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A Note on Complex Combined Boundary Value Problem for the Nonhomogeneous Tri-Analytic Equation

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ABSTRACT: In this paper, a combination of boundary value problems for the nonhomogeneous analytic equations have been studied. The aim of this paper is to find a solution for the Schwarz-(Dirichlet-Neumann) problem and obtain its solvability conditions.

Key Words: Schwarz problem, Dirichlet-Neumann problem, combined boundary value problems, tri-analytic equations.

Contents

1. Introduction

In complex analysis there are three basic boundary value problems: Schwarz, Dirichlet and Neumann problems. These problems are investigated in particular domains such as unit disc and half plane and explicit solutions are produced. In addition, these problems were investigated for different domains.

In the unit disc of the complex plane $\mathbb{C}$, for the inhomogeneous Cauchy-Riemann equation the Schwarz problem is well-posed but the Dirichlet and Neumann problems are overdetermined.Combinations of these boundary value problems are proper to determine solutions for higher order equations. However, not all of them are well-posed problems and solvability conditions have to determined.

In this article, a new boundary value problem is studied by combining different boundary conditions: Schwarz condition, Dirichlet and Neumann conditions. We use the iteration method presented in [1] following the ideas in [2]. We extended the binary combinations problems studied in [2,3] to triple combinations of value boundary problems. All studies in this paper are restricted to the inhomogeneous tri-analytic equation.

In this paper, the results of the complex analysis as of Gauss Theorem, Cauchy Theorem and Cauchy-Pompeiu formula [1,4,5,6] are used.

The proofs of the basic boundary value problems that we use in this paper and their proofs can be found in [1,5].

Theorem 1.1. *[1] The Schwarz problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\bar{z}} = f \quad in\ \mathbb{D}, \quad Re w = \gamma \ on\ \partial\mathbb{D}, \quad Im w(0) = c$$

for $f \in L_1(\mathbb{D};\mathbb{C})$ and $\gamma \in C(\partial\mathbb{D};\mathbb{C}), c \in \mathbb{R}$ is uniquely solvable by the Cauchy-Schwarz-Pompeiu formula

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + ic - \frac{1}{2\pi}\iint_{\mathbb{D}} \left(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right) d\xi d\eta. \tag{1.1}$$

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Here and in the future, we put $\zeta = \xi + \eta$, $\xi, \eta \in \mathbb{R}$, $L_1(\mathbb{D};\mathbb{C}) := \{f : f$ is complex-valued, measurable and $\iint\limits_{\mathbb{D}} |f| d\mu < \infty\}$ and $C(\partial\mathbb{D};\mathbb{C})$ is the set of continous complex-valued functions on the boundary of the unit disc $\mathbb{D}$.

Theorem 1.2. *[1] The Dirichlet-Neumann problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\bar{z}\bar{z}} = f \quad \text{in } \mathbb{D}, \quad w = \gamma_0 \text{ on } \partial\mathbb{D}, \quad \partial_\mu w_{\bar{z}} = \gamma_1 \text{ on } \partial\mathbb{D}, \qquad w_{\bar{z}}(0) = c$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C}) \cap C(\partial\mathbb{D};\mathbb{C})$, $\gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C}), c \in \mathbb{C}$ if and only if

$$c - \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_0(\zeta)\frac{d\zeta}{1-\bar{z}\zeta} + \frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{1-|\zeta|^2}{1-\bar{z}\zeta}\frac{f(\zeta)}{\zeta} d\xi d\eta = 0 \tag{1.2}$$

and

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} (\gamma_1(\zeta) - \bar{\zeta} f(\zeta))\frac{d\zeta}{\zeta(1-\bar{z}\zeta)} + \frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{\bar{z} f(\zeta)}{(1-\bar{z}\zeta)^2} d\xi d\eta = 0. \tag{1.3}$$

The solution then is given by

$$w(z) = c\bar{z} + \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \frac{\gamma_0(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} (\gamma_1(\zeta) - \bar{\zeta} f(\zeta)) \log(1 - z\bar{\zeta}) \frac{1-|z|^2}{z}\frac{d\zeta}{\zeta} + \frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{|\zeta|^2 - |z|^2}{\zeta - z}\frac{f(\zeta)}{\zeta} d\xi d\eta. \tag{1.4}$$

2. Computations of Some Integrals

To solve the combined problem we introduce the following lemmas.

