0,T]$ the mass of the Rocket which moves from an initial position $M_0 = (x_{10}, x_{20})$ to a final position $M_f = (x_{1f}, x_{2f})$, where x_{1f} is free.
$x(t) = (x_1(t), x_2(t))$ and $v(t) = (v_1(t), v_2(t))$ are respectively the position and speed of the rocket at the instant t. The motion equations are given by:

$$\begin{cases} \dot{x}_1(t) = v_1(t), \\ \dot{x}_2(t) = v_2(t), \quad t \in [0,T]. \end{cases} \tag{2.1}$$

Let $T_p(t)$ and $\theta(t)$ are respectively the thrust and the flight bath angle of the rocket. We have

$$\vec{T}_p(t) = u_{\max} \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}, \tag{2.2}$$

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where $u_{\max} = |\overrightarrow{T}_p(t)|$. The earth is supposed to be flat. Let consider aerodynamic forces $\overrightarrow{F}_a$ and the acceleration of the gravity g is constant ($g = 9.80665 m.s^{-2}$). We obtain the following equations:

$$F_a(x_2(t), v_1(t), v_2(t)) = \alpha(v_1(t)^2 + v_2(t)^2)e^{-\beta x_2(t)} \begin{pmatrix} \cos(\phi(t)) \\ \sin(\phi(t)) \end{pmatrix}, \; \overrightarrow{g} = \begin{pmatrix} 0 \\ -g \end{pmatrix}, \tag{2.3}$$

where α, β and ϕ the parameters of the aerodynamic forces with:

- α: is the product of the drag coefficient: the surface of the machine and the atmospheric density.

- β: is the inverse of the altitude scale.
- ϕ: is the angle of attack that the speed vector makes with the horizontal, hence:

$$\cos(\phi(t)) = \frac{v_1(t)}{\sqrt{v_1(t)^2 + v_2(t)^2}}, \tag{2.4}$$

and

$$\sin(\phi(t)) = \frac{v_2(t)}{\sqrt{v_1(t)^2 + v_2(t)^2}}, \tag{2.5}$$

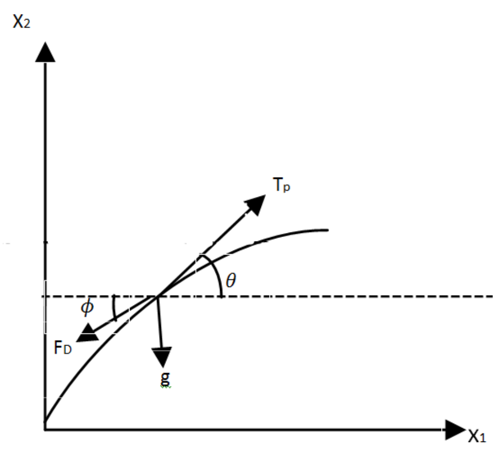

Figure 1: Forces applied

The dynamics are given by law of quantity of the movement theorem is given by the following formulas:

$$\frac{d\overrightarrow{p}}{dt} = \frac{(m + dm)(\overrightarrow{v} + d\overrightarrow{v}) - dm(\overrightarrow{v} + \overrightarrow{k}) - m\overrightarrow{v}}{dt} = \overrightarrow{R}. \tag{2.6}$$

By simplifying the formula (2.6), after eliminating the quadratic term to the differentials, we obtain:

$$\frac{d\overrightarrow{p}(t)}{dt} = m(t)\frac{d\overrightarrow{v}(t)}{dt} - \overrightarrow{k}\frac{dm(t)}{dt} = \overrightarrow{R}(t), \tag{2.7}$$

therefore

$$m(t)\frac{d\overrightarrow{v}}{dt} - \overrightarrow{k}\frac{dm(t)}{dt} = m(t)\overrightarrow{g} + \overrightarrow{F}_a(x_2(t), v_1(t), v_2(t)), \tag{2.8}$$

where $\overrightarrow{R} = m(t)\overrightarrow{g} + \overrightarrow{F}_a(x_2(t), v_1(t), v_2(t))$.
We obtain the following equation:

$$\frac{d\overrightarrow{v}(t)}{dt} = \overrightarrow{g} + \frac{\overrightarrow{T}_p(t)}{m(t)} + \frac{\overrightarrow{F}_a(x_2(t), v_1(t), v_2(t))}{m(t)}. \tag{2.9}$$

where

$$\overrightarrow{T}_p = \overrightarrow{k}\frac{dm}{dt}, \frac{dm}{dt} = -bu_{\max} \text{ and } \| \overrightarrow{k} \| = \frac{1}{b}. \tag{2.10}$$

