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Conformal Bi-Slant $\xi^{\perp}$-Riemannian Submersion: A Note in Contact Geometry

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abstract: This study explores conformal bi-slant $\xi^{\perp}$-Riemannian submersion where the total manifold is a contact metric manifold, more specifically a Sasakian manifold. To illustrate this study, few non-trivial examples are discussed. Meanwhile, we addressed the conditions for integrability of anti-invariant and slant distributions and determine the conditions for the map that must be met in order to be totally geodesic. Furthermore, some decomposition theorems for the fibres as well as for the total space are discussed.

Key Words: Conformal bi-slant submersions, conformal submersions, integrability conditions, Sasakian manifold, contact metric manifold.

Contents

1. Introduction

Isometric immersion and Riemannian submersion, which is a double-fold concept of isometric immersion, are the two fundamental maps in Riemannian geometry. Riemannian submersion was induced by B. O'Neill [23] in 1966. Many authors have studied the Riemannian submersion from various perspectives like Riemannian submersion [23], slant submersion ([10], [30]), contact complex submersion [18], semi-invariant submersion [29], h-semi-invariant submersion [25] etc. Riemannian submersions have applications in physics and mathematics, such as in supergravity and superstring theories [20, 22], Kaluza-Klein theory ([19], [22]), and the Yang Mills theory ([8], [34]). Also, Frejlich and Dunn et al. ([11], [14]) obtained submersions of Lie algebra.
Fuglede [15] and Ishihara [21], studied the horizontally conformal maps as special case of Riemannian submersions with $\lambda = 1$. Meanwhile, Akyol and Sahin defined conformal anti-invariant submersions [5], conformal semi-invariant submersions [6], conformal slant submersion [4], and conformal semi-slant submersions [2]. Also, a variety of researchers have examined the geometry of conformal submersions ([17], [23]). Prasad et al. [26] investigated Quasi bi-slant submersion of Kenmotsu manifold whereas Sezin [32] studied bi-slant submersions from contact manifold with taking ξ as horizontal vector field.

As a generalization of conformal slant,conformal [4], conformal semi-slant [3] and conformal hemi-slant submersions [27], we consider conformal bi-slant $\xi^{\perp}$-Riemannian submersion with total space, a Sasakian manifold, where the vector field ξ is considered in horizontal space of the submersion. The paper has the following structure. Section 2 presents the fundamental information and definitions of contact metric manifolds, particularly Sasakian manifolds with properties relevant to this paper. Section 3 presents all of main results of this study. We define the conformal bi-slant $\xi^{\perp}$-Riemannian submersion. Along with some basic findings, we explore the condition of integrability for distributions and totally geodesicness. We prove Decomposition theorems for the total space and the fibres as well.

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Throughout paper, we will use some abbreviations as follows:

Riemannian Manifold	RM
Riemannian Manifolds	RMs
Sasakian Manifold	SM
Sasakian Manifolds	SMs
conformal bi-slant-$\xi^{\perp}$ submersion	CBSS

2. Preliminaries

A $(2n+1)$-dimensional manifold M which having an almost contact structures (ϕ, ξ, η), where a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \ \phi\xi = 0,\ \ \eta \circ \phi = 0,\ \ \eta(\xi) = 1 \tag{2.1}$$

where I is the identity tensor. If $N \oplus d\eta \otimes \xi = 0$, with Nijenhuis tensor N related to ϕ then almost contact structure turns into normal. There is also a Riemannian metric g which holds

$$g(\phi\mathbb{U}, \phi\mathbb{V}) = g(\mathbb{U}, \mathbb{V}) - \eta(\mathbb{U})\eta(\mathbb{V}), \eta(\mathbb{U}) = g(U, \xi). \tag{2.2}$$

Then (ϕ, ξ, η, g)-structure is called an almost contact metric structure. A normal contact metric structure is called a Sasakian structure, which satisfies

$$(\nabla_{\mathbb{U}}\phi)\mathbb{V} = g(\mathbb{U}, \mathbb{V})\xi - \eta(\mathbb{V})\mathbb{U} \tag{2.3}$$

where ∇ is the Levi-Civita connection of g. From above formula, we have for Sasakian manifold (SM)

$$\nabla_{\mathbb{U}}\xi = -\phi\mathbb{U}. \tag{2.4}$$

Now, we provide a definition for conformal submersion and discuss some useful results that help us to achieve our main results.

Definition 2.1. *A smooth map φ from (M_1, g_1) onto (M_2, g_2) where M_1 and M_2 are Riemannian manifolds (RMs) with m_1 and m_2 be the dimensions of manifolds respectively, is called horizontally weakly conformal or semi-conformal at $p \in M_1$ if, either*

i. $d\varphi_p = 0$ or

ii. $d\varphi_p$ is surjective and there always have a number $\Omega(p) \neq 0$ satisfying

$$g_2(d\varphi_p\mathbb{U}, d\varphi_p\mathbb{V}) = \Omega(p)g_1(\mathbb{U}, \mathbb{V})$$

for $\mathbb{U}, \mathbb{V} \in \Gamma(ker(d\varphi))^{\perp}$.

In this case, we label a point p satisfying type (i) as a critical point and rank of $d\varphi_p$ is 0 at this point and type (ii) as a regular point at which the rank of $d\varphi_p$ is m_2. Also, the number $\Omega(p)$ is called the square dilation. Its square root $\sqrt{\lambda(p)} = \sqrt{\Omega(p)}$ is called the dilation. If the map φ is horizontally weakly conformal at each point on M_1, it is referred to as horizontally weakly or semi-conformal on M_1. If φ has no critical point, it is said to be a (horizontally) conformal submersion.

Let $\varphi : M_1 \to M_2$ be a submersion. A vector field $\mathbb{X}$ on M_1 is called a basic vector field if $\mathbb{X} \in \Gamma((\ker\varphi_*)^{\perp})$ and φ-related with a vector field $\mathbb{X}$ on M_2 i.e $\varphi_*(\mathbb{X}(q)) = \mathbb{X}\varphi(q)$ for $q \in M_1$.

The formulas provide the two $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ are

$$\mathcal{T}(E_1, E_2) = \mathcal{T}_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{V}E_1}\mathbb{V}E_2 + \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}E_2 \tag{2.5}$$

$$\mathcal{A}(E_1, E_2) = \mathcal{A}_{E_1}E_2 = \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}E_2 + \mathcal{H}\nabla_{\mathcal{H}E_1}\mathbb{V}E_2 \tag{2.6}$$

for $E_1, E_2 \in \Gamma(TM_1)$.

Note that a Riemannian submersion $\varphi : M_1 \to M_2$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. From equations 2.5 and 2.6, we can deduce

$$\nabla_{U_1}V_1 = \mathcal{T}_{U_1}V_1 + \mathcal{V}\nabla_{U_1}V_1 \tag{2.7}$$

$$\nabla_{U_1}X_1 = \mathcal{T}_{U_1}X_1 + \mathcal{H}\nabla_{U_1}X_1 \tag{2.8}$$

$$\nabla_{X_1}U_1 = \mathcal{A}_{X_1}U_1 + \mathcal{V}\nabla_{X_1}U_1 \tag{2.9}$$

$$\nabla_{X_1}Y_1 = \mathcal{H}\nabla_{X_1}Y_1 + \mathcal{A}_{X_1}Y_1 \tag{2.10}$$

for any $U_1, V_1 \in \Gamma(kerF_*)$ and $X_1, Y_1 \in \Gamma(kerF_*)^{\perp}$.
It is clear that $\mathcal{T}$ and $\mathcal{A}$ are skew-symmetric, i.e.,

$$g(\mathcal{A}_{\mathbb{X}}E_1, E_2) = -g(E_1, \mathcal{A}_{\mathbb{X}}E_2),\ \ g(\mathcal{T}_{\mathbb{V}}E_1, E_2) = -g(E_1, \mathcal{T}_{\mathbb{V}}E_2) \tag{2.11}$$

for all $E_1, E_2 \in T_pM_1$. The following results holds for the particular case where φ is horizontally conformal.