Lemma 2.1. *For $|z| < 1$, $|\tilde{\zeta}| < 1$ we get*

1. $$\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta + z}{\zeta - z}\bar{\zeta} d\xi d\eta = -\bar{z}^2.$$

2. $$\frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta + z}{\zeta - z}\frac{1}{\tilde{\zeta} - \zeta} d\xi d\eta = \frac{(\bar{\tilde{\zeta}} - \bar{z})(\tilde{\zeta} + z)}{2\tilde{\zeta}(\tilde{\zeta} - z)} - \frac{\bar{z}}{2\tilde{\zeta}}.$$

3. $$\frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta + z}{\zeta - z}\frac{(1 - |\zeta|^2)}{\zeta} \log(1 - \zeta\bar{\tilde{\zeta}}) d\xi d\eta = \frac{|z|^4 - 2|z|^2 + 1}{2z^2} \log(1 - z\bar{\tilde{\zeta}}) + \frac{\bar{\tilde{\zeta}}}{2z} + \frac{\bar{\tilde{\zeta}}^2}{8}.$$

4. $$\frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta + z}{\zeta - z}\frac{(|\tilde{\zeta}|^2 - |\zeta|^2)}{\tilde{\zeta} - \zeta} d\xi d\eta = \frac{\bar{\tilde{\zeta}}|\tilde{\zeta}|^2 - 2\bar{z}|\tilde{\zeta}|^2 + z\bar{\tilde{\zeta}}^2 - 2|z|^2\bar{\tilde{\zeta}} + 2\bar{z}|z|^2}{4(\tilde{\zeta} - z)} - \frac{\bar{z}\bar{\tilde{\zeta}}}{2}.$$

Proof. **1.** We write the given integral as

$$\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta + z}{\zeta - z}\bar{\zeta} d\xi d\eta = \frac{2}{\pi}\int\limits_{\mathbb{D}} \frac{\bar{\zeta}}{\zeta - z} d\xi d\eta + \frac{-1}{\pi}\int\limits_{\mathbb{D}} \frac{\bar{\zeta}}{\zeta} d\xi d\eta.$$

Using the Cauchy- Pompeiu formula we get

$$\frac{1}{\pi}\int\limits_{\mathbb{D}} \frac{\bar{\zeta}}{\zeta - z} d\xi d\eta = \frac{-\bar{z}^2}{2} + \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \frac{\bar{\zeta}^2}{2(\zeta - z)} d\zeta.$$

Since

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\bar{\zeta}^2}{\zeta - z}d\zeta = 0 \text{ and } \frac{1}{\pi}\int_{\mathbb{D}} \frac{\bar{\zeta}}{\zeta}d\xi d\eta = 0$$

we obtain the result.

2. We separate the integral as

$$\frac{1}{2\pi}\iint_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}\frac{1}{\tilde{\zeta}-\zeta}d\xi d\eta$$
$$= \frac{z}{\tilde{\zeta}}\frac{1}{2\pi}\iint_{\mathbb{D}} \frac{1}{\zeta(\zeta-z)}d\xi d\eta - \frac{\tilde{\zeta}+z}{\tilde{\zeta}}\frac{1}{2\pi}\iint_{\mathbb{D}} \frac{1}{(\zeta-z)(\zeta-\tilde{\zeta})}d\xi d\eta.$$

By Gauss Theorem and Cauchy Pompeiu Formula we obtain the followings

$$\frac{1}{\pi}\iint_{\mathbb{D}} \frac{1}{(\zeta-z)(\zeta-\tilde{\zeta})}d\xi d\eta = \frac{\bar{z}-\bar{\tilde{\zeta}}}{\tilde{\zeta}-z}\cdot$$

and substituting $\tilde{\zeta}=0$ we have

$$\frac{1}{\pi}\iint_{\mathbb{D}} \frac{1}{\zeta(\zeta-z)}d\xi d\eta = \frac{-\bar{z}}{z}\cdot$$

3. As

$$\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}\frac{(1-|\zeta|^2)}{\zeta} = \frac{\bar{\zeta}}{\zeta} - \frac{1}{\zeta^2} - \frac{2\bar{\zeta}}{\zeta-z} + \frac{2}{\zeta(\zeta-z)}$$

the given integral can be written as

$$\frac{1}{2\pi}\iint_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}\frac{(1-|\zeta|^2)}{\zeta}\log(1-\zeta\bar{\tilde{\zeta}})d\xi d\eta = \frac{1}{2\pi}\iint_{\mathbb{D}} \frac{\bar{\zeta}}{\zeta}\log(1-\zeta\bar{\tilde{\zeta}})d\xi d\eta$$
$$-\frac{1}{2\pi}\iint_{\mathbb{D}} \frac{\log(1-\zeta\bar{\tilde{\zeta}})}{\zeta^2}d\xi d\eta - \frac{1}{\pi}\iint_{\mathbb{D}} \frac{\bar{\zeta}}{\zeta-z}\log(1-\zeta\bar{\tilde{\zeta}})d\xi d\eta + \frac{1}{\pi}\iint_{\mathbb{D}} \frac{\log(1-\zeta\bar{\tilde{\zeta}})}{\zeta(\zeta-z)}d\xi d\eta.$$

Gauss Theorem and Cauchy Pompeiu formula yield the expected result.

