Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–17.
ISSN-0037-8712 in press
doi:10.5269/bspm.65211

Solving Bilevel Quasimonotone Variational Inequality Problem In Hilbert Spaces

D. O. Peter, A. A. Mebawondu*, G. C. Ugwunnadi, P. Pillay and O. K. Narain

ABSTRACT: In this paper, we propose and study a Bilevel quasimonotone Variational Inequality Problem (BVIP) in the framework of Hilbert space. We introduce a new modified inertial iterative technique with self-adaptive step size for approximating a solution of the BVIP. In addition, we established a strong convergence result of the proposed iterative technique with adaptive step-size conditions without prior knowledge of Lipschitz's constant of the cost operators as well as the strongly monotonicity coefficient under some standard mild assumptions. Finally, we provide some numerical experiments to demonstrate the efficiency of our proposed methods in comparison with some recently announced results in the literature.

Key Words: Variational inequality problem, inertial technique, quasimonotone, Hilbert space.

Contents

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, C a nonempty closed convex subset of H and $F : H \to H$ be a nonlinear operator. The classical Variational Inequality Problem (VIP) is formulated as:

$$\text{Find } x \in C \text{ such that } \langle Fx, y - x \rangle \geq 0 \;\forall\; y \in C. \tag{1.1}$$

The notion of VIP was introduced independently by Stampacchia [36] and Fichera [13,14] for modeling problems arising from mechanics and for solving Signorini problem. It is well-known that many problems in economics, mathematical sciences, and mathematical physics can be formulated as VIP. We denote the solution set of a VIP by Ω. Due to the fruitful applications of the VIP, many researchers in this area have developed different iterative techniques to solve VIP (1.1). In particular, Goldsten in [20] introduced an iterative technique defined as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C(x_n - \lambda F x_n), \end{cases} \tag{1.2}$$

for all $n \in \mathbb{N}$, where $\lambda \in (0, \frac{2\alpha}{L^2})$, F is α-strongly monotone and L-Lipschitz continuous and P_C is a metric projection defined from H onto C. The author established that the iterative method (1.2) converges to the solution set of VIP (1.1). However, it was observed that if F monotone and L-Lipschitzian continuous, the iterative method (1.2) may not converge to the solution set of VIP (1.1), see [22] and the reference therein for details. In addition, computing the value of λ may be very difficult or impossible. In the light

2010 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted September 29, 2022. Published May 01, 2023

of these draw back, Korpelevich in [24] introduced and studied the Extragradient Method (EM) iterative technique defined as follows:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda_n F x_n), \\ x_{n+1} = P_C(x_n - \lambda_n F y_n), \end{cases} \tag{1.3}$$

for all $n \geq 1$, where $\lambda_n \in (0, \frac{1}{L})$, F is monotone and L-Lipschitz continuous and P_C is a metric projection defined from H onto C. This method was able to provide an affirmative answer to the question of weakening the cost operator, however, the computation of λ_n remains a challenge. More so, another set back of this technique (1.3) is that it requires two projections onto the feasible set C per iteration, which is costly when C is not a simple structure. Since the inception of EM, many authors have introduced, modified and studied different EM in which the cost operator F is monotone and pseudomonotonicity. For example, He et at. [23], Apostol et al. [2], He et al. [22], Ceng et al. [4], Ceng et al [5], Nadezhkina and Takahashi [27] and many others. In addition, the notion of VIP (1.1) has also been extended and generalized by many authors. For example, Mainge in [28] introduced and studied the notion of Bilevel Variational Inequality Problems (BVIP). The BVIP is defined as follows:

$$\text{Find } \ x^* \in \Omega \text{ such that } \langle Gx^*, x - x^* \rangle \geq 0 \ \forall \ x \in \Omega. \tag{1.4}$$

where $G : H \to H$ is L-Lispschitz continuous and γ-strongly monotone. It is easy to see that the BVIP (1.4) is a problem that is made up of the VIP (1.1) as a constraint. He proposed the following extragradient technique:

$$\begin{cases} u_0 \in C \\ v_n = P_C(u_n - \lambda_n F u_n) \\ t_n = P_C(u_n - \lambda_n F v_n) \\ u_{n+1} = t_n - \alpha_n G t_n, \end{cases} \tag{1.5}$$

where $\{\lambda_n\} \subset [a, b] \in \left(0, \frac{1}{L}\right)$ and $\alpha_n \subset (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$. It was established that the sequence generated by $\{x_n\}$ converges strongly to a unique solution of problem BVIP (1.4). It is easy to see that the iterative technique (1.5) has at least two setbacks, for example $\{\lambda_n\} \subset [a, b] \subset \left(0, \frac{1}{L}\right)$, and the double metric projection (P_C). In order to overcome this setbacks, researchers have introduced the Tseng type iterative technique, the projection contraction iterative technique and the subgradient extragradient iterative technique that are self adaptive, see [35,37,38,39,42] and the reference therein for details. In particular, Tan et al., [38] introduced and studied the following iterative technique;

$$\begin{cases} x_0, x_1 \in C, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n F w_n), \\ T_n = \{x \in H : \langle w_n - \lambda_n F w_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = P_{T_n}(w_n - \lambda_n F y_n), \\ x_{n+1} = z_n - \alpha_n \gamma G z_n, \end{cases} \tag{1.6}$$

for all $n \in \mathbb{N}$, and

$$\begin{cases} x_0, x_1 \in C, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n F w_n), \\ z_n = y_n - \lambda_n(F_1 w_n - F y_n), \\ x_{n+1} = z_n - \alpha_n \gamma G z_n, \end{cases} \tag{1.7}$$

for all $n \in \mathbb{N}$, where F is L-Lipschitz continuous, pseudomonotone and sequentially weakly continuous, G is α-strongly monotone and L_1-Lipschitz continuous. They established that the iterative techniques (1.2) and (1.7) converge strongly to a unique solution of the BVIP (1.4) using some standard assumptions. The question is still wide open, if some (all) of the mentioned iterative techniques can further be improved. One of the ways in which these iterative techniques have been improved over the years is the introduction of the inertial extrapolation technique. In 1964, Polyak in [32] introduced an inertial extrapolation as an acceleration technique process for solving the smooth convex minimization problem. Since then, this technique has been employed by research to improve their iterative techniques. The inertial technique requires the first two initial terms of the iterative technique and the next iterate is defined by making use of the previous two iterates. Since inception of the inertial extrapolation, many authors have modified, extended and generalized the technique, see [15,16,35] and the references therein. However, it has always been a question of interest, if the extrapolation technique can further be improved. The importance of the BVIP cannot be overemphasized. The BVIP (1.4) has been applied to different areas of mathematical sciences, engineering, physics and so on. For example,the BVIP (1.4) has applications in equilibrium constraints, bilevel convex programming models, minimum-norm problems with the solution set of variational inequalities, bilevel linear programming, image restoration and many more see ([1,12,18,19,26,40,41]) and the references therein. Due to these applications, many authors have introduced different iterative techniques for solving the BVIP in the framework of Hilbert spaces (see, [1,15,16,17,28,29,39] and the references therein). It is well-known that the underlying cost operators have crucial roles to play in real applications of these iterative methods. In the light of this introducing an iterative technique with weaker cost operators and better rate of convergence is highly sorted after. Having consider the above discussed literatures and the references therein, it is natural to ask the following question:

1. Can we construct an efficient inertial type iterative technique that does require the knowledge of the Lipschitz constant during implementation of the algorithm?