Proposition 2.2. *Let $\varphi : M_1 \to M_2$ be horizontally conformal submersion with dilation λ and $\mathbb{X}, \mathbb{Y} \in \Gamma((ker\varphi_*)^{\perp})$, then*

$$A_{\mathbb{X}}\mathbb{Y} = \frac{1}{2}(\mathbb{V}[\mathbb{X}, \mathbb{Y}] - \lambda^2 g_1(\mathbb{X}, \mathbb{Y})G_{\mathcal{V}}(\frac{1}{\lambda^2})) \tag{2.12}$$

where gradient of function denoted by G.

The second fundamental form of smooth map φ given by the formula

$$(\nabla\varphi_*)(\mathbb{X}, \mathbb{Y}) = \nabla^{\varphi}_{\mathbb{X}}\varphi_*\mathbb{Y} - \varphi_*\nabla_{\mathbb{X}}\mathbb{Y} \tag{2.13}$$

and the map be totally geodesic if $(\nabla\varphi_*)(\mathbb{X}, \mathbb{Y}) = 0$ for all $\mathbb{X}, \mathbb{Y} \in \Gamma(T_pM_1)$ where ∇ and $\nabla\varphi$ are Levi-Civita and pullback connections.

Lemma 2.3. *Let M_1 and M_2 be RMs and φ be horizontal conformal submersion. Then, for any vector fields $\mathbb{X}_1, \mathbb{Y}_1 \in \Gamma((ker\varphi_*)^{\perp})$ and $\mathbb{U}_1, \mathbb{V}_1 \in \Gamma(ker\varphi_*)$, we have*

(i) $(\nabla\varphi_*)(X_1, Y_1) = X_1(ln\lambda)\varphi_*(Y_1) + Y_1(ln\lambda)\varphi_*(X_1) - g(X_1, Y_1)\varphi_*(grad\ ln\lambda)$,

(ii) $(\nabla\varphi_*)(U_1, V_1) = -\varphi_*(\mathcal{T}_{U_1}V_1)$,

(iii) $(\nabla\varphi_*)(X_1, U_1) = -\varphi_*(\nabla_{X_1}U_1) = -\varphi_*(\mathcal{A}_{X_1}U_1)$.

3. Conformal bi-slant $\xi^{\perp}$-Riemannian submersions

Definition 3.1. *Consider φ is a conformal submersion from a SM $(M_1, \phi, \xi, \eta, g_1)$ onto RM (M_2, g_2). Then φ is defined a conformal bi-slant $\xi^{\perp}$-Riemannian submersion (CBSS) if $D_1^{\bar{\theta}}$ and $D_2^{\bar{\theta}}$ are slant distributions with corresponding slant angles $\bar{\theta}_1$ and $\bar{\theta}_2$, such that $ker\varphi_* = D_1^{\bar{\theta}} \oplus D_2^{\bar{\theta}}$. If $\bar{\theta}_1, \bar{\theta}_2$ are neither equal to 0 nor $\frac{\pi}{2}$, then φ is proper.*

If m and n are dimension of $D_1^{\bar{\theta}}$ and $D_2^{\bar{\theta}}$ respectively, then we can say

i If $m = 0$ and $\bar{\theta}_2 = \frac{\pi}{2}$ then φ is a conformal anti-invariant submersion,

ii If $m, n \neq 0$, $\bar{\theta}_1 = 0$ and $\bar{\theta}_2 = \frac{\pi}{2}$ then φ is a conformal semi-invariant submersion.

iii If $m, n \neq 0$, $\bar{\theta}_1 = 0$ and $0 < \bar{\theta}_2 < \frac{\pi}{2}$ then φ is a conformal semi-slant submersion.

iv If $m, n \neq 0$, $\bar{\theta}_1 = \frac{\pi}{2}$ and $0 < \bar{\theta}_2 < \frac{\pi}{2}$ then φ is a conformal hemi-slant submersion.

Note that $\mathbb{R}^{2n+1}$ denote a SM with the structure (ϕ, ξ, η, g) defined as

$$\phi\left(\sum_{i=1}^{n}\left(X_i\frac{\partial}{\partial x^i}+Y_i\frac{\partial}{\partial y^i}\right)+Z\frac{\partial}{\partial z}\right)=\sum_{i=1}^{n}\left(Y_i\frac{\partial}{\partial x^i}-X_i\frac{\partial}{\partial y^i}\right),$$
$$\eta=\frac{1}{2}\left(dz-\sum_{i=1}^{n}y^i dx^i\right), \xi=2\frac{\partial}{\partial z}$$
$$g=\eta\otimes\eta+\frac{1}{4}\sum_{i=1}^{n}\left(dx^i\otimes dx^i+dy^i\otimes dy^i\right),$$

where $\left(x^1,\ldots,x^n,y^1,\ldots,y^n,z\right)$ are the cartesian coordinates.

Now, taking into account the definition above, we can provide the following examples:.

Example 3.2. *Let* $F:(\mathbb{R}^9, g_{\mathbb{R}^9})\longrightarrow(\mathbb{R}^5, g_{\mathbb{R}^5})$ *be a conformal submersion defined by*

$$F\left(x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4,z\right)$$
$$=\pi^6\left(\cos\alpha x_1-\sin\alpha x_3,\frac{x_2+x_4}{\sqrt{2}},\sin\beta y_3+\cos\beta y_4,y_1,z\right)$$

then

$$\ker F_* =< V_1=\sin\alpha\partial x_1+\cos\alpha\partial x_3, V_2=\frac{1}{\sqrt{2}}\left(\partial x_2-\partial x_4\right)$$
$$V_3=\cos\beta\partial y_3-\sin\beta\partial y_4, V_4=\partial y_2> \ \textit{and}$$
$$\left(\ker F_*\right)^{\perp} =< H_1=\cos\alpha\partial x_1-\sin\alpha\partial x_3, H_2=\frac{1}{\sqrt{2}}\left(\partial x_2+\partial x_4\right)$$
$$H_3=\sin\beta\partial y_3+\cos\beta\partial y_4, H_4=\partial y_1, H_5=\xi=\partial z>$$

Thus, the submersion F *is conformal bi-slant* $\xi^{\perp}$*-Riemannian submersion with* $\mathbb{D}_1=\langle V_1,V_3\rangle$ *with slant angle* $\bar{\theta}_1$ *such that* $\cos\bar{\theta}_1=\sin(\beta-\alpha)$ *and* $\mathbb{D}_2=\langle V_2,V_4\rangle$ *with the slant angle* $\bar{\theta}_2=\dfrac{\pi}{4}$.

Example 3.3. *Let* $F:(\mathbb{R}^9, g_{\mathbb{R}^9})\longrightarrow(\mathbb{R}^5, g_{\mathbb{R}^5})$ *such that*

$$(x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4,z)=e^5\left(\frac{x_1+\sqrt{3}x_2}{2},\sin\alpha x_3+\cos\alpha x_4,y_1,y_2,z\right).$$

Then it follows that

$$D_1 =< V_1=\frac{1}{2}\left(\sqrt{3}\partial x_1-\partial x_2\right), V_3=\partial y_3>,$$
$$D_2=\operatorname{span}\langle V_2=\cos\alpha\partial x_3-\sin\alpha\partial x_4, V_4=\partial y_4\rangle \ \textit{and}$$
$$\left(\ker F_*\right)^{\perp} = < H_1=\frac{1}{2}(\partial x_1+\sqrt{3}\partial x_2), H_2=\sin\alpha\partial x_3+\cos\alpha\partial x_4,$$
$$H_3=\partial y_1, H_4=\partial y_2, H_5=\xi=\partial z>.$$

Hence, F *is a conformal bi-slant* $\xi^{\perp}$*-Riamannian submersion with the slant angles* $\bar{\theta}_1=\dfrac{\pi}{3}$ *and* $\bar{\theta}_2=\alpha$*, respectively and* $\lambda=e^{\sqrt{5}}$.