4. We split the given integral in two integrals:

$$\frac{1}{2\pi}\iint_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}\frac{(|\tilde{\zeta}|^2-|\zeta|^2)}{\tilde{\zeta}-\zeta}d\xi d\eta$$
$$= \frac{1}{2\pi}\iint_{\mathbb{D}} \frac{|\tilde{\zeta}|^2-|\zeta|^2}{(\zeta-z)(\tilde{\zeta}-\zeta)}d\xi d\eta + \frac{z}{2\pi}\iint_{\mathbb{D}} \frac{|\tilde{\zeta}|^2-|\zeta|^2}{\zeta(\zeta-z)(\tilde{\zeta}-\zeta)}d\xi d\eta.$$

In similar method in (**2.**) we obtain the value of the integrals. □

Lemma 2.2. *For $|z|<1$, $|\tilde{\zeta}|<1$ we get*

1. $\displaystyle \frac{1}{\pi}\iint_{\mathbb{D}} \frac{1}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\zeta d\xi d\eta = z^2.$

2. $\frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\frac{1}{\overline{\tilde{\zeta}-\zeta}}d\xi d\eta = \frac{\tilde{\zeta}}{2\bar{\tilde{\zeta}}} - \frac{z(z-\tilde{\zeta})}{1-z\bar{\tilde{\zeta}}}\cdot$

3. $\frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\frac{(1-|\zeta|^2)}{\bar{\zeta}}\log(1-\bar{\zeta}\tilde{\zeta})d\xi d\eta = \frac{-z\tilde{\zeta}}{2}\cdot$

4. $\frac{-1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\frac{(|\tilde{\zeta}|^2-|\zeta|^2)}{\overline{\tilde{\zeta}-\zeta}}d\xi d\eta = \frac{4z^2|\tilde{\zeta}|^2-\tilde{\zeta}^2-2z^2-z\tilde{\zeta}|\tilde{\zeta}|^2}{4(1-z\bar{\tilde{\zeta}})}\cdot$

Proof. **1.** We separate the given integral into two integrals as

$$\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{1}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\zeta d\xi d\eta = \frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{\zeta}{\bar{\zeta}(1-z\bar{\zeta})}d\xi d\eta + \frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{z\zeta}{1-z\bar{\zeta}}d\xi d\eta.$$

We observe that

$$\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{\zeta}{\bar{\zeta}}d\xi d\eta = 0$$

and

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \frac{z\zeta^2}{1-z\bar{\zeta}}d\bar{\zeta} = z^2.$$

2. Since

$$\frac{1}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\frac{1}{\overline{\tilde{\zeta}-\zeta}} = \frac{2z}{(1-z\bar{\zeta})(\overline{\tilde{\zeta}-\zeta})} + \frac{1}{\bar{\zeta}(\overline{\tilde{\zeta}-\zeta})}$$

Then

$$\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{z}{(1-z\bar{\zeta})(\overline{\tilde{\zeta}-\zeta})}d\xi d\eta = -\frac{z(z-\tilde{\zeta})}{1-z\bar{\tilde{\zeta}}}$$

and

$$\frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\bar{\zeta}(\overline{\tilde{\zeta}-\zeta})}d\xi d\eta = \frac{\tilde{\zeta}}{2\bar{\tilde{\zeta}}}\cdot$$

3. Similar to (**3.**) in Lemma1.

4. We separate the integral as

$$\frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\frac{(|\tilde{\zeta}|^2-|\zeta|^2)}{\overline{\zeta-\tilde{\zeta}}}d\xi d\eta$$

$$= \frac{|\tilde{\zeta}|^2}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\bar{\zeta}(1-z\bar{\zeta})(\overline{\zeta-\tilde{\zeta}})}d\xi d\eta + \frac{-1}{2\pi}\iint\limits_{\mathbb{D}} \frac{\bar{\zeta}}{(1-z\bar{\zeta})(\overline{\zeta-\tilde{\zeta}})}d\xi d\eta$$

$$+\frac{z}{2\pi}\iint\limits_{\mathbb{D}} \frac{|\tilde{\zeta}|^2-|\zeta|^2}{(1-z\bar{\zeta})(\overline{\zeta-\tilde{\zeta}})}d\xi d\eta$$

From the Cauchy Pompeiu formula we obtain the expected result.