2. Can we construct an iterative technique for a BVIP (1.4) in which the cost operators are quasimonotone and α-strongly monotone in the framework of infinite dimensional Hilbert spaces and obtain strong convergence?

3. Can we further explore the iterative techniques (1.7) and (1.5), in which the cost operator is quasimonotone and strongly monotone and the knowledge of the Lipschitz constant needed not to be known during implementation?

The purpose of this work is to provide and affirmative answer to the above questions by introducing a modified inertial iterative technique with self-adaptive step size for approximating the solution of quasimonotone BVIP (1.4). In addition, we use a modified inertial technique to accelerate the rate of convergence of our proposed methods. In addition, our numerical experiments justify that our method is better than other methods in the literature for solving the BVIP (1.4).

The rest of this paper is organized as follows: In Section 2, we recall some useful definitions and results that are relevant for our study. In Section 3, we present our proposed method. In Section 4, we establish strong convergence of our method and in Section 5, we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite dimensional Hilbert spaces. Lastly in Section 6, we give the conclusion of the paper.

2. Preliminaries

In this section, we begin by recalling some known and useful results which are needed in the sequel. Let H be a real Hilbert space. The set of fixed points of a nonlinear mapping $T : H \to H$ will be denoted by $F(T)$, that is $F(T) = \{x \in H : Tx = x\}$. We denotes strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. For any $x, y \in H$ and $\alpha \in [0, 1]$, it is well-known that

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2. \tag{2.1}$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2. \tag{2.2}$$

$$\|x-y\|^2 \leq \|x\|^2 + 2\langle y, x-y\rangle. \tag{2.3}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.4}$$

Definition 2.1. *Let $T : H \to H$ be an operator. Then T is called*

(a) L-Lipschitz continuous if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|,$$

for all $x, y \in H$. If $L = 1$, then T is called nonexpansive. If $y \in F(T)$, and

$$\|Tx - y\| \leq \|x - y\|,$$

for all $x \in H$. Then T is called quasinonexpansive.

(b) monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \ \forall x, y \in H;$$

(c) pseudomonotone if

$$\langle Tx, y - x\rangle \geq 0 \Rightarrow \langle Ty, y - x\rangle \geq 0, \ \forall x, y \in H;$$

(d) α- strongly monotone if there exists $\alpha > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \ \forall \ x, y \in H;$$

(e) quasimonotone

$$\langle Tx, x - y\rangle > 0 \Rightarrow \langle Ty, x - y\rangle \geq 0 \ \forall \ x, y \in H;$$

(f) sequentially weakly continuous if for each sequence $\{x_n\}$, we obtain $\{x_n\}$ converges weakly to x implies that Tx_n converges weakly to Tx.

Remark 2.1. *It is well-known that α-strongly monotone $\Rightarrow$ monotone $\Rightarrow$ pseudomonotone $\Rightarrow$ quasi-monotone. However, the converses are not generally true.*

Let C be a nonempty, closed and convex subset of H. For any $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\| \ \forall y \in C.$$

The operator P_C is called the metric projection of H onto C. It is well-known that P_C is a nonexpansive mapping and that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{2.5}$$

for all $x, y \in H$. Furthermore, P_C is characterized by the property

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$$

and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.6}$$

for all $x \in H$ and $y \in C$.

Lemma 2.2. *[21,43] Let C be a nonempty, closed and convex subset of a Hilbert space H and $A : H \to H$ ba L-Lipschitzian and quasimonotone operator. Suppose that $y \in C$ and for some $p \in C$, we have $\langle Ay, p - y\rangle \geq 0$, then at least on of the following hold*

$$\langle Ap, p - y\rangle \geq 0 \text{ or } \langle Ay, q - y\rangle \leq 0$$

for all $q \in C$.

Lemma 2.3. *[34] Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition

$$\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim\limits_{k\to\infty} a_n = 0$.

Lemma 2.4. *[1] Let C be nonempty closed convex subset of a real Hilbert space H. For any $x \in H$ and $z \in C$, we have $z = P_C x$ if and only if $\langle x - z, y - z\rangle \leq 0 \;\forall\; y \in C$.*

Lemma 2.5. *[1] Let H be a Hilbert space and $F : H \to H$ be a τ-strongly monotone and L-Lipschitz continuous operator on H. Let $\alpha \in (0, 1)$ and $\gamma \in (0, \frac{2\tau}{L^2})$. Then for any nonexpansive operator $T : H \to H$, we can associate $T^\gamma : H \to H$ defined by $T^\gamma x = (I - \alpha\gamma F)Tx$ for all $x \in H$. Then, T^γ is a contraction. That is*

$$\|T^\gamma x - T^\gamma y\| \leq (1 - \alpha\nu)\|x - y\|$$

for all $x, y \in H$, where $\nu = 1 - \sqrt{1 - \gamma(2\tau - \gamma L^2)} \in (0, 1)$.

3. Proposed Algorithm

In this section, we present our proposed method for solving a bilevel quasimonotone variational inequality problem.

Assumption 1.

Condition A. Suppose

1. The feasible sets C is nonempty set, closed and convex subsets of the real Hilbert space H..
2. $\{S_n\}$ is a sequence of nonexpansive mapping on H.
3. $F : H \to H$ is quasimonotone, sequentially weakly continuous an Lipschitz continuous with Lipschitz constant $L > 0$.
4. $G : H \to H$ is τ-strongly monotone and Lipschitz continuous with Lipschitz constant $L_1 > 0$.
5. The solution set $\Phi := \{x \in \Omega : \langle Gx^*, x - x^*\rangle \geq 0 \;\forall\; x \in \Omega\}$ is nonempty, where Ω is the solution of the classical VIP (1.1).

Condition B.

1. $\{\beta_n\} \subset (0, 1)$, $\lim\limits_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$.
2. $\alpha \in (0, \frac{2\tau}{L_1^2}), \lambda_0 > 0, \nu \in (0, 1)$ and choose the nonnegative real sequence $\{p_n\}$ such that $\sum_{n=1}^{\infty} p_n < \infty$.

We present the following iterative algorithm.

Algorithm 1. Initialization Step:

Step 1: *Choose* $x_0, x_1 \in H$, *given the iterates* x_{n-1} *and* x_n *for all* $n \in \mathbb{N}$, *choose* θ_n *such that* $0 \leq \theta_n \leq \bar{\theta}_n$, *where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{n-1}{n+\beta-1}, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \frac{n-1}{n+\beta-1}, & \text{otherwise,} \end{cases} \tag{3.1}$$

with θ *been a positive constant and* $\{\epsilon_n\}$ *is a positive sequence such that* $\epsilon_n = \circ(\beta_n)$.