Let us consider φ is CBSS from SM (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2), with taking $\mathbb{U}\in ker\varphi_*$, we can write

$$\mathbb{U}=P_1\mathbb{U}+P_2\mathbb{U} \tag{3.1}$$

where $P_1\mathbb{U} \in \Gamma(D_1^{\bar\theta})$ and $P_2\mathbb{U} \in \Gamma(D_2^{\bar\theta})$.

Also, for $\mathbb{U} \in \Gamma(ker\varphi_*)$

$$\phi\mathbb{U} = \delta\mathbb{U} + \zeta\mathbb{U} \tag{3.2}$$

where $\delta\mathbb{U} \in \Gamma(ker\varphi_*)$ and $\zeta\mathbb{U} \in \Gamma((ker\varphi_*)^{\perp})$. For $\mathbb{X} \in \Gamma((ker\varphi_*)^{\perp})$, we have

$$\phi\mathbb{X} = t\mathbb{X} + f\mathbb{X} \tag{3.3}$$

where $t\mathbb{X} \in \Gamma(ker\varphi_*)$ and $f\mathbb{X} \in \Gamma((ker\varphi_*)^{\perp})$.

The horizontal distribution $(ker\varphi_*)^{\perp}$ is decomposed as

$$(ker\varphi_*)^{\perp} = \zeta D_1^{\bar\theta} \oplus \zeta D_2^{\bar\theta} \oplus \mu \tag{3.4}$$

where $\mu = \phi\mu \oplus \xi$ such that μ is distribution which is complementary to $\zeta D_1^{\bar\theta} \oplus \zeta D_2^{\bar\theta}$ in $(ker\varphi_*)^{\perp}$.

Theorem 3.4. *Let $\varphi : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ be CBSS from SM onto a RM with slant angles $\bar\theta_1$ and $\bar\theta_2$. Then we have*

$$\delta^2\mathbb{U}_1 = -\cos^2\bar\theta_1\mathbb{U}_1 \tag{3.5}$$

$$\delta^2\mathbb{U}_2 = -\cos^2\bar\theta_2\mathbb{U}_2 \tag{3.6}$$

for $\mathbb{U}_1 \in \Gamma(D_1^{\bar\theta})$ and $\mathbb{U}_2 \in \Gamma(D_2^{\bar\theta})$.

Theorem 3.5. *Let φ be the CBSS from the SM $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2) with slant angles $\bar\theta_1$ and $\bar\theta_2$. Then*

(i) the distribution $D_1^{\bar\theta}$ is integrable if and only if

$$\begin{aligned}\lambda^{-2}g_2((\nabla\varphi_*)(\mathbb{U}, \zeta\mathbb{V}), \varphi_*\zeta\mathbb{W}) =& g_1(T_{\mathbb{V}}\zeta\delta\mathbb{U} - T_{\mathbb{U}}\zeta\delta\mathbb{V}, \mathbb{W})\\ &+ g_1(T_{\mathbb{U}}\zeta\mathbb{V} - T_{\mathbb{V}}\zeta\mathbb{U}, \delta\mathbb{W})\\ &+ \lambda^{-2}g_2((\nabla\varphi_*)(\mathbb{V}, \zeta\mathbb{U}), \varphi_*\zeta\mathbb{W})\end{aligned}$$

(ii) the distribution $D_2^{\bar\theta}$ is integrable if and only if

$$\begin{aligned}\lambda^{-2}g_2((\nabla\varphi_*)(\mathbb{U}, \zeta\mathbb{V}), \varphi_*\zeta\mathbb{V}) =& g_1(T_{\mathbb{Z}}\zeta\delta\mathbb{W} - T_{\mathbb{W}}\zeta\delta\mathbb{Z} < \mathbb{U})\\ &+ g_1(T_{\mathbb{W}}\zeta\mathbb{Z} - T_{\mathbb{Z}}\zeta\mathbb{W}, \delta\mathbb{U})\\ &+ \lambda^{-2}g_2((\nabla\varphi_*)(\mathbb{Z}, \zeta\mathbb{W}), \varphi_*\zeta\mathbb{U})\end{aligned}$$

for $\mathbb{U}, \mathbb{V} \in \Gamma(D_1^{\bar\theta})$ and $\mathbb{Z}, \mathbb{W} \in \Gamma(D_2^{\bar\theta})$.

Proof. (i). For any vector fields $\mathbb{U}, \mathbb{V} \in \Gamma(D_1^{\bar\theta})$ and $\mathbb{W} \in \Gamma(D_2^{\bar\theta})$ and on using equations (2.2), (2.3) and from (3.2), we have

$$\begin{aligned}g_1([\mathbb{U}, \mathbb{V}], \mathbb{W}) =& g_1(\nabla_{\mathbb{V}}\delta^2\mathbb{U}, \mathbb{W} - g_1(\nabla_{\mathbb{V}}\delta^2\mathbb{V}, \mathbb{W})\\ &- g_1(\nabla_{\mathbb{U}}\zeta\delta\mathbb{V}, \mathbb{W}) + g_1(\nabla_{\mathbb{V}}\zeta\delta\mathbb{U}, \mathbb{W})\\ &+ g_1(\nabla_{\mathbb{U}}\zeta\mathbb{V}, \phi\mathbb{W}) - g_1(\nabla_{\mathbb{V}}\zeta\mathbb{U}, \phi\mathbb{W}).\end{aligned}$$

Considering Theorem 3.4, we have

$$\begin{aligned}\sin^2\bar\theta_1 g_1([\mathbb{U}, \mathbb{V}], \mathbb{W}) =& -g_1(\nabla_{\mathbb{U}}\zeta\delta\mathbb{V}, \mathbb{W}) + g_1(\nabla_{\mathbb{V}}\zeta\delta\mathbb{U}, \mathbb{W})\\ &+ g_1(\nabla_{\mathbb{U}}\zeta\mathbb{V}, \phi\mathbb{W}) - g_1(\nabla_{\mathbb{V}}\zeta\mathbb{U}, \phi\mathbb{W}).\end{aligned}$$

On using equation (2.8), we obtained

$$\begin{aligned}\sin^2\bar\theta_1 g_1([\mathbb{U}, \mathbb{V}], \mathbb{W}) =& g_1(\mathcal{T}_{\mathbb{V}}\zeta\delta\mathbb{U} - \mathcal{T}_{\mathbb{U}}\zeta\delta\mathbb{V}, \mathbb{W})\\ &- g_1(\mathcal{T}_{\mathbb{U}}\zeta\mathbb{V} - \mathcal{T}_{\mathbb{V}}\zeta\mathbb{U}, \delta\mathbb{W})\\ &+ g_1(\mathcal{H}\nabla_{\mathbb{U}}\zeta\mathbb{V} - \mathcal{H}\nabla_{\mathbb{V}}\zeta\mathbb{U}, \zeta\mathbb{W}).\end{aligned}$$