□

3. The Schwarz - (Dirichlet-Neumann) problem

In this part, a solution to the Schwarz - (Dirichlet-Neumann) problem is presented. After computations of some boundary and area integrals we can give the main result now.

Theorem 3.1. *The Schwarz - (Dirichlet-Neumann) problem for the inhomogeneous tri-analytic equation in the unit disc*

$$\partial_{\bar z}^3 w = f \quad in\ \mathbb{D}, \quad Re w = \gamma \quad on\ \partial\mathbb{D}, \quad Im w(0) = c, \quad w_{\bar z} = \gamma_0 \quad on\ \partial\mathbb{D}, \quad \partial_\nu w_{\bar z\bar z} = \gamma_1 \quad on\ \partial\mathbb{D}, \quad w_{\bar z\bar z}(0) = c_1,$$

for $f \in C^\alpha(\overline{\mathbb{D}};\mathbb{C})$, $0<\alpha<1$, $\gamma,\gamma_0,\gamma_1 \in C(\partial\mathbb{D};\mathbb{C})$, $c, c_1 \in \mathbb{R}$ is uniquely solvable if and only if for $|z|<1$,

$$c_1 - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\zeta)\frac{d\zeta}{1-\bar z\zeta} + \frac{1}{\pi}\iint_{\mathbb{D}} \frac{1-|\zeta|^2}{1-\bar z\zeta}\frac{f(\zeta)}{\zeta}d\xi d\eta = 0 \tag{3.1}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\gamma_1(\zeta) - \bar\zeta f(\zeta))\frac{d\zeta}{\zeta(1-\bar z\zeta)} + \frac{1}{\pi}\iint_{\mathbb{D}} \frac{\bar z f(\zeta)}{(1-\bar z\zeta)^2}d\xi d\eta = 0. \tag{3.2}$$

The solution then is given by

$$\begin{aligned} w(z) = {}& ic + \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} + \frac{c_1\bar z^2}{2} - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\zeta)\left(\frac{\overline{\zeta-z}}{\zeta-z} - \frac{\bar\zeta}{2\zeta}\right)d\zeta \\ & - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\gamma_1(\zeta) - \bar\zeta f(\zeta))\left(\frac{|z|^4 - 2|z|^2 + 1}{2z^2}\log(1-z\bar\zeta) + \frac{\bar\zeta}{2z} + \frac{\bar\zeta^2}{8}\right)\frac{d\zeta}{\zeta} \\ & - \frac{1}{\pi}\iint_{\mathbb{D}} \frac{f(\zeta)}{\zeta}\left(\frac{\bar\zeta|\zeta|^2 - 4\bar z|\zeta|^2 + 2\bar z|z|^2 + z\bar\zeta^2}{4(\zeta-z)}\right)d\xi d\eta - \frac{c_1 z^2}{2} - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\zeta)\left(\frac{z(z-\zeta)}{1-z\bar\zeta} - \frac{\zeta}{2\bar\zeta}\right)d\bar\zeta \\ & - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\gamma_1(\zeta) - \zeta f(\zeta))\frac{z\zeta}{2}\frac{d\bar\zeta}{\bar\zeta} + \frac{1}{\pi}\iint_{\mathbb{D}} \frac{f(\zeta)}{\bar\zeta}\left(\frac{4z^2|\zeta|^2 - \zeta^2 - 2z^2 - z\zeta|\zeta|^2}{4(1-z\bar\zeta)}\right)d\xi d\eta \end{aligned}$$

Proof. The problem is decomposed into the system

$$\partial_{\bar z} w = \varphi \quad in\ \mathbb{D}, \quad Re w = \gamma \quad on\ \partial\mathbb{D}, \quad Im w(0) = c,$$

and

$$\partial_{\bar z^2}\varphi = f \ \ in\ \mathbb{D}, \quad \varphi = \gamma_0 \quad on\ \partial\mathbb{D}, \quad \partial_\nu \varphi_{\bar z} = \gamma_1 \quad on\ \partial\mathbb{D}, \quad \varphi_{\bar z}(0) = c_1$$