Step 2. *Set*

$$w_n = x_n + \theta_n(S_n x_n - S_n x_{n-1}).$$

Then, compute

$$y_n = P_C(w_n - \lambda_n F w_n), \tag{3.2}$$
$$z_n = y_n + \lambda_n(F w_n - F y_n), \tag{3.3}$$

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\nu\|w_n - y_n\|}{\|F w_n - F y_n\|}, \lambda_n + p_n\right\}, & \text{if } F w_n \neq F y_n \\ \\ \lambda_n + p_n, & \text{otherwise.} \end{cases} \tag{3.4}$$

Step 4. *Compute*

$$x_{n+1} = z_n - \beta_n \alpha G z_n \tag{3.5}$$

4. Convergence Analysis

Lemma 4.1. *The step-size* λ_{n+1} *in Algorithm 1 is well defined.*

Proof The proof that λ_{n+1}, is well defined follows similar approach as in Lemma 3.1 of [25], thus we omit it.

Lemma 4.2. *Let* $\{x_n\}$ *be a sequence generated by Algorithm 1 under Assumption 3. Then,* $\{x_n\}$ *is bounded.*

Proof Let $p \in \Phi$. Since $\lim_{n\to\infty} \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \leq N_1$, for all $n \in \mathbb{N}$. Then using Algorithm 1, we have

$$\begin{aligned} \|w_n - p\| &= \|x_n + \theta_n(S_n x_n - S_n x_{n-1}) - p\| \\ &\leq \|x_n - p\| + \theta_n\|S_n x_n - S_n x_{n-1}\| \\ &\leq \|x_n - p\| + \beta_n \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \\ &\leq \|x_n - p\| + \beta_n N_1. \end{aligned} \tag{4.1}$$

Also, using Algorithm 1 and (2.2)

$$\begin{aligned} \|z_n - p\|^2 &= \|y_n + \lambda_n(F w_n - F y_n) - p\|^2 \\ &= \|y_n - \lambda_n(F y_n - F w_n) - p\|^2 \\ &= \|y_n - p\|^2 + \lambda_n^2\|F y_n - F w_n\|^2 - 2\lambda_n\langle F y_n - F w_n, y_n - p\rangle \\ &= \|y_n - w_n + w_n - p\|^2 + \lambda_n^2\|F y_n - F w_n\|^2 - 2\lambda_n\langle F y_n - F w_n, y_n - p\rangle \\ &= \|w_n - p\|^2 + \|y_n - w_n\|^2 + 2\langle y_n - w_n, w_n - p\rangle + \lambda_n^2\|F y_n - F w_n\|^2 \qquad (4.2) \\ &\quad - 2\lambda_n\langle F y_n - F w_n, y_n - p\rangle \\ &= \|w_n - p\|^2 + \|y_n - w_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle + 2\langle y_n - w_n, y_n - p\rangle \\ &\quad + \lambda_n^2\|F y_n - F w_n\|^2 - 2\lambda_n\langle F y_n - F w_n, y_n - p\rangle. \end{aligned} \tag{4.3}$$

Since $y_n = P_C(w_n - \lambda_n A w_n)$ and $p \in \Phi \subseteq C$, and by Lemma 2.4, we have

$$\langle y_n - w_n + \lambda_n F w_n, y_n - p\rangle \leq 0.$$

It then follows that $\langle y_n - w_n, y_n - p\rangle \leq -\lambda_n \langle F w_n, y_n - p\rangle$. Using the fact that $y_n \in C$ and $p \in \Phi$, we have $\langle F y_n, y_n - p\rangle \geq 0$. Using the above facts and (3.4),we have (4.2) becomes

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|w_n - p\|^2 - \|y_n - w_n\|^2 - 2\lambda_n\langle Fw_n, y_n - p\rangle + \lambda_n^2\|Fy_n - Fw_n\|^2 \qquad (4.4)\\
&\quad - 2\lambda_n\langle Fy_n - Fw_n, y_n - p\rangle\\
&= \|w_n - p\|^2 - \|y_n - w_n\|^2 + \lambda_n^2\|Fy_n - Fw_n\|^2 - 2\lambda_n\langle Fy_n, y_n - p\rangle\\
&\leq \|w_n - p\|^2 - \|y_n - w_n\|^2 + \lambda_n^2\|Fy_n - Fw_n\|^2\\
&\leq \|w_n - p\|^2 - (1 - \frac{\lambda_n^2\nu^2}{\lambda_{n+1}^2})\|y_n - w_n\|^2\\
&\leq \|w_n - p\|^2. \qquad (4.5)
\end{aligned}$$

It follows that

$$\|z_n - p\| \leq \|w_n - p\|. \tag{4.6}$$

In addition, using Algorithm 1, Lemma 2.5 and the fact that $\gamma = 1 - \sqrt{1 - \alpha(2\tau - \alpha L_1^2)} \in (0, 1)$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|z_n - \beta_n\alpha G z_n - p\|\\
&= \|(1 - \beta_n\alpha G)z_n - (1 - \beta_n\alpha G)p - \beta_n\alpha Gp\|\\
&\leq (1 - \gamma\beta_n)\|z_n - p\| + \beta_n\alpha\|Gp\|\\
&\leq (1 - \gamma\beta_n)\|w_n - p\| + \beta_n\alpha\|Gp\|\\
&\leq (1 - \gamma\beta_n)[\|x_n - p\| + \beta_n N_1] + \beta_n\alpha\|F_2 p\|\\
&\leq (1 - \gamma\beta_n)\|x_n - p\| + \gamma\beta_n[\frac{N_1 + \alpha\|Gp\|}{\gamma}]\\
&\leq \max\{\|x_n - p\|, \frac{[N_1 + \alpha\|Gp\|}{\gamma}\}\\
&\ \ \vdots \qquad (4.7)\\
&\leq \max\{\|x_1 - p\|, \frac{[N_1 + \alpha\|Gp\|}{\gamma}\}. \qquad (4.8)
\end{aligned}$$

Thus, we have that $\{x_n\}$ is bounded.