Now considering Lemma 2.3 and equation (2.13), we yields

$$\begin{aligned}\sin^2\bar{\theta}_1 g_1([\mathbb{U},\mathbb{V}],\mathbb{W}) = {} & g_1(\mathcal{T}_{\mathbb{V}}\zeta\delta\mathbb{U} - \mathcal{T}_{\mathbb{U}}\zeta\delta\mathbb{V},\mathbb{W}) \\ & - g_1(\mathcal{T}_{\mathbb{U}}\zeta\mathbb{V} - \mathcal{T}_{\mathbb{V}}\zeta\mathbb{U},\delta\mathbb{W}) \\ & - \lambda^{-2} g_2((\nabla\varphi_*)(\mathbb{U},\zeta\mathbb{V}),\varphi_*\zeta\mathbb{W}) \\ & + \lambda^{-2} g_2((\nabla\varphi_*)(\mathbb{V},\zeta\mathbb{U}),\varphi_*\zeta\mathbb{W})\end{aligned}$$

For part (ii) the calculation is same as (i). □

Theorem 3.6. *Let φ be the CBSS from the SM (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2) with slant angles $\bar{\theta}_1$ and $\bar{\theta}_2$. Then the distribution $D_1^{\bar{\theta}}$ defines totally geodesic foliation if and only if*

$$\lambda^{-2} g_2((\nabla\varphi_*)(\zeta\mathbb{V},\mathbb{U}),\varphi_*\zeta\mathbb{Z}) = g_1(\mathcal{T}_{\mathbb{U}}\zeta\mathbb{V},\delta\mathbb{Z}) - g_1(\mathcal{T}_{\mathbb{U}}\zeta\delta\mathbb{V},\mathbb{Z}) \tag{3.7}$$

and

$$\begin{aligned}\lambda^{-2} g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\mathbb{U},\varphi_*\zeta\mathbb{V}) = {} & \sin^2\bar{\theta}_1 g_1([\mathbb{U},\mathbb{X}],\mathbb{V}) + g_1(\mathcal{A}_{\mathbb{X}}\zeta\delta\mathbb{U},\mathbb{V}) \\ & + g_1(G(\ln\lambda),\mathbb{X}) g_1(\zeta\mathbb{U},\zeta\mathbb{V}) \\ & + g_1(G(\ln\lambda),\zeta\mathbb{U}) g_1(\mathbb{X},\zeta\mathbb{V}) \\ & - g_1(G(\ln\lambda),\zeta\mathbb{V}) g_1(\mathbb{X},\zeta\mathbb{U}) \\ & - g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{U},\delta\mathbb{V})\end{aligned} \tag{3.8}$$

for $\mathbb{U},\mathbb{V}\in\Gamma(D_1^{\bar{\theta}})$, $\mathbb{Z}\in\Gamma(D_2^{\bar{\theta}})$ and $\mathbb{X}\in((ker\varphi_*)^{\perp})$.

Proof. for $\mathbb{U},\mathbb{V}\in\Gamma(D_1^{\bar{\theta}})$ and $\mathbb{Z}\in\Gamma(D_2^{\bar{\theta}})$ with using equation (2.2), (2.3) and (3.2), we have

$$g_1([\mathbb{U},\mathbb{V}],\mathbb{Z})) = g_1(\nabla_{\mathbb{U}}\zeta\mathbb{V},\phi\mathbb{Z}) - g_1(\nabla_{\mathbb{U}}\zeta\delta\mathbb{V},\mathbb{Z}) - g_1(\nabla_{\mathbb{U}}\delta^2\mathbb{V},\mathbb{Z}).$$

From Theorem 3.4, we can write

$$\sin^2\bar{\theta}_1 g_1(\nabla_{\mathbb{U}}\mathbb{V},\mathbb{Z}) = -g_1(\nabla_{\mathbb{U}}\zeta\delta\mathbb{V},\mathbb{Z}) + g_1(\nabla_{\mathbb{U}}\zeta\mathbb{V},\phi\mathbb{Z})$$

On using (2.8), we have

$$\begin{aligned}\sin^2\bar{\theta}_1 g_1(\nabla_{\mathbb{U}}\mathbb{V},\mathbb{Z}) = {} & g_1(\mathcal{T}_{\mathbb{U}}\zeta\mathbb{V},\delta\mathbb{Z}) - g_1(\mathcal{T}_{\mathbb{U}}\zeta\delta\mathbb{V},\mathbb{Z})] \\ & + g_1(\mathcal{H}\nabla_{\mathbb{U}}\zeta\mathbb{V},\zeta\mathbb{Z}).\end{aligned}$$

Considering equation (2.13) and Lemma 2.3, we obtain

$$\begin{aligned}\sin^2\bar{\theta}_1 g_1(\nabla_{\mathbb{U}}\mathbb{V},\mathbb{Z}) = {} & g_1(\mathcal{T}_{\mathbb{U}}\zeta\mathbb{V},\delta\mathbb{Z}) - g_1(\mathcal{T}_{\mathbb{U}}\zeta\delta\mathbb{V},\mathbb{Z})] \\ & - \lambda^{-2} g_2((\nabla\varphi_*)(\zeta\mathbb{V},\mathbb{U}),\varphi_*\zeta\mathbb{Z})\end{aligned}$$

which is the equation first in Theorem 3.6.

On the other hand, $\mathbb{U},\mathbb{V}\in\Gamma(D_1)$ and $\mathbb{X}\in\Gamma(ker\varphi_*)^{\perp}$ with using (2.2), (2.3) and (3.2), we can write

$$g_1(\nabla_{\mathbb{U}}\mathbb{V},\mathbb{X}) = -g_1([\mathbb{U},\mathbb{X}],\mathbb{V}) + g_1(\phi\nabla_{\mathbb{X}}\delta\mathbb{U},\mathbb{V}) - g_1(\nabla_{\mathbb{X}}\zeta\mathbb{U},\phi\mathbb{V}).$$

Considering Theorem 3.4, we obtained

$$\sin^2\bar{\theta}_1 g_1(\nabla_{\mathbb{U}}\mathbb{V},\mathbb{X}) = -g_1([\mathbb{U},\mathbb{V}],\mathbb{X}) + g_1(\nabla_{\mathbb{X}}\zeta\delta\mathbb{U},\mathbb{V}) - g_1(\nabla_{\mathbb{X}}\zeta\mathbb{U},\phi\mathbb{V}).$$

On using equation (2.10), we have

$$\begin{aligned}\sin^2\bar{\theta}_1 g_1(\nabla_{\mathbb{U}}\mathbb{V},\mathbb{X}) = {} & \sin^2\bar{\theta}_1 g_1([\mathbb{U},\mathbb{X}],\mathbb{V}) + g_1(\mathcal{A}_{\mathbb{X}}\zeta\delta\mathbb{U},\mathbb{V}) \\ & - g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{U},\delta\mathbb{V}) - \lambda^{-2} g_2(\varphi_*\nabla_{\mathbb{X}}\zeta\mathbb{U},\varphi_*\zeta\mathbb{U}).\end{aligned}$$

Using Lemma 2.3, we yields

$$\begin{aligned}\sin^2\bar{\theta}_1 g_1(\nabla_{\mathbb{U}}\mathbb{V},\mathbb{X}) = {} & \sin^2\bar{\theta}_1 g_1([\mathbb{U},\mathbb{X}],\mathbb{V}) + g_1(\mathcal{A}_{\mathbb{X}}\zeta\delta\mathbb{U},\mathbb{V})\\ & - \lambda^{-2} g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\mathbb{U},\varphi_*\zeta\mathbb{V})\\ & + g_1(G(\ln\lambda),\mathbb{X})g_1(\zeta\mathbb{U},\zeta\mathbb{V})\\ & + g_1(G(\ln\lambda),\zeta\mathbb{U})g_1(\mathbb{X},\zeta\mathbb{V})\\ & - g_1(G(\ln\lambda),\zeta\mathbb{V})g_1(\mathbb{X},\zeta\mathbb{U})\\ & - g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{U},\delta\mathbb{V})\end{aligned}$$