By projecting on the $x-$axis, we obtains:

$$\frac{dv_1(t)}{dt} = \frac{u_{\max}}{m(t)}\cos(\theta(t)) - \frac{\alpha}{m(t)}v_1(t)\sqrt{v_1(t)^2 + v_2(t)^2}e^{-\beta x_2(t)}, \tag{2.11}$$

and

$$\frac{dv_2(t)}{dt} = \frac{u_{\max}}{m(t)}\sin(\theta(t)) - \frac{\alpha}{m(t)}v_2(t)\sqrt{v_1(t)^2 + v_2(t)^2}e^{-\beta x_2(t)} - g. \tag{2.12}$$

We formulate our problem by the optimal control problem whose the control represents the flight bath angle and our goal is to bring back the orbit rocket of an altitude known a priori with a maximum lateral offset. The mathematical model is given as follows whose used the following data [19]:

$$\begin{cases} \text{Minimize } J(\theta, T) = -x_1(T), \\ \dot{x}_1(t) = v_1(t), \\ \dot{x}_2(t) = v_2(t), \\ \dot{v}_1(t) = \dfrac{u_{\max}}{m(t)}\cos(\theta(t)) - \dfrac{\alpha v_1(t)\sqrt{v_1(t)^2 + v_2(t)^2}}{m(t)}e^{-\beta x_2(t)}, \\ \dot{v}_2(t) = \dfrac{u_{\max}}{m(t)}\sin(\theta(t)) - \dfrac{\alpha v_2(t)\sqrt{v_1(t)^2 + v_2(t)^2}}{m(t)}e^{-\beta x_2(t)} - g, \\ \dot{m}(t) = -bu_{\max}, \\ x_1(0) = 0, x_2(0) = 0.005,\ v_1(0) = 0,\ v_2(0) = 0.01,\ m(0) = m_0 \\ x_2(T) = h,\ v_1(T) = v_c,\ v_2(T) = 0, \\ \theta(t) \in \mathbb{R},\ t \in [0, T],\ T \text{ free}, \end{cases} \tag{2.13}$$

where

$$\alpha = 2.164,\ \beta = 0.113,\ v_c = 7.905km.s^{-1},\ h = 180km,\ m_0 = 122176.39kg.$$

3. Solution of Problem by using Indirect Method

The analytical solution of the problem is very complicated. For this, the problem is solved numerically with the indirect shooting method based on the Pontryagin's maximum Principle.

3.1. Necessary conditions of optimality

The theoretical solution of the problem is based on the Pontryagin's maximum principle, which gives a necessary condition of optimality. The Hamiltonian of the problem (2.13) is defined as follows:

$$\begin{aligned} H : \mathbb{R}^5 \times \mathbb{R}^5 \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (x(t), p(t), \theta(t)) &\longmapsto H(x(t), p(t), \theta(t)), \end{aligned} \tag{3.1}$$

where

$$x(t) = (x_1(t), x_2(t), v_1(t), v_2(t), m(t))$$

and

$$p(t) = (p_j(t),\ j = 1 \ldots 5).$$

$$\begin{aligned} H = \ & p_1(t)v_1(t) + p_2(t)v_2(t) \\ & + p_3(t)\left(\frac{u_{\max}}{m(t)}\cos(\theta(t)) - \frac{\alpha}{m(t)}v_1(t)\sqrt{v_1(t)^2 + v_2(t)^2}e^{-\beta x_2(t)}\right) \\ & + p_4(t)\left(\frac{u_{\max}}{m(t)}\sin(\theta(t)) - \frac{\alpha}{m(t)}v_2(t)\sqrt{v_1(t)^2 + v_2(t)^2}e^{-\beta x_2(t)} - g\right) \\ & - bp_5(t)u_{\max}. \end{aligned} \tag{3.2}$$

where $p(t)$, is the adjoint vector, it is solution of the following system:

$$\begin{cases} \dot{p}_1(t) = 0, \\ \dot{p}_2(t) = (p_3(t)v_1(t) + p_4(t)v_2(t))\dfrac{-\alpha\beta\sqrt{v_1(t)^2 + v_2(t)^2}}{m(t)}e^{-\beta x_2(t)}, \\ \dot{p}_3(t) = -p_1(t) + \alpha\dfrac{p_3(t)(2v_1(t)^2 + v_2(t)^2) + p_4(t)v_1(t)v_2(t)}{m(t)\sqrt{v_1(t)^2 + v_2(t)^2}}e^{-\beta x_2(t)}, \\ \dot{p}_4(t) = -p_2(t) + \alpha\dfrac{p_3(t)v_1(t)v_2(t) + p_4(t)(v_1(t)^2 + 2v_2(t)^2)}{m(t)\sqrt{v_1(t)^2 + v_2(t)^2}}e^{-\beta x_2(t)}, \\ \dot{p}_5(t) = \frac{u_{\max}}{m(t)^2}(p_3(t)\cos(\theta(t)) + p_4(t)\sin(\theta(t))) \\ \qquad -\dfrac{\alpha\sqrt{v_1(t)^2 + v_2(t)^2}e^{-\beta x_2(t)}}{m(t)^2}(p_3(t)v_1(t) + p_4(t)v_2(t)),\ t \in [0, T]. \end{cases} \tag{3.3}$$

The first order optimality condition gives:

$$\frac{\partial H}{\partial \theta} = -\frac{u_{\max}}{m(t)}p_3(t)\sin(\theta) + \frac{u_{\max}}{m(t)}p_4(t)\cos(\theta) = 0, \tag{3.4}$$

Since $p_3(t) \neq 0, \forall t \in [0, T]$, the equation (3.4) is equivalent to:

$$\tan(\theta(t)) = \frac{p_4(t)}{p_3(t)}, \text{for all } t \in [0, T], \tag{3.5}$$

So

$$\theta^*(t) = \arctan\left[\frac{p_4(t)}{p_3(t)}\right],\ t \in [0, T]. \tag{3.6}$$

The equation (3.5) is also equivalent to:

$$\cos(\theta(t)) = \frac{p_3(t)}{\sqrt{p_3(t)^2 + p_4(t)^2}}, \tag{3.7}$$

$$\sin(\theta(t)) = \frac{p_4(t)}{\sqrt{p_3(t)^2 + p_4(t)^2}},\ t \in [0, T]. \tag{3.8}$$

Furthermore, we have

$$\begin{aligned} \frac{\partial^2 H}{\partial \theta^2} &= -\frac{u_{\max}}{m(t)}p_3(t)\cos(\theta) - \frac{u_{\max}}{m(t)}p_4(t)\sin(\theta) \\ &= -\frac{u_{\max}}{m(t)}\left(p_3(t)\frac{p_3(t)}{\sqrt{p_3(t)^2 + p_4(t)^2}}\right) \\ &\quad -\frac{u_{\max}}{m(t)}\left(p_4(t)\frac{p_4(t)}{\sqrt{p_3(t)^2 + p_4(t)^2}}\right) \\ &= -\frac{u_{\max}}{m(t)}\sqrt{p_3(t)^2 + p_4(t)^2} < 0. \end{aligned} \tag{3.9}$$

Since $u_{\max}$ and $m(t), \forall t \in [0, T]$, are nonnegative, we get $\dfrac{\partial^2 H}{\partial \theta^2} < 0$.
Therefore, the second order necessary optimality condition is checked. The dynamic system is autonomous, so the Hamiltonian is constant for all $t \in [0, T]$. Therefore, we will have

$$H(t, x(t), p(t), \theta(t)) = c. \tag{3.10}$$

3.1.1. The tranversality conditions. Since T is free, according to the condition of transversality, we have:

$$H(T, x(T), p(T), \theta(T)) = 0. \tag{3.11}$$

From relationships (3.10) and (3.11), we get

$$H(t, x(t), p(t), \theta(t)) = 0, \quad \forall t \in [0, T]. \tag{3.12}$$

The tranversality conditions can be used to calculate the first and the last components of $p(T)$.

3.2. Numerical solution by the Shooting method

The Shooting method is based on the Pontryagin's maximum principle. It consists to find a zero of the Shooting function by using Newton method's. The shooting method defined in the three following steps [1,2,12,17]:

- Step 1: Form a boundary value problem using the model equations and the adjoint vectors equations as well as the transversality conditions.
- Step 2: Determine the shooting function.
- Step 3: Solve a system of nonlinear equations.