Lemma 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 1 under Assumption 3 and suppose that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $x^* \in H$ and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$. Then, $x^* \in \Omega$.*

Proof Since $y_{n_k} = P_C(w_{n_k} - \lambda_{n_k} F w_{n_k})$, then from the characteristic of the metric projection, we have

$$\langle w_{n_k} - \lambda_{n_k} F w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \leq 0 \ \forall \ x \in C, \tag{4.9}$$

which implies that

$$\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle - \lambda_{n_k}\langle F w_{n_k}, x - y_{n_k}\rangle \leq 0, \tag{4.10}$$

which implies that

$$\begin{aligned}
\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle &\leq \lambda_{n_k}\langle F w_{n_k}, x - y_{n_k}\rangle \qquad (4.11)\\
&= \lambda_{n_k}\langle F w_{n_k}, w_{n_k} - y_{n_k}\rangle + \lambda_{n_k}\langle F w_{n_k}, x - w_{n_k}\rangle. \qquad (4.12)
\end{aligned}$$

Since $\lambda_{n_k} > 0$, we have

$$\frac{1}{\lambda_{n_k}}\langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle Fw_{n_k}, y_{n_k} - w_{n_k}\rangle \leq \langle Fw_{n_k}, x - w_{n_k}\rangle. \tag{4.13}$$

Using our hypothesis and the fact that $\lim\limits_{k\to\infty} \lambda_{n_k} > 0$, , we have

$$0 \leq \liminf_{k\to\infty}\langle Fw_{n_k}, x - w_{n_k}\rangle \leq \limsup_{k\to\infty}\langle Fw_{n_k}, x - w_{n_k}\rangle. \tag{4.14}$$

Now, observe that

$$\begin{aligned}\langle Fy_{n_k}, x - y_{n_k}\rangle &= \langle Fy_{n_k}, x - w_{n_k}\rangle + \langle Fy_{n_k}, w_{n_k} - y_{n_k}\rangle \\ &= \langle Fy_{n_k} - Fw_{n_k}, x - w_{n_k}\rangle + \langle Fw_{n_k}, x - w_{n_k}\rangle + \langle Fy_{n_k}, w_{n_k} - y_{n_k}\rangle. \end{aligned} \tag{4.15}$$

Since F is Lischitz continuous on H and our hypothesis, we have

$$\lim_{k\to\infty} \|Fw_{n_k} - Fy_{n_k}\| \leq L \lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0. \tag{4.16}$$

Combining (4.14), (4.15) and (4.16), we have

$$0 \leq \liminf_{k\to\infty}\langle Fy_{n_k}, x - y_{n_k}\rangle \leq \limsup_{k\to\infty}\langle Fy_{n_k}, x - y_{n_k}\rangle. \tag{4.17}$$

In what follows, we now establish that $x^* \in \Omega$. To start with, we consider the case in which $\limsup_{k\to\infty}\langle Fy_{n_k}, x - y_{n_k}\rangle > 0$ for all $x \in C$. Then there exists a subsequence $\{y_{n_{k_m}}\}$ of sequence $\{y_{n_k}\}$ such that $\limsup_{m\to\infty}\langle Fy_{n_{k_m}}, x - y_{n_{k_m}}\rangle > 0$ for all $x \in C$. It follows that we can find N_0 such that

$$\langle Fy_{n_{k_m}}, x - y_{n_{k_m}}\rangle > 0 \ \forall m > N_0. \tag{4.18}$$

Since A is quasimonotone, it follows that

$$\langle Fx, x - y_{n_{k_m}}\rangle > 0 \ \forall m > N_0. \tag{4.19}$$

Now observe that

$$\|w_{n_{k_m}} - x_{n_{k_m}}\| = \alpha_{n_{k_m}} \frac{\theta_{n_{k_m}}}{\alpha_{n_{k_m}}}\|S_{n_{k_m}} x_{n_{k_m}} - S_{n_{k_m}} x_{n_{k_m}-1}\| \to 0 \text{ as } m \to \infty. \tag{4.20}$$

Since, the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is weakly convergent to a point $x^* \in H$. Hence, using the fact that $\lim\limits_{n\to\infty} \|w_{n_{k_m}} - y_{n_{k_m}}\| = 0$, we have that $\{y_{n_{k_m}}\}$ also converges to x^*. Now passing the limit as $m \to \infty$ in (4.19), we have

$$\lim_{m\to\infty}\langle Fx, x - y_{n_{k_m}}\rangle = \langle Fx, x - x^*\rangle > 0. \tag{4.21}$$

Hence, $x^* \in \Omega$.

Secondly, we consider the case in which $\limsup_{k\to\infty}\langle Fy_{n_k}, x - y_{n_k}\rangle = 0$ for $x \in C$. Let $\{\delta_k\}$ be a non-increasing positive sequence defined by

$$\delta_k = |\langle Fy_{n_k}, x - y_{n_k}\rangle| + \frac{1}{k+1}. \tag{4.22}$$

By our hypothesis, it is easy to see that

$$\lim_{k\to\infty} \delta_k = \lim_{k\to\infty}\langle Fy_{n_k}, x - y_{n_k}\rangle + \lim_{k\to\infty} \frac{1}{k+1} = 0. \tag{4.23}$$

By our hypothesis and (4.22), we have

$$\langle Fy_{n_k}, x - y_{n_k}\rangle + \delta_k > 0 \tag{4.24}$$

for each $k \geq 1$, since $\{y_{n_k}\} \subset C$, it implies that $\{Fy_{n_k}\}$ is strictly non-zero and $\liminf_{k\to\infty} \|Fy_{n_k}\| = N_0 > 0$. We therefore deduce that

$$\|Fy_{n_k}\| > \frac{N_0}{2} \tag{4.25}$$

In addition, let $\{\epsilon_{n_k}\}$ be a sequence defined by $\epsilon_{n_k} = \frac{Fy_{n_k}}{\|Fy_{n_k}\|^2}$. It implies that

$$\langle Fy_{n_k}, \epsilon_{n_k}\rangle = 1. \tag{4.26}$$

Combining (4.24) and (4.26), we have

$$\langle Fy_{n_k}, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle > 0. \tag{4.27}$$

By quasimonotonicity of the operator F on H, we get that

$$\langle F(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle \geq 0. \tag{4.28}$$

Now, observe that

$$\langle Fx, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle = \langle Fx - F(x + \delta_k\epsilon_{n_k}) + F(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle \tag{4.29}$$

$$= \langle Fx - F(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle + \langle F(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle \tag{4.30}$$

Combining (4.28), (4.29) and applying the well known Cauchy Schwartz inequality, we have

$$\langle Fx, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle \geq \langle Fx - F(x + \delta_k\epsilon_{n_k}), x + \delta_k\epsilon_{n_k} - y_n\rangle \tag{4.31}$$

$$\geq -\|Fx - F(x + \delta_k\epsilon_{n_k})\|\|x + \delta_k\epsilon_{n_k} - y_{n_k}\|. \tag{4.32}$$

Since F is Lipschitz continuous, we have

$$\langle Fx, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle + L\|\delta_k\epsilon_{n_k}\|\|x + \delta_k\epsilon_{n_k} - y_{n_k}\| \geq 0 \tag{4.33}$$

Combining (4.25) and (4.33) and using the definition of $\{\epsilon_{n_k}\}$, we have

$$\langle Fx, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle + \frac{2L}{N_0}\delta_k\|x + \delta_k\epsilon_{n_k} - y_{n_k}\| \geq 0. \tag{4.34}$$

Since, the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is weakly convergent to a point $x^* \in H$. Hence, using the fact that $\lim\limits_{n\to\infty} \|w_{n_k} - y_{n_k}\| = 0$, we have that $\{y_{n_k}\}$ also converges to x^*. Taking limit as $k \to \infty$, since $\delta_k \to 0$, we have

$$\lim_{k\to\infty}\left[\langle Fx, x + \delta_k\epsilon_{n_k} - y_{n_k}\rangle + \frac{2L}{N_0}\delta_k\|x + \delta_k\epsilon_{n_k} - y_{n_k}\|\right] = \langle Fx, x - x^*\rangle > 0. \tag{4.35}$$

Hence $x^* \in \Omega$.