This completes the proof of the Theorem. □

Theorem 3.7. *Let φ be the CBSS from the SM (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2) with slant angles $\bar{\theta}_1$ and $\bar{\theta}_2$. Then the distribution $D_2^{\bar{\theta}}$ defines totally geodesic foliation if and only if*

$$\lambda^{-2} g_2((\nabla\varphi_*)(\zeta\mathbb{Z},\mathbb{W}),\varphi_*\zeta\mathbb{U}) = -g_1(\mathcal{T}_{\mathbb{W}}\zeta\delta\mathbb{Z},\mathbb{U}) + g_1(\mathcal{T}_{\mathbb{W}}\zeta\mathbb{Z},\delta\mathbb{U}) \tag{3.9}$$

and

$$\begin{aligned}\lambda^{-2} g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\mathbb{W},\varphi_*\zeta\mathbb{Z}) = {} & \sin^2\bar{\theta}_2 g_1([\mathbb{W},\mathbb{X}],\mathbb{Z}) + g_1(\mathcal{A}_X\zeta\delta\mathbb{W},\mathbb{Z})\\ & + g_1(G(\ln\lambda),\mathbb{X})g_1(\zeta\mathbb{W},\zeta\mathbb{Z})\\ & + g_1(G(\ln\lambda),\zeta\mathbb{W})g_1(\mathbb{X},\zeta\mathbb{Z})\\ & - g_1(G(\ln\lambda),\zeta\mathbb{Z})g_1(\mathbb{X},\zeta\mathbb{W})\\ & - g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{W},\delta\mathbb{Z})\end{aligned} \tag{3.10}$$

for $\mathbb{Z},\mathbb{W}\in\Gamma(D_2^{\bar{\theta}}),\mathbb{U}\in\Gamma(D_1^{\bar{\theta}})$ and $\mathbb{X}\in((ker\varphi_*)^{\perp})$.

Proof. The proof of this Theorem is same as Theorem 3.6. □

Theorem 3.8. *Let φ be the CBSS from the SM (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2) with slant angles $\bar{\theta}_1$ and $\bar{\theta}_2$. Then vertical distribution $(ker\varphi_*)$ is locally Riemannian product $M_{1_{D_1^{\bar{\theta}}}}\times M_{D_2^{\bar{\theta}}}$ if and only if equation (3.7)-(3.10) holds where $M_{1_{D_1^{\bar{\theta}}}}$ and $M_{1_{D_2^{\bar{\theta}}}}$ are integral manifolds of distributions $D_1^{\bar{\theta}}$ and $D_2^{\bar{\theta}}$ respectively.*

Theorem 3.9. *Let φ be the CBSS from the SM (M_1,ϕ,ξ,η,g_1) onto a RM (M_2,g_2) with slant angles $\bar{\theta}_1$ and $\bar{\theta}_2$. Then horizontal distribution $((ker\varphi_*)^{\perp})$ defines totally geodesic foliation if and only if*

$$\begin{aligned}\lambda^{-2} g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\mathbb{U},\varphi_* f\mathbb{Y}) = {} & -g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{U},t\mathbb{Y}) - \eta(\mathbb{Y})g_1(\phi\mathbb{X},\mathbb{U})\\ & + \lambda^{-2} g_2(G(\ln\lambda),\mathbb{X})g_1(\zeta\mathbb{U},f\mathbb{Y})\\ & + g_1(G(\ln\lambda),\zeta\mathbb{U})g_1(\mathbb{X},f\mathbb{Y})\\ & - g_1(G(\ln\lambda),f\mathbb{Y})g_1(\mathbb{X},\zeta\mathbb{U})\\ & + g_1(G(\ln\lambda),\mathbb{X})g_1(\zeta\delta\mathbb{U},\mathbb{Y})\\ & + g_1(G(\ln\lambda),\zeta\delta\mathbb{U})g_1(\mathbb{X},f\mathbb{Y})\\ & - g_1(G(\ln\lambda),\mathbb{Y})g_1(\mathbb{X},\zeta\delta\mathbb{U})\\ & - \lambda^{-2} g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\delta\mathbb{U},\varphi_*\mathbb{Y}).\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
\lambda^{-2}g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\mathbb{V},\varphi_*f\mathbb{Y}) = & -g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{V},t\mathbb{Y}) - \eta(\mathbb{Y})g_1(\phi\mathbb{X},\mathbb{V}) \\
& + \lambda^{-2}g_2(G(\ln\lambda),\mathbb{X})g_1(\zeta\mathbb{V},f\mathbb{Y}) \\
& + g_1(G(\ln\lambda),\zeta\mathbb{V})g_1(\mathbb{X},f\mathbb{Y}) \\
& - g_1(G(\ln\lambda),f\mathbb{Y})g_1(\mathbb{X},\zeta\mathbb{V}) \\
& + g_1(G(\ln\lambda),\mathbb{X})g_1(\zeta\delta\mathbb{V},\mathbb{Y}) \\
& + g_1(G(\ln\lambda),\zeta\delta\mathbb{V})g_1(\mathbb{X},f\mathbb{Y}) \\
& - g_1(G(\ln\lambda),\mathbb{Y})g_1(\mathbb{X},\zeta\delta\mathbb{V}) \\
& - \lambda^{-2}g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\delta\mathbb{V},\varphi_*\mathbb{Y}).
\end{aligned} \tag{3.12}
$$

for $\mathbb{X},\mathbb{Y}\in\Gamma((ker\varphi_*)^{\perp}),\mathbb{U}\in\Gamma(D_1^{\bar{\theta}})$ *and* $\mathbb{V}\in\Gamma(D_2^{\bar{\theta}})$.

Proof. for $\mathbb{X},\mathbb{Y}\in\Gamma((ker\varphi_*)^{\perp})$ and $\mathbb{U}\in\Gamma(D_1^{\bar{\theta}})$ with using (2.2), (2.3) and (3.2), we have

$$g_1(\nabla_{\mathbb{X}}\mathbb{Y},\mathbb{U}) = g_1(\nabla_{\mathbb{X}}\phi\delta\mathbb{U},\mathbb{Y}) - g_1(\nabla_{\mathbb{X}}\zeta\mathbb{U},\mathbb{Y}) - \eta(\mathbb{Y})g_1(\phi\mathbb{X},\mathbb{U}).$$

Taking account the fact from Theorem 3.4, we can write

$$
\begin{aligned}
& \sin^2\bar{\theta}_1 g_1(\nabla_{\mathbb{X}}\mathbb{Y},\mathbb{U}) \\
& = -g_1(\nabla_{\mathbb{X}}\zeta\delta\mathbb{U},\mathbb{Y}) - g_1(\nabla_{\mathbb{X}}\zeta\mathbb{U},\mathbb{Y}) - \eta(\mathbb{Y})g_1(\phi\mathbb{X},\mathbb{U}).
\end{aligned}
$$

From (2.10), we can obtained

$$
\begin{aligned}
\sin^2\bar{\theta}_1 g_1(\nabla_{\mathbb{X}}\mathbb{Y},\mathbb{U}) = & -g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{U},t\mathbb{Y}) - \eta(\mathbb{Y})g_1(\phi\mathbb{X},\mathbb{U}) \\
& - \lambda^{-2}g_2(\varphi_*\nabla_{\mathbb{X}}\zeta\mathbb{U},\varphi_*f\mathbb{Y}) \\
& - \lambda^{-2}g_2(\varphi_*\nabla_{\mathbb{X}}\zeta\delta\mathbb{U},\varphi_*\mathbb{Y}).
\end{aligned}
$$