Based on the Pontryagin's maximum principle, optimality equations are given as follows:

$$\begin{cases}
\dot{x}_1(t) = x_3(t), \\
\dot{x}_2(t) = x_4(t), \\
\dot{x}_3(t) = \dfrac{u_{\max}}{x_5(t)} \dfrac{p_3(t)}{\sqrt{p_3(t)^2 + p_4(t)^2}} - \dfrac{\alpha x_3(t)\sqrt{x_3(t)^2 + x_4(t)^2}}{x_5(t)} e^{-\beta x_2(t)}, \\
\dot{x}_4(t) = \dfrac{u_{\max}}{x_5(t)} \dfrac{p_4(t)}{\sqrt{p_3(t)^2 + p_4(t)^2}} - \dfrac{\alpha x_4(t)\sqrt{x_3(t)^2 + x_4(t)^2}}{x_5(t)} e^{-\beta x_2(t)} - g, \\
\dot{x_5}(t) = -b u_{\max}, \\
\dot{p}_1(t) = 0, \\
\dot{p}_2(t) = -(p_3(t)x_3(t) + p_4(t)x_4(t)) \dfrac{\alpha\beta\sqrt{x_3(t)^2 + x_4(t)^2}}{x_5(t)} e^{-\beta x_2(t)}, \\
\dot{p}_3(t) = -p_1(t) + \alpha \dfrac{p_3(t)(2x_3(t)^2 + x_4(t)^2) + p_4(t)x_3(t)v_2(t)}{x_5(t)\sqrt{x_3(t)^2 + x_4(t)^2}} e^{-\beta x_2(t)}, \\
\dot{p}_4(t) = -p_2(t) + \alpha \dfrac{p_3(t)x_3(t)x_4(t) + p_4(t)(x_3(t)^2 + 2x_4(t)^2)}{x_5(t)\sqrt{x_3(t)^2 + x_4(t)^2}} e^{-\beta x_2(t)}, \\
\dot{p}_5(t) = \dfrac{u_{\max}}{x_5(t)^2} \sqrt{p_3(t) + p_4(t)} - \dfrac{\alpha\sqrt{x_3(t)^2 + x_4(t)^2} e^{-\beta x_2(t)}}{x_5(t)^2} (p_3(t)x_3(t) + p_4(t)x_4(t)), \\
x_1(0) = 0,\ x_2(0) = 0.005, x_3(0) = 0,\ x_4(0) = 0.01,\ x_5(0) = m_0,\ p_1(T) = 1, p_5(T) = 0, \\
x_2(T) = h, x_4(T) = v_c, x_4(T) = 0,\ t \in [0, T],
\end{cases} \tag{3.13}$$

where

$$x(t) = (x_j(t),\ j = 1 \ldots 5) = (x_1(t), x_2(t), v_1(t), v_2(t), m(t)).$$

We construct the Shooting function:

$$\begin{array}{rll} G: & \mathbb{R}^{10} & \longrightarrow \mathbb{R}^{10} \\ & (p(0), p(T)) & \longrightarrow G(p(0), p(T)), \end{array} \tag{3.14}$$

and

$$
G(p(0),p(T)) = \begin{pmatrix} p_1(0)-1 \\ p_2(0)-\lambda_1 \\ p_3(0)-\lambda_2 \\ p_4(0)-\lambda_3 \\ p_5(0)-\lambda_4 \\ p_1(T)-1 \\ p_2(T)-\lambda_5 \\ p_3(T)-\lambda_6 \\ p_4(T)-\lambda_7 \\ p_5(T) \end{pmatrix}.
$$

Therefore, the problem (3.13) is equivalent to:

$$
\begin{cases}
\dot{z}_1(t) = z_3(t), \\
\dot{z}_2(t) = z_4(t), \\
\dot{z}_3(t) = \dfrac{u_{\max}}{z_5(t)} \dfrac{z_8(t)}{\sqrt{z_8(t)^2+z_9(t)^2}} - \dfrac{\alpha z_3(t)\sqrt{z_3(t)^2+z_4(t)^2}}{z_5(t)} e^{-\beta z_2(t)}, \\
\dot{z}_4(t) = \dfrac{u_{\max}}{z_5(t)} \dfrac{z_9(t)}{\sqrt{z_8(t)^2+z_9(t)^2}} - \dfrac{\alpha z_4(t)\sqrt{z_3(t)^2+z_4(t)^2}}{z_5(t)} e^{-\beta z_2(t)} - g, \\
\dot{z}_5(t) = -b u_{\max}, \\
\dot{z}_6(t) = 0, \\
\dot{z}_7(t) = -(z_8(t)z_3(t)+z_9(t)z_4(t)) \dfrac{\alpha\beta\sqrt{z_3(t)^2+z_4(t)^2}}{z_5(t)} e^{-\beta z_2(t)}, \\
\dot{z}_8(t) = -z_6(t) + \alpha \dfrac{z_8(t)(2z_3(t)^2+z_4(t)^2)+z_9(t)z_3(t)z_4(t)}{x_5(t)\sqrt{z_3(t)^2+z_4(t)^2}} e^{-\beta z_2(t)}, \\
\dot{z}_9(t) = -z_7(t) + \alpha \dfrac{z_8(t)z_3(t)z_4(t)+z_9(t)(z_3(t)^2+2z_4(t)^2)}{z_5(t)\sqrt{z_3(t)^2+z_4(t)^2}} e^{-\beta z_2(t)}, \\
\dot{z}_{10}(t) = \dfrac{u_{\max}}{z_5(t)^2}\sqrt{z_8(t)^2+z_9(t)^2} - \dfrac{\alpha\sqrt{z_3(t)^2+z_4(t)^2}e^{-\beta z_2(t)}}{z_5(t)^2}(z_8(t)z_3(t)+z_9(t)z_4(t)), \\
z_1(0)=0,\ z_2(0)=0.005,\ z_3(0)=0,\ z_4(0)=0.01,\ z_5(0)=m_0,\ z_6(T)=1,\ z_{10}(T)=0, \\
z_2(T)=h,\ z_4(T)=v_c,\ z_4(T)=0,\ t\in[0,T],
\end{cases}
\tag{3.15}
$$

where $z(t) = (z_i(t),\ i=1,\dots,10) = (x(t),p(t))$.
The numerical solution of the nonlinear system $G(p(0),p(T))=0$ can be calculated using the Newton's method.

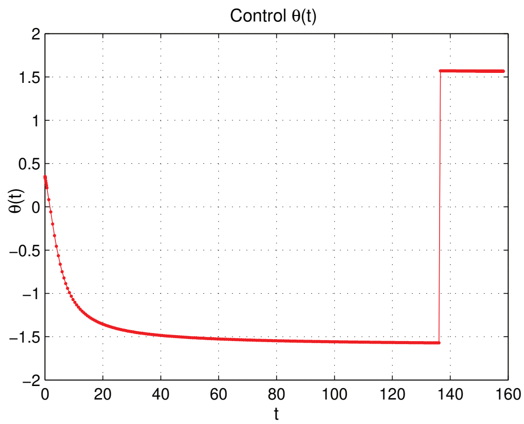

Figure 2: Variation of the flight bath angle according to the time

• From Figure 2, we note that the angle $\theta(t)$ decreases as a function of time from $\theta(0) = 0.35\pi$ to $\theta(t_c) = -1.57\pi$, where $t_c = 136.1s$ (t_c is the commutation time), then increase quickly in the interval $[t_c, t_c + \epsilon]$, where $\epsilon = 0.5$, until it get to the angle $\theta(t_c + \epsilon) = 1.57\pi$, then it settles at this time in the interval $[t_c + \epsilon, T]$. In the reality of these results, The rocket change the direction at time $t_c = 136.1s$.

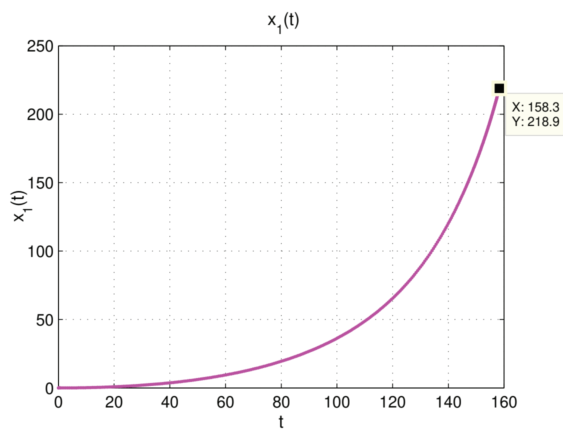

Figure 3: Variation of the lateral offset according to the time

• From Figure 3, we remark that $x_1(t)$ is rapidly increases with time, it reaches the optimal lateral offset $x_1^*(T) = 218.9km$. The angle is proportional than lateral offset, for this last, lateral increases.