Theorem 4.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 1 under Assumption 3. Then, $\{x_n\}$ converges strongly to $p \in \Phi$.*

Proof Let $p \in \Phi$, observe that

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n + \theta_n(S_n x_n - S_n x_{n-1}) - p\|^2 \\
&= \|x_n - p\|^2 + 2\theta_n\langle x_n - p, S_n x_n - S_n x_{n-1}\rangle + \theta_n^2\|S_n x_n - S_n x_{n-1}\|^2 \\
&\leq \|x_n - p\|^2 + 2\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|] \\
&= \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \beta_n\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\|] \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \beta_n N_1] \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2, \qquad (4.36)
\end{aligned}$$

for some $N_2 > 0$.

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|z_n - \beta_n\alpha G z_n - p\|^2 \\
&= \|(1 - \beta_n\alpha G)z_n - (1 - \beta_n\alpha G)p - \beta_n\alpha Gp\|^2 \\
&\leq \|(1 - \beta_n\alpha G)z_n - (1 - \beta_n\alpha G)p\|^2 - 2\beta_n\alpha\langle Gp, x_{n+1} - p\rangle \\
&\leq (1 - \gamma\beta_n)^2\|z_n - p\|^2 + 2\beta_n\alpha\langle Gp, p - x_{n+1}\rangle \\
&\leq (1 - \gamma\beta_n)\|w_n - p\|^2 + 2\beta_n\alpha\langle Gp, p - x_{n+1}\rangle \\
&\leq (1 - \gamma\beta_n)[\|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2] + 2\beta_n\alpha\langle Gp, p - x_{n+1}\rangle \\
&= (1 - \gamma\beta_n)\|x_n - p\|^2 + \gamma\beta_n\left[\frac{\theta_n}{\gamma\beta_n}\|x_n - x_{n-1}\|N_2 + 2\frac{\alpha}{\gamma}\langle Gp, p - x_{n+1}\rangle\right] \\
&= (1 - \gamma\beta_n)\|x_n - p\|^2 + \gamma\beta_n\Psi_n \qquad (4.37)
\end{aligned}$$

where $\Psi_n = \frac{\theta_n}{\gamma\beta_n}\|x_n - x_{n-1}\|N_2 + 2\frac{\alpha}{\gamma}\langle Gp, p - x_{n+1}\rangle$. According to Lemma 2.3, to conclude our proof, it is sufficient to establish that $\limsup_{k\to\infty}\Psi_n \leq 0$ for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ satisfying the condition:

$$\liminf_{k\to\infty}\{\|x_{n_k+1} - p\| - \|x_{n_k} - p\|\} \geq 0. \qquad (4.38)$$

To establish that $\limsup_{k\to\infty}\Psi_{n_k} \leq 0$, we suppose that for every subsequence $\{\|x_{n_k} - p\|\}$ of $\{\|x_n - p\|\}$ such that (4.38) holds. Then,

$$\begin{aligned}
&\liminf_{k\to\infty}\{\|x_{n_k+1} - p\|^2 - \|x_{n_k} - p\|^2\} \\
&= \liminf_{k\to\infty}\{(\|x_{n_k+1} - p\| - \|x_{n_k} - p\|)(\|x_{n_k+1} - p\| + \|x_{n_k} - p\|)\} \geq 0. \qquad (4.39)
\end{aligned}$$

It is easy to see from (4.37),

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|z_n - p\|^2 + 2\beta_n\alpha\langle Gp, p - x_{n+1}\rangle \\
&\leq \|w_n - p\|^2 - (1 - \frac{\lambda_n^2\nu^2}{\lambda_{n+1}^2})\|y_n - w_n\|^2 + 2\beta_n\alpha\langle Gp, p - x_{n+1}\rangle \\
&\leq \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2 - (1 - \frac{\lambda_n^2\nu^2}{\lambda_{n+1}^2})\|y_n - w_n\|^2 + 2\beta_n\alpha\langle Gp, p - x_{n+1}\rangle, \qquad (4.40)
\end{aligned}$$

which implies that

$$\begin{aligned}
&\limsup_{k\to\infty}\left((1-\frac{\lambda_{n_k}^2\nu^2}{\lambda_{n_k+1}^2})\|y_{n_k}-w_{n_k}\|^2\right)\\
&\le \limsup_{k\to\infty}\left[\|x_{n_k}-p\|^2+\beta_{n_k}\frac{\theta_{n_k}}{\beta_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2+2\beta_{n_k}\alpha\langle Gp,p-x_{n_k+1}\rangle-\|x_{n_k+1}-p\|^2\right]\\
&\le -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\le 0.
\end{aligned}$$

Thus, we have

$$\lim_{k\to\infty}\|y_{n_k}-w_{n_k}\|=0. \tag{4.41}$$

It is easy to see that, as $k\to\infty$, we have

$$\|w_{n_k}-x_{n_k}\|=\theta_{n_k}||S_nx_{n_k}-S_nx_{n_k-1}||=\beta_{n_k}\cdot\frac{\theta_{n_k}}{\beta_{n_k}}||S_nx_{n_k}-S_nx_{n_k-1}||\to 0. \tag{4.42}$$

$$\|z_{n_k}-w_{n_k}\|\le\|y_{n_k}-w_{n_k}\|+\frac{\lambda_{n_k}\nu}{\lambda_{n_k+1}}\|y_{n_k}-w_{n_k}\|\ \to 0 \text{ as } k\to\infty. \tag{4.43}$$

$$\|y_{n_k}-x_{n_k}\|\le\|y_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\ \to 0 \text{ as } k\to\infty. \tag{4.44}$$

$$\|z_{n_k}-x_{n_k}\|\le\|z_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\ \to 0 \text{ as } k\to\infty. \tag{4.45}$$

$$\|x_{n_k+1}-z_{n_k}\|\le\|z_{n_k}-\beta_{n_k}\alpha Gz_{n_k}-z_{n_k}\|=\beta_{n_k}\|\alpha Gz_{n_k}\|\ \to 0 \text{ as } k\to\infty. \tag{4.46}$$

$$\|x_{n_k+1}-x_{n_k}\|\le\|x_{n_k+1}-z_{n_k}\|+\|z_{n_k}-x_{n_k}\|\ \to 0 \text{ as } k\to\infty. \tag{4.47}$$

Now, since $\{x_{n_k}\}$ is bounded, then, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $x^*\in H$. In addition, using (4.41), and Lemma 4.3, we obtain that $x^*\in\Phi$. Furthermore, since $x_{n_{k_j}}$ converges weakly to x^*, we obtain that

$$\limsup_{k\to\infty}\langle Gp,p-x_{n_k}\rangle=\lim_{j\to\infty}\langle Gp,p-x_{n_{k_j}}\rangle=\langle Gp,p-x^*\rangle. \tag{4.48}$$

Hence, since p is a unique solution of Φ, it follows that

$$\limsup_{k\to\infty}\langle Gp,p-x_{n_k}\rangle=\langle Gp,p-x^*\rangle\le 0, \tag{4.49}$$

we have obtain from (4.49) and (4.47)

$$\limsup_{k\to\infty}\langle Gp,p-x_{n_k+1}\rangle\le 0. \tag{4.50}$$

Using our assumption and (4.49), we have that $\lim_{k\to\infty}\Psi_{n_k}=\lim_{k\to\infty}\left(\frac{\theta_{n_k}}{\gamma\beta_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2+2\frac{\alpha}{\gamma}\langle Gp,p-x_{n_k+1}\rangle\right)\le 0$. Thus, From Lemma 2.3, we have that $\lim_{n\to\infty}\|x_n-p\|=0$.