Considering equation Lemma 2.3, we have

$$
\begin{aligned}
\sin^2\bar{\theta}_1 g_1(\nabla_{\mathbb{X}}\mathbb{Y},\mathbb{U}) = & -g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{U},t\mathbb{Y}) - \eta(\mathbb{Y})g_1(\phi\mathbb{X},\mathbb{U}) \\
& - \lambda^{-2}g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\mathbb{U},\varphi_*f\mathbb{Y}) \\
& + \lambda^{-2}g_2(G(\ln\lambda),\mathbb{X})g_1(\zeta\mathbb{U},f\mathbb{Y}) \\
& + g_1(G(\ln\lambda),\zeta\mathbb{U})g_1(\mathbb{X},f\mathbb{Y}) \\
& - g_1(G(\ln\lambda),f\mathbb{Y})g_1(\mathbb{X},\zeta\mathbb{U}) \\
& + g_1(G(\ln\lambda),\mathbb{X})g_1(\zeta\delta\mathbb{U},\mathbb{Y}) \\
& + g_1(G(\ln\lambda),\zeta\delta\mathbb{U})g_1(\mathbb{X},f\mathbb{Y}) \\
& - g_1(G(\ln\lambda),\mathbb{Y})g_1(\mathbb{X},\zeta\delta\mathbb{U}) \\
& - \lambda^{-2}g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\delta\mathbb{U},\varphi_*\mathbb{Y}).
\end{aligned}
$$

Similarly, for $\mathbb{X},\mathbb{Y}\in\Gamma((ker\varphi_*)^{\perp})$ and $\mathbb{V}\in\Gamma(D_2)$, we have

$$
\begin{aligned}
\sin^2\bar{\theta}_2 g_1(\nabla_{\mathbb{X}}\mathbb{Y},\mathbb{V}) == & -g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{V},t\mathbb{Y}) - \eta(\mathbb{Y})g_1(\phi\mathbb{X},\mathbb{V}) \\
& - \lambda^{-2}g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\mathbb{V},\varphi_*f\mathbb{Y}) \\
& + \lambda^{-2}g_2(G(\ln\lambda),\mathbb{X})g_1(\zeta\mathbb{V},f\mathbb{Y}) \\
& + g_1(G(\ln\lambda),\zeta\mathbb{V})g_1(\mathbb{X},f\mathbb{Y}) \\
& - g_1(G(\ln\lambda),f\mathbb{Y})g_1(\mathbb{X},\zeta\mathbb{V}) \\
& + g_1(G(\ln\lambda),\mathbb{X})g_1(\zeta\delta\mathbb{V},\mathbb{Y}) \\
& + g_1(G(\ln\lambda),\zeta\delta\mathbb{V})g_1(\mathbb{X},f\mathbb{Y}) \\
& - g_1(G(\ln\lambda),\mathbb{Y})g_1(\mathbb{X},\zeta\delta\mathbb{V}) \\
& - \lambda^{-2}g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\delta\mathbb{V},\varphi_*\mathbb{Y}).
\end{aligned}
$$

□

Theorem 3.10. *Let φ be the CBSS from the SM $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2) with slant angles $\bar{\theta}_1$ and $\bar{\theta}_2$. Then vertical distribution $(ker\varphi_*)$ defines totally geodesic foliation if and only if*

$$\begin{aligned}\lambda^{-2} g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\mathbb{U}, \varphi_*\zeta\mathbb{V}) = &(\cos^2\bar{\theta}_1 - \cos^2\bar{\theta}_2) g_1(\nabla_{\mathbb{X}} P_2\mathbb{U}, \mathbb{V})\\ &+ g_1(\mathcal{A}_{\delta}\mathbb{V}, \zeta\mathbb{U}) - g_1(\mathcal{A}_{\mathbb{X}}\mathbb{V}, \zeta\delta\mathbb{U})\\ &+ g_1(G(\ln\lambda), \mathbb{X}) g_1(\zeta\mathbb{U}, \zeta\mathbb{V})\\ &+ g_1(G(\ln\lambda), \zeta\mathbb{U}) g_1(\mathbb{X}, \zeta\mathbb{V})\\ &- g_1(G(\ln\lambda), \zeta\mathbb{V}) g_1(\mathbb{X}, \zeta\mathbb{U})\\ &- \sin^2\bar{\theta}_1\ g_1([\mathbb{U}, \mathbb{X}], \mathbb{V})\end{aligned} \tag{3.13}$$

for $\mathbb{U}, \mathbb{V} \in \Gamma(ker\varphi_*)$ and $\mathbb{X} \in \Gamma((ker\varphi_*)^{\perp})$.

Proof. On taking $\mathbb{U}, \mathbb{V} \in \Gamma(ker\varphi_*)$ and $\mathbb{X} \in \Gamma((ker\varphi_*)^{\perp})$ with using (2.2), (2.3) and (3.2), we have

$$g_1(\nabla_{\mathbb{U}}\mathbb{V}, \mathbb{X}) = -g_1([\mathbb{U}, \mathbb{X}], \mathbb{V}) + g_1(\nabla_{\mathbb{X}}\phi\delta\mathbb{U}, \mathbb{V}) - g_1(\nabla_{\mathbb{X}}\zeta\mathbb{U}, \phi\mathbb{V}).$$

On using decomposition (3.1) and Theorem 3.4, we obtained

$$\begin{aligned}g_1(\nabla_{\mathbb{U}}\mathbb{V}, \mathbb{X}) = &-g_1([\mathbb{U}, \mathbb{X}], \mathbb{V}) - \cos^2\bar{\theta}_1\ g_1(\nabla_{\mathbb{X}} P_1\mathbb{U}, \mathbb{V})\\ &- \cos^2\bar{\theta}_2\ g_1(\nabla_{\mathbb{X}} P_2\mathbb{U}, \mathbb{V}) + g_1(\nabla_{\mathbb{X}}\zeta\delta\mathbb{U}, \mathbb{V})\\ &- g_1(\nabla_{\mathbb{X}}\zeta\mathbb{U}, \delta\mathbb{V}) - g_1(\nabla_{\mathbb{X}}\zeta\mathbb{U}, \zeta\mathbb{V}).\end{aligned}$$

With taking account the fact of equation (2.10), we can write

$$\begin{aligned}\sin^2\bar{\theta}_1 g_1(\nabla_{\mathbb{U}}\mathbb{V}, \mathbb{X}) = &(\cos^2\bar{\theta}_1 - \cos^2\bar{\theta}_2) g_1(\nabla_{\mathbb{X}} P_2\mathbb{U}, \mathbb{V})\\ &- \sin^2\bar{\theta}_1\ g_1([\mathbb{U}, \mathbb{X}], \mathbb{V}) + g_1(\mathcal{A}_{\mathbb{X}}\delta\mathbb{V}, \zeta\mathbb{U})\\ &- g_1(\mathcal{A}_{\mathbb{X}}\mathbb{V}, \zeta\delta\mathbb{U}) - g_1(\mathcal{H}\nabla_{\mathbb{X}}\zeta\mathbb{U}, \zeta\mathbb{V}).\end{aligned}$$

Using equation (2.13), we yields

$$\begin{aligned}\sin^2\bar{\theta}_1\ g_1(\nabla_{\mathbb{U}}\mathbb{V}, \mathbb{X}) = &(\cos^2\bar{\theta}_1 - \cos^2\bar{\theta}_2) g_1(\nabla_{\mathbb{X}} P_2\mathbb{U}, \mathbb{V}) - \sin^2\bar{\theta}_1\ g_1([\mathbb{U}, \mathbb{X}], \mathbb{V})\\ &+ g_1(\mathcal{A}_{\mathbb{X}}\delta\mathbb{V}, \zeta\mathbb{U}) - g_1(\mathcal{A}_{\mathbb{X}}\mathbb{V}, \zeta\delta\mathbb{U})\\ &+ \lambda^{-2} g_2((\nabla\varphi_*)(\mathbb{X}, \zeta\mathbb{U}), \varphi_*\zeta\mathbb{V})\\ &- \lambda^{-2} g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\mathbb{U}, \varphi_*\zeta\mathbb{V}).\end{aligned}$$