From Theorem 1.2, the solvability conditions for the Dirichlet-Neumann problem are

$$c_1 - \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \gamma_0(\zeta)\frac{d\zeta}{1-\bar z\zeta} + \frac{1}{\pi}\iint_{\mathbb{D}} \frac{1-|\zeta|^2}{1-\bar z\zeta}\frac{f(\zeta)}{\zeta}d\xi d\eta = 0 \tag{3.3}$$

and

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\gamma_1(\zeta) - \bar\zeta f(\zeta))\frac{d\zeta}{\zeta(1-\bar z\zeta)} + \frac{1}{\pi}\iint_{\mathbb{D}} \frac{\bar z f(\zeta)}{(1-\bar z\zeta)^2}d\xi d\eta = 0. \tag{3.4}$$

which yields the unique solution

$$\varphi(z) = c\bar{z} + \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \frac{\gamma_0(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} (\gamma_1(\zeta) - \bar{\zeta}f(\zeta))\log(1 - z\bar{\zeta})\frac{1-|z|^2}{z}\frac{d\zeta}{\zeta} + \frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{|\zeta|^2 - |z|^2}{\zeta - z}\frac{f(\zeta)}{\zeta}d\xi d\eta. \tag{3.5}$$

After Theorem 1.1, the unique solution for the Schwarz problem is

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} + ic - \frac{1}{2\pi}\iint\limits_{\mathbb{D}} \left(\frac{\varphi(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z} + \frac{\overline{\varphi(\zeta)}}{\bar{\zeta}}\frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}}\right)d\xi d\eta. \tag{3.6}$$

Replacing (3.5) into (3.6) we obtain

$$w(z) = ic + \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma(\zeta)\frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta} - \frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta + z}{\zeta - z}\left\{c_1\bar{\zeta}\right\}d\xi d\eta - \frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta + z}{\zeta - z}\left\{\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \frac{\gamma_0(\tilde{\zeta})}{\tilde{\zeta} - \zeta}d\tilde{\zeta}\right\}d\xi d\eta$$

$$- \frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta + z}{\zeta - z}\left\{\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} (\gamma_1(\tilde{\zeta}) - \bar{\tilde{\zeta}}f(\tilde{\zeta}))\log(1 - \zeta\bar{\tilde{\zeta}})\frac{1-|\zeta|^2}{\zeta}\frac{d\tilde{\zeta}}{\tilde{\zeta}}\right\}d\xi d\eta$$

$$- \frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\zeta}\frac{\zeta + z}{\zeta - z}\left\{\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{(|\tilde{\zeta}|^2 - |\zeta|^2)}{\tilde{\zeta} - \zeta}\frac{f(\tilde{\zeta})}{\tilde{\zeta}}d\tilde{\xi}d\tilde{\eta}\right\}d\xi d\eta$$

$$- \frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\bar{\bar{\zeta}}}\frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}}\left\{c_1\zeta\right\}d\xi d\eta - \frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\bar{\bar{\zeta}}}\frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}}\left\{\frac{-1}{2\pi i}\int\limits_{\partial\mathbb{D}} \frac{\overline{\gamma_0(\tilde{\zeta})}}{\bar{\tilde{\zeta}} - \zeta}d\bar{\tilde{\zeta}}\right\}d\xi d\eta$$

$$- \frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\bar{\bar{\zeta}}}\frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}}\left\{\frac{-1}{2\pi i}\int\limits_{\partial\mathbb{D}} (\gamma_1(\tilde{\zeta}) - \tilde{\zeta}f(\tilde{\zeta}))\log(1 - \bar{\zeta}\tilde{\zeta})\frac{1-|\zeta|^2}{\bar{\zeta}}\frac{d\bar{\tilde{\zeta}}}{\bar{\tilde{\zeta}}}\right\}d\xi d\eta$$

$$- \frac{1}{2\pi}\iint\limits_{\mathbb{D}} \frac{1}{\bar{\bar{\zeta}}}\frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}}\left\{\frac{1}{\pi}\iint\limits_{\mathbb{D}} \frac{(|\tilde{\zeta}|^2 - |\zeta|^2)}{\overline{\tilde{\zeta} - \zeta}}\frac{f(\tilde{\zeta})}{\bar{\tilde{\zeta}}}d\tilde{\xi}d\tilde{\eta}\right\}d\xi d\eta.$$

Using the results of Lemma 2.1 and Lemma 2.2, we obtain the unique solution for the combined problem.

□

4. Conclusion

A combination of boundary value problems is considered in this paper. The uniqueness and solvability conditions of the Schwarz-(Dirichlet-Neumann) problem are considered. In the future, that problem will be investigated in the polydisc.

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