5. Numerical Example

In this section, we will give some numerical examples which will show the applicability and the efficiency of our proposed iterative technique in comparison to Algorithm 1.7, and Algorithm 1.6.

Example 5.1. *Let $H = L_2([0,1])$ be equipped with the inner product*

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt \ \forall \ x, y \in L_2([0,1]) \ \textit{and} \ \|x\|^2 := \int_0^1 |x(t)|^2 dt \ \forall x, y, \in L_2([0,1]).$$

Let $F, G : L_2([0,1]) \to L_2([0,1])$ be defined by

$$Fx(t) = \max\{0, x(t)\}, \ t \in [0,1],$$
$$Gx(t) = \frac{x(t)}{2}, \ t \in [0,1].$$

*It is easy to see that F is 1-Lipschitz continuous and monotone, and G τ-strongly monotone. We used this example due to Remark **??** so that we can compare. Let $S_n : L_2([0,1]) \to L_2([0,1])$ be defined by*

$$S_n x(t) = \sin x(t).$$

Let C be defined by $C = \{x \in L_2 : \langle a, x\rangle = b\}$ where $a \neq 0$ and $b = 2$. Thus, we have

$$P_C(\bar{x}) = \max\left\{0, \frac{b - \langle a, \bar{x}\rangle}{\|a\|^2}\right\} a + \bar{x}.$$

We choose $\beta_n = \frac{2}{200n+5}, \theta_n = \overline{\theta}, \alpha = 0.2, \nu = 0.3, , \lambda_0 = \frac{1}{3}, \lambda_{n+1} = \frac{100}{(n+1)^{1.3}}, \epsilon_n = \frac{\alpha_n}{n^{0.01}}$, for all $n \in \mathbb{N}$. Also if we consider $\epsilon = \|x_n - x_{n_1}\| \leq 10^{-5}$ as the stopping criterion and choose the following as starting points:

$$\textit{Case I: } x_0(t) = 2t^2 + t + 2, \ x_1(t) = t;$$
$$\textit{Case II: } x_0(t) = 2t^2 + e^{2t} + 1, \ x_1(t) = 3t^3 + 3;$$
$$\textit{Case III: } x_0(t) = t + 2, \ x_1(t) = \cos(t);$$

Table 1: Computation result for Example **??**.

		Algorithm 1	Algorithm 1.7	Algorithm 1.6
Case I	No of Iter.	25	28	26
	CPU time (sec)	0.0231	0.0345	0.0245
Case II	No of Iter.	25	30	27
	CPU time (sec)	0.0251	0.0321	0.0291
Case III	No of Iter.	27	30	27
	CPU time (sec)	0.0244	0.0358	0.0271

Example 5.2. *Let $H_1 = H_2 = l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \cdots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $\|x\| = (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$ for all $x \in l_2(\mathbb{R})$. Suppose the operators $F; G : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ are defined by*

$$Gx - X - X_0,$$
$$Fx = (5 - \|x\|)x \ \forall \ x \in l_2(\mathbb{R}).$$

It is easy to see that F ar quasimonotone, Lipschitzain continuous and weakly sequentially continuous on $l_2(\mathbb{R})$, and G is τ-strongly monotone, see [25]. Let $C = Q = \{x \in l_2(\mathbb{R}) : \|x\| \leq 3\}$. Clearly, C is nonempty, closed and convex subsets of $l_2(\mathbb{R})$. Hence, we have

$$P_C(x) = \begin{cases} x & \textit{if } \|x\| \leq 3, \\ \frac{3x}{\|x\|}, & \textit{if otherwise}. \end{cases} \tag{5.1}$$

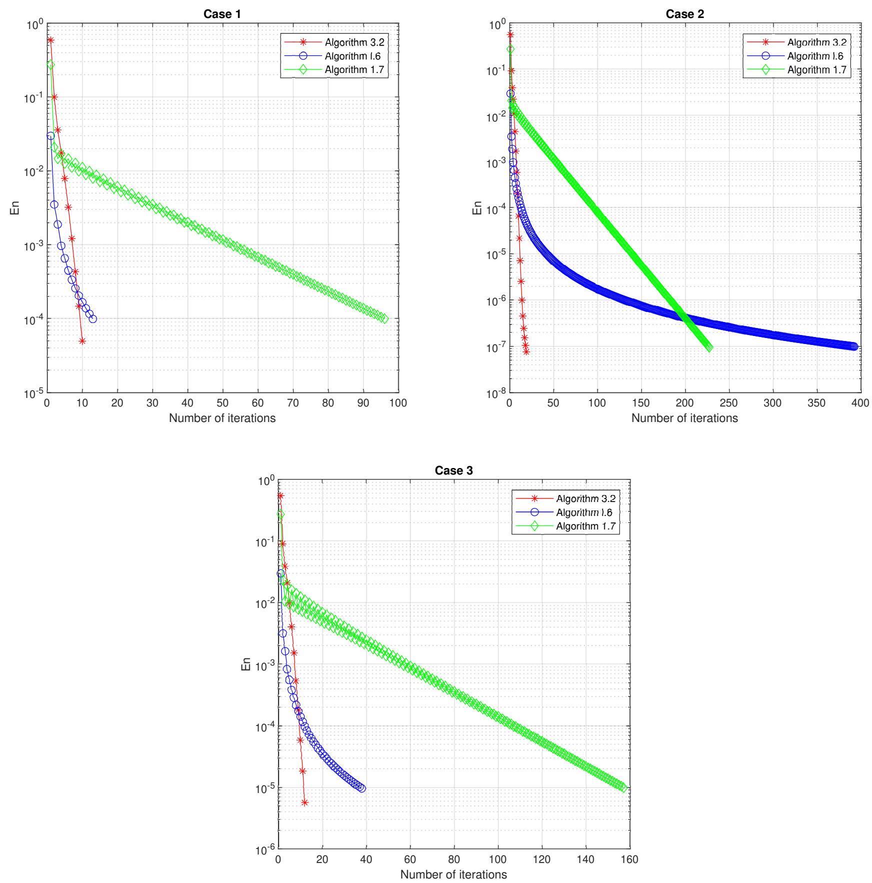

Figure 1: Example 5.1, Top Left: Case I; Top Right: Case II; Bottom Centered: case III.