Considering Lemma 2.3, have

$$\begin{aligned}\sin^2\bar{\theta}_1\ g_1(\nabla_{\mathbb{U}}\mathbb{V}, \mathbb{X}) = &(\cos^2\bar{\theta}_1 - \cos^2\bar{\theta}_2) g_1(\nabla_{\mathbb{X}} P_2\mathbb{U}, \mathbb{V}) - \sin^2\bar{\theta}_1\ g_1([\mathbb{U}, \mathbb{X}], \mathbb{V})\\ &+ g_1(\mathcal{A}_{\mathbb{X}}\delta\mathbb{V}, \zeta\mathbb{U}) - g_1(\mathcal{A}_{\mathbb{X}}\mathbb{V}, \zeta\delta\mathbb{U})\\ &+ g_1(G(\ln\lambda), \mathbb{X}) g_1(\zeta\mathbb{U}, \zeta\mathbb{V})\\ &+ g_1(G(\ln\lambda), \zeta\mathbb{U}) g_1(\mathbb{X}, \zeta\mathbb{V})\\ &- g_1(G(\ln\lambda), \zeta\mathbb{V}) g_1(\mathbb{X}, \zeta\mathbb{U})\\ &- \lambda^{-2} g_1(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\mathbb{U}, \varphi_*\zeta\mathbb{V}).\end{aligned}$$

This completes the proof of the Theorem. □

We can provide these decomposition Theorems by taking into consideration the prior Theorems.:

Theorem 3.11. *Let φ be the CBSS from the SM $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2) with slant angles $\bar{\theta}_1$ and $\bar{\theta}_2$. Then the total space $M_{1_{D_1^{\bar{\theta}}}} \times M_{1_{D_2^{\bar{\theta}}}} \times M_{1_{((ker\varphi_*)^{\perp})}}$ is locally product if and only if equation (3.7)-(3.12) are holds where $M_{1_{D_1^{\bar{\theta}}}}, M_{1_{D_2^{\bar{\theta}}}}$ and $M_{1_{((ker\varphi_*)^{\perp})}}$ are integral manifolds of the distributions $D_1^{\bar{\theta}}$, $D_2^{\bar{\theta}}$ and $((ker\varphi_*)^{\perp})$ respectively.*

Theorem 3.12. *Let φ be the CBSS from the SM $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2) with slant angles $\bar{\theta}_1$ and $\bar{\theta}_2$. Then the total space $M_{1_{ker\varphi_*}} \times M_{1_{((ker\varphi_*)^{\perp})}}$ is locally product if and only if equation (3.11)-(3.13) are holds where $M_{1_{ker\varphi_*}}$ and $M_{1_{((ker\varphi_*)^{\perp})}}$ are integral manifolds of the distributions $(ker\varphi_*)$ and $((ker\varphi_*)^{\perp})$ respectively.*

Theorem 3.13. *Let φ be the CBSS from the SM $(M_1, \phi, \xi, \eta, g_1)$ onto a RM (M_2, g_2) with slant angles $\bar{\theta}_1$ and $\bar{\theta}_2$. Then φ is totally geodesic map if and only if*

i

$$\begin{aligned}\lambda^{-2} g_2(\nabla^{\varphi}_{\mathbb{U}} \varphi_* \zeta\delta\mathbb{V}, \varphi_*\mathbb{X}) &= (\cos^2\bar{\theta}_1 - \cos^2\bar{\theta}_2) g_1(\nabla_{\mathbb{U}} P_2\mathbb{V}, \mathbb{X}) \\ &\quad - g_1(\mathcal{T}_{\mathbb{U}}\zeta\mathbb{V}, \mathbb{X}) \\ &\quad + g_1(G(\ln\lambda), \mathbb{U}) g_1(\zeta\delta\mathbb{V}, \mathbb{X}) \\ &\quad - g_1(G(\ln\lambda), \mathbb{U}) g_1(\zeta\mathbb{V}, f\mathbb{X}) \\ &\quad - \lambda^{-2} g_2(\nabla^{\varphi}_{\mathbb{U}} \varphi_*\zeta\delta\mathbb{V}, \varphi_*\mathbb{X})\end{aligned}$$

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$$\begin{aligned}\lambda^{-2} g_2(\nabla^{\varphi}_{\mathbb{X}} \varphi_* \zeta\mathbb{U}, \varphi_* f\mathbb{Y}) &= (\cos^2\bar{\theta}_2 - \cos^2\bar{\theta}_1) g_1(\mathcal{A}_{\mathbb{X}} P_1\mathbb{U}, \mathbb{Y}) \\ &\quad + g_1(t\mathbb{X}, \mathbb{U})\eta(\mathbb{Y}) + g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{U}, \zeta\mathbb{Y}) \\ &\quad - g_1(G(\ln\lambda), \mathbb{X}) g_1(\zeta\delta\mathbb{U}, \mathbb{Y}) \\ &\quad - g_1(G(\ln\lambda), \zeta\delta\mathbb{U}) g_1(\mathbb{X}, \mathbb{Y}) \\ &\quad + g_1(G(\ln\lambda), \mathbb{Y}) g_1(\mathbb{X}, \zeta\delta\mathbb{U}) \\ &\quad + g_1(G(\ln\lambda), \mathbb{X}) g_1(\zeta\mathbb{U}, f\mathbb{Y}) \\ &\quad + g_1(G(\ln\lambda), \zeta\mathbb{U}) g_1(\mathbb{X}, f\mathbb{Y}) \\ &\quad + g_1(G(\ln\lambda), f\mathbb{Y}) g_1(\mathbb{X}, \zeta\mathbb{U}) \\ &\quad + g_2(\nabla^{\varphi}_{\mathbb{X}} \varphi_*\zeta\delta\mathbb{U}, \varphi_*\mathbb{Y})\end{aligned}$$

for $\mathbb{U}, \mathbb{V} \in \Gamma(ker\varphi_*)$ and $\mathbb{X}, \mathbb{Y} \in \Gamma((ker\varphi_*)^{\perp})$.

Proof. for $\mathbb{U}, \mathbb{V} \in \Gamma(ker\varphi_*)$ and $\mathbb{X} \in \Gamma((ker\varphi_*)^{\perp})$ with using equation (2.13), we can write

$$\lambda^{-2} g_2((\nabla\varphi_*)(\mathbb{U}, \mathbb{V}), \varphi_*\mathbb{X}) = g_1(\nabla_{\mathbb{U}}\mathbb{V}, \mathbb{X}) \tag{3.14}$$

From equations (2.2), (2.3) and (3.2), we have

$$\begin{aligned}g_1(\nabla_{\mathbb{U}}\mathbb{V}, \mathbb{X}) &= g_1(\nabla_{\mathbb{U}}\zeta\mathbb{V}, \phi\mathbb{X}) - g_1(\nabla_{\mathbb{U}}\phi\delta\mathbb{V}, \mathbb{X}) \\ &\quad + g_1(\mathbb{U}, \delta\mathbb{V})\eta(\mathbb{X}) + g_1(\mathbb{U}, \delta\mathbb{V})\eta(\mathbb{X})\end{aligned}$$

Considering Theorem 3.4 and decomposition (3.1), we obtained

$$\begin{aligned}&\sin^2\bar{\theta}_1\ g_1(\nabla_{\mathbb{U}}\mathbb{V}, \mathbb{X}) \\ &= (\cos^2\bar{\theta}_2 - \cos^2\bar{\theta}_1) g_1(\nabla_{\mathbb{U}} P_2\mathbb{V}, \mathbb{X}) \\ &\ + g_1(\nabla_{\mathbb{U}}\zeta\delta\mathbb{V}, \mathbb{X}) + g_1(\nabla_{\mathbb{U}}\zeta\mathbb{V}, \phi\mathbb{X}).\end{aligned}$$