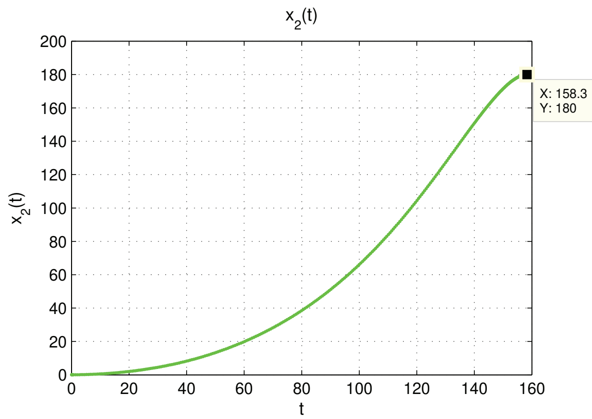

Figure 4: Variation of the altitude according to the time

• Furthermore, we see that $x_2(t)$ reaches the altitude target $180km$. While the rocket is in movement, the altitude increase (See Figure 4) .

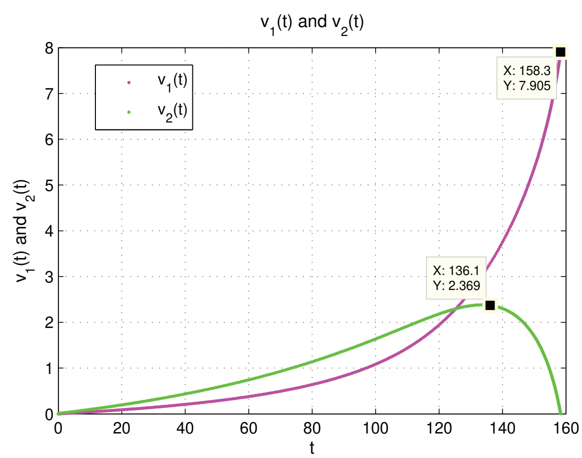

Figure 5: Variation of the speed components according to the time

• The velocity $v_1(t)$ is also increasing with time, it varies from $v_1(0) = 0$ to $v_1(T) = v_c = 7.904km.s^{-1}$, $v_1(t)$ is the speed which used to follows trajectory and to prepare to enter orbit, $v_2(t)$ grows from $v_2(0) = 0.005km.s^{-1}$ to $v_2(t_c + \epsilon) = 2.359km.s^{-1}$ where $t_c + \epsilon = 136.6s$, then decreases from $v_2(t_c + \epsilon)$ to $v_2(T) = 0$, $v_2(t)$ is the speed which guarantees its arrival in orbit (See Figure 5).

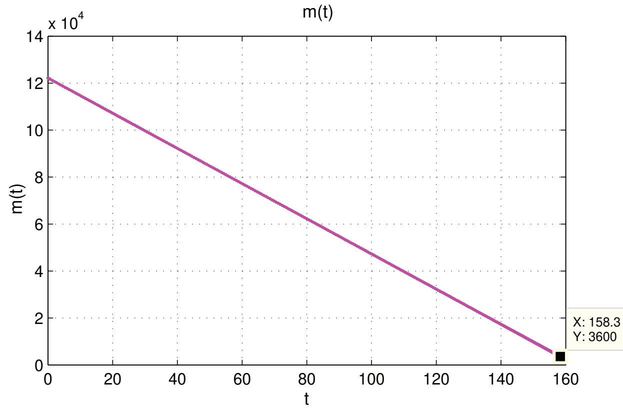

Figure 6: Mass of the rocket

• From Figure 6, we remark that $m(t)$ of the vehicle aerospatial decreases in the form of a linear function from $m(0) = 122176.39kg$ to $m(T) = 3600kg$ within a period $T = 158.3s$, more the rocket in motion, the mass decreases.

• Finally, we note that the execution time of the Shooting method necessary to find the optimal solution is $1.87s$.

4. Conclusion

In this work, we have modelled a practical problem arising in aerospace field as a nonlinear optimal control problem. First, the Pontryagin's maximum principle has been used which gives a necessary condition of optimality. Then the considered problem is solved numerically with the shooting method. The choice is carried the shooting method in the numerical resolution for its precision and its speed. The results found coincide with our reality: when the time t increases, $x_2(t)$ increases and $v_2(t)$ then decreases at time t_c until it vanishes, to avoid the explosion of gear at the orbital entrance.

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