In addition, we define $S_n, S : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ *are defined by* $S_n x = (0, x_1, x_2, x_3, \cdots)$ *and* $Sx = (0, \frac{x_1}{2}, \frac{x_2}{2}, \cdots)$. *We choose* $\beta_n = \frac{2}{200n+5}, \theta_n = \overline{\theta}, \alpha = 0.2, \nu = 0.3, , \lambda_0 = \frac{1}{3}, \lambda_{n+1} = \frac{100}{(n+1)^{1.3}}, \epsilon_n = \frac{\alpha_n}{n^{0.01}}$, *for all* $n \in \mathbb{N}$. *Also if we consider* $\epsilon = \|x_n - x_{n_1}\| \leq 10^{-5}$ *as the stopping criterion and choose the following as starting points:*

$$\textit{Case I: } x_0 = (2, 2, 2, \cdots),\ x_1 = (0.5, 0.5, 0.5, \cdots);$$

$$\textit{Case II: } x_0 = (1, 2, 3, 4, \cdots),\ x_1 = (1, 1, 1, \cdots);$$

$$\textit{Case III: } x_0 = (0.1, 0.2, 0.3, \cdots),\ x_1 = (2, 4, 6, \cdots).$$

Table 2: Computation result for Example 5.2.

		Algorithm 1	Algorithm 1 without $\{S_n\}$
Case I	No of Iter.	30	39
	CPU time (sec)	0.0390	0.0709
Case II	No of Iter.	20	44
	CPU time (sec)	0.0212	0.0592
Case III	No of Iter.	24	30
	CPU time (sec)	0.0326	0.0581

6. Conclusion

A modified Tseng with an inertial extrapolation step is introduced and studied for solving the BVIP (1.4) in infinite dimensional real Hilbert space when the cost operators are quasimonotone, and τ-strongly monotone and Lipschitz continuous. In addition, we established that the proposed iterative method converges strongly to the solution set of BVIP (1.4). Our method uses stepsizes that are generated at each iteration by some simple computations, which allows it to be easily implemented without the prior knowledge of the operator norm or the coefficient of an underlying operator. . In addition, we present some examples and numerical experiment to show the efficiency and implementation of our method in the framework of infinite and finite dimensional Hilbert spaces. We emphasize that one of the novelty of this work is in the use of a weaker operator (quasimonotone), modified inertial term introduced and the method of proof of the strong convergence of our iterative algorithm to the solution of problem (1.4).

References

1. P. K. Anh, T. V. Anh, and L. D. Muu,, On bilevel split pseudomonotone variational inequality problems with applications, *Acta. Math. Vietnam,* **42**, (2017), 413–429.

2. R. Y. Apostol, A. A. Grynenko and V. V. Semenov, Iterative algorithms for monotone bilevel variational inequalities *J. Comput. Appl Math.,* **107**, (2012), 3–14.

3. C. Byrne , A unified treatment for some iterative algorithms in signal processing and image reconstruction, *Inv. Probl.,* **20**, (2004), 103–120.

4. L. C. Ceng, N. Hadjisavvas, and N. C. Wong, Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems, *Glob. Optim.,* **46**, (2020), 635–646.

5. L. C. Ceng, A. Gibali, and S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert space, *J. Optim. Theory Appl.,* **148**, (2011), 318–335.

6. Y. Censor, A. Gibali, and S. Reich, Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space, *Optimization,* **61**, (2011), 1119–1132.

7. Y. Censor, A. Gibali, and S. Reich, The split variational inequality problem, *The Technion-Israel Institute of Technology, Haifa,* **59**, (2012), 301–323.

8. Y. Censor, A. Gibali, and S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert space, *J. Optim. Theory Appl.,* **48**, (2011), 318–335.

9. Y. Censor, A. Gibali, and S. Reich, Algorithms for the split variational inequality problem, *Numer. Algorithms,* **59**, (2012), 301–323.

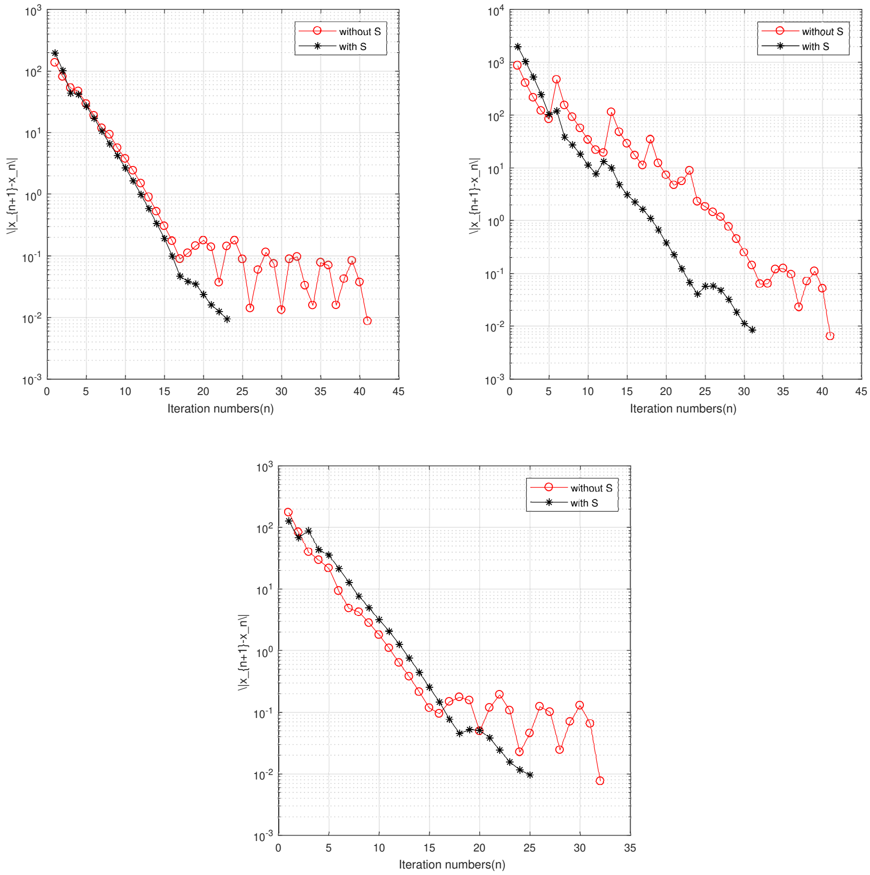

Figure 2: Example 5.2, Top Left: Case I; Top Right: Case II; Bottom Centered: case III.

10. Y. Censor, T. Bortfeld, B. Martin, and A. Trofimov, A unified approach for inversion problems in intensity modulated radiation therapy, *Phys. Med. Biol.,* **51**, (2006), 2353–2365.

11. Y. Censor, and T. Elfving, A multiprojection algorithm using Bregman projections in product space, *Numer. Algorithms,* (1994), 221–239.

12. P. Daniele, F. Giannessi, and A. Maugeri, Equilibrium problems and variational models, *Dordrecht: Kluwer Academic,* (2003).

13. G. Ficher, Sul pproblem elastostatico di signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei Rend. *Cl. Sci. Fis. Mat. Natur*, **34**, (1963), 138–142.

14. G. Ficher, Problemi elastostatici con vincoli unilaterali: il problema di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincci, *Cl. Sci. Fis. Mat. Nat., Sez.,* **7**, (1964), 91–140.

15. F. Akutsha, A. A. Mebawondu, H. A. Abass and O. K. Narain, A self adaptive method for solving a class of bilevel variational inequalities with split variational inequlaity and composed fixed point problem constraints in Hilbert spaces, *Numerical Algebra, Control and Optimization,* (2023), 1–22, DOI: 10.3934/naco.2021046.

16. F. Akutsha, A A. A. Mebawondu, G. C. Ugwunnadi and O. K. Narain, Inertial extrapolation method with regularization for solving monotone bilevel variation inequalities and fixed point problems in real Hilbert space, *Journal of Nonlinear Functional Analysis,* (2022), 1–20.

17. F. Akutsha, A. A. Mebawondu, G. C. Ugwunnadi, P. Pillay and O. K. Narain, IInertial extrapolation method with regularization for solving a new class of bilevel problem in real Hilbert spaces, *SeMA Journal*, 2022, 1–22.

18. F. Giannessi, A. Maugeri, and P. M. Pardalos, H-monotone operator and resolvent operator technique for variational inclusion, *Dordrecht: Kluwer Academic,* (2004).

19. J. Glackin, J. G. Ecker, and M. Kupferschmid, Solving bilevel linear programs using multiple objective linear programming, *J. Optim Theory Appl.,* **140**, (2009), 197–212.

20. A. A. Goldstein, Convex programming in Hilbert space *Bull. Am. Math. Soc.,* **70**, (1964), 709 –710.

21. N. Hadjisavvas, and S. Schaible, Quasimonotone variational inequalities in Banach spaces, *J. Optim. Theory Appl.,* **90**, (1996), 95–111.

22. B. S. He and L. Z. Liao, Improvements of some projection methods for monotone nonlinear variational inequalities *Optim Theory Appl.,* **112**, (2002), 111–128.

23. H. He, C. Ling, and H. K. Xu, A relaxed projection method for split variational inequalities, *J. Optim. Theory Appl.,* **166**, (2015), 213–233.

24. G. M. Korpelevich, An extragradient method for finding saddle points and for other problems, *Ekon. Mat. Metody.,* **12**, (1976), 747–756.

25. H. Luiu and J. Yang, Weak convergence of iterative methods for solving quasimonotone variational inequalities, *Computation Optimization and Applications,* **77**, (2020), 491-508.

26. Z. Q. Luo, J. S. Pang, and D. Ralph, Mathematical programs with equilibrium constraints, *Cambridge, Cambridge University Press,* 1996.

27. N. Nadezhkina, and W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings *J. Optim. Theory Appl.,* **128**, (2006), 191–201.

28. P. E. Mainge ,A hybrid extragradient-viscosity method for monotone operators and fixed point problems, *SIAM J Control Optim.*, **12**, (2008), 1499–1515.

29. N. H. Minh, L. H. M. Van and T. V. Anh, An algorithm for a class of bilevel variational inequality with split variational inequality and fixed point problem constraints, *Acta Mathematica Vietnamica,* https://doi.org/10.1007/s40306-020-00389-9.

30. N. Nadezhkina, and W. Takahashi, Inertial methods for finding minimum-norm solutions of the split variational inequality problem beyond monotonicity *Numerical Alg.,* **88**, (2021), 1419–1456.

31. V. H. Pham, D. H. Nguyen, and T. V. Anh, A strongly convergent modified Halpern subgradient extragradient method for solving the split variational inequality problem, *Vietnam J. Math.* **48**, (2020), 187–204.

32. B. T. Polyak, Some methods of speeding up the convergence of iteration methods, *Politehn Univ Buchar Sci Bull Ser A Appl Math Phys.,* **4**, no. 5, (1964), 1–17

33. S. Reich and T. M. Tuyen, A new algorithm for solving the split common null point problem in Hilbert spaces, *Numer. Algorithms,* **83**, (2020), 789–805.

34. S. Saejung and P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces, *Nonlinear Anal.*, **75**, (2012), 742–750.

35. Y. Shehu, P. T. Vuong, A. Zemkoho,, An inertial extrapolation method for convex simple bilevel optimization, *Optim Methods Softw.,* https://doi.org/10.1080/10556788.2019.1619729, 2019.

36. G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, *C. R. Math. Acad. Sci.*, **258**, (1964), 4413–4416.

37. M. Tian, and B. N. Jiang, Weak convergence theorem for a class of split variational inequality problems and applications in Hilbert space, *J. Inequal. Appl.,* **123**, (2017), 1–20.

38. B. Tan, L. Liu, and X. Qin,,elf adaptive inertial extragradient algorithms for solving bilevel pseudomonotone variational inequality problems, *Applied Math.,* https://doi.org/10.1007/s13160-020-00450-y, 2020.

39. D. V. Thong, X-H. Li, Q. L. Dong, Y. J. Cho, T. M. Rassias, A projection and contraction method with adaptive step sizes for solving bilevel pseudomonotone variational inequality problems, *Optimization*, (2020), DOI:10.1080/02331934.2020.1849206

40. C. R. Trujillo and S. Zlobec, Bilevel convex programming models, *Optimization* **58**, (2009), 1009–1028.

41. Y. Yao, Muglia, and G. L.Marino, A modified Korpelevichs method convergent to the minimum-norm solution of a variational inequality, *Optimization,* **63**, (2014), 559–569.

42. S. E. Yimer, P. Kuman, A. G. Gebrie, R. Wangkeeree, R., Inertial method for bilevel variational inequality problems with fixed point and minimizer point constraints, *MDPI, Mathematics,* **841**, no. 7, (2019), doi:10.3390/math7090841.

43. L. Zheng, A double projection algorithm for quasimonotone variational inequalities in Banach spaces, *Inequal. Appl.,* **123**, (2018), 1–20.

D. O. Peter,
Department of Mathematics,
University of KwaZulu-Natal,
South Africa
.
E-mail address: 222126781@stu.ukzn.ac.za

and

A. A. Mebawondu,
Department of Computer Sciences and Mathematics,
Mountain Top University,
Nigeria. University of KwaZulu-Natal,
South Africa.
E-mail address: aamebawondu@mtu.edu.ng, dele@aims.ac.za

and

G. C. Ugwunnadi,
Department of Mathematics and Applied Mathematics,
Sefako Makgatho Health Sciences University,
South Africa. University of Eswatini, Private Bag 4, Kwaluseni,
Eswatini.
E-mail address: [3]ugwunnadi4u@yahoo.com, gcugwunnadi@uniswa.sz

and

P. Pillay,
Department of Mathematics,
University of KwaZulu-Natal,
South Africa.
E-mail address: [4]pillaypi@ukzn.ac.za

and

O. K. Narain,
Department of Mathematics,
University of KwaZulu-Natal,
South Africa.
E-mail address: [5]naraino@ukzn.ac.za