From equation (3) and (3.14), we get

$$\begin{aligned}&\sin^2\bar\theta_1\ \lambda^{-2}g_2((\nabla\varphi_*)(\mathbb{U},\mathbb{V}),\varphi_*\mathbb{X})\\&=(\cos^2\bar\theta_2-\cos^2\bar\theta_1)g_1(\nabla_{\mathbb{U}}P_2\mathbb{V},\mathbb{X})\\&\ +g_1(\nabla_{\mathbb{U}}\zeta\delta\mathbb{V},\mathbb{X})+g_1(\nabla_{\mathbb{U}}\zeta\mathbb{V},\phi\mathbb{X}).\end{aligned}$$

On using equations (2.7) and (2.8), we have

$$\begin{aligned}&\sin^2\bar\theta_1\ \lambda^{-2}g_2((\nabla\varphi_*)(\mathbb{U},\mathbb{V}),\varphi_*\mathbb{X})\\&=(\cos^2\bar\theta_2-\cos^2\bar\theta_1)g_1(\mathcal{T}_{\mathbb{U}}P_2\mathbb{V},\mathbb{X})\\&\ +g_1(\mathcal{H}\nabla_{\mathbb{U}}\zeta\delta\mathbb{V},\mathbb{X})+g_1(\mathcal{T}_{\mathbb{U}}\zeta\mathbb{V},t\mathbb{X})\\&\ +g_1(\mathcal{H}\nabla_{\mathbb{U}}\zeta\mathbb{V},f\mathbb{X}).\end{aligned}$$

From equation (2.13) and Lemma 2.3, we get

$$\begin{aligned}&\sin^2\bar\theta_1\ \lambda^{-2}g_2((\nabla\varphi_*)(\mathbb{U},\mathbb{V}),\varphi_*\mathbb{X})\\&=\lambda^{-2}g_2(\mathbb{U}(\ln\lambda)\varphi_*\zeta\delta\mathbb{V},\varphi_*\mathbb{X})\\&\ +\lambda^{-2}g_2(\nabla^{\varphi}_{\mathbb{U}}\varphi_*\zeta\delta\mathbb{V},\varphi_*\mathbb{X})\\&\ +\lambda^{-2}g_2(\mathbb{U}(\ln\lambda)\varphi_*\zeta\mathbb{V},\varphi_*f\mathbb{X})\\&\ +\lambda^{-2}g_2(\nabla^{\varphi}_{\mathbb{U}}\varphi_*\zeta\mathbb{V},\varphi_*f\mathbb{X}).\end{aligned}$$

On the other hand, for $\mathbb{U}\in\Gamma(ker\varphi_*)$ and $\mathbb{X},\mathbb{Y}\in\Gamma((ker\varphi_*)^{\perp})$, we get

$$\lambda^{-2}g_2((\nabla\varphi_*)(\mathbb{X},\mathbb{U}),\varphi_*\mathbb{Y})=g_1(\nabla_{\mathbb{X}}\mathbb{U},\mathbb{Y}).\tag{3.15}$$

On using (2.2), (2.3) and (3.2), we have

$$g_1(\nabla_{\mathbb{X}}\mathbb{U},\mathbb{Y})=-g_1(\nabla_{\mathbb{X}}\phi\delta\mathbb{U},\mathbb{Y})+g_1(\nabla_{\mathbb{X}}\zeta\mathbb{U},\phi\mathbb{Y})+g_1(t\mathbb{X},\mathbb{U})\eta(\mathbb{Y})$$

With the help of decomposition (3.8) and Theorem 3.4, we obtain

$$\begin{aligned}&\sin^2\bar\theta_1\ g_1(\nabla_{\mathbb{X}}\mathbb{U},\mathbb{Y})\\&=(\cos^2\bar\theta_2-\cos^2\bar\theta_1)g_1(\nabla_{\mathbb{X}}P_1\mathbb{U},\mathbb{Y})\\&\ +g_1(\nabla_{\mathbb{X}}\zeta\mathbb{U},\phi\mathbb{Y})+g_1(t\mathbb{X},\mathbb{U})\eta(\mathbb{Y})\\&\ -g_1(\nabla_{\mathbb{X}}\zeta\delta\mathbb{U},\mathbb{Y})\end{aligned}\tag{3.16}$$

From (3.15) and (3.16), we can write

$$\begin{aligned}&\sin^2\bar\theta_1\ \lambda^{-2}g_2((\nabla\varphi_*)(\mathbb{X},\mathbb{U}),\varphi_*\mathbb{Y})\\&=(\cos^2\bar\theta_2-\cos^2\bar\theta_1)g_1(\nabla_{\mathbb{X}}P_1\mathbb{U},\mathbb{Y})\\&\ +g_1(\nabla_{\mathbb{X}}\zeta\mathbb{U},\phi\mathbb{Y})+g_1(t\mathbb{X},\mathbb{U})\eta(\mathbb{Y})\\&\ -g_1(\nabla_{\mathbb{X}}\zeta\delta\mathbb{U},\mathbb{Y}).\end{aligned}$$

From equation (2.9) and (2.10), we have

$$\begin{aligned}&\sin^2\bar\theta_1\ \lambda^{-2}g_2((\nabla\varphi_*)(\mathbb{X},\mathbb{U}),\varphi_*\mathbb{Y})\\&=(\cos^2\bar\theta_2-\cos^2\bar\theta_1)g_1(\mathcal{A}_{\mathbb{X}}P_1\mathbb{U},\mathbb{Y})\\&\ +g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{U},t\mathbb{Y})+g_1(t\mathbb{X},\mathbb{U})\eta(\mathbb{Y})\\&\ -g_1(\mathcal{H}\nabla_{\mathbb{X}}\zeta\delta\mathbb{U},\mathbb{Y})+g_1(\mathcal{H}\nabla_{\mathbb{X}}\zeta\mathbb{U},\mathbb{Y}).\end{aligned}$$

Considering Lemma 2.3 with equation (2.13), we yields

$$\begin{aligned}
&\sin^2\bar{\theta}_1\lambda^{-2}g_2((\nabla\varphi_*)(\mathbb{X},\mathbb{U}),\varphi_*\mathbb{Y})\\
&=(\cos^2\bar{\theta}_2-\cos^2\bar{\theta}_1)g_1(\mathcal{A}_{\mathbb{X}}P_1\mathbb{U},\mathbb{Y})\\
&+g_1(\mathcal{A}_{\mathbb{X}}\zeta\mathbb{U},t\mathbb{Y})+g_1(t\mathbb{X},\mathbb{U})\eta(\mathbb{Y})\\
&-g_1(G(\ln\lambda),\mathbb{X})g_1(\zeta\delta\mathbb{U},\mathbb{Y})\\
&-g_1(G(\ln\lambda),\zeta\delta\mathbb{U})g_1(\mathbb{X},\mathbb{Y})\\
&+g_1(G(\ln\lambda),\mathbb{Y})g_1(\mathbb{X},\zeta\delta\mathbb{U})\\
&+g_1(G(\ln\lambda),\mathbb{X})g_1(\zeta\mathbb{U},f\mathbb{Y})\\
&+g_1(G(\ln\lambda),\zeta\mathbb{U})g_1(\mathbb{X},f\mathbb{Y})\\
&-g_1(G(\ln\lambda),f\mathbb{Y})g_1(\mathbb{X},\zeta\mathbb{U})\\
&+\lambda^{-2}g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\delta\mathbb{U})\\
&-\lambda^{-2}g_2(\nabla^{\varphi}_{\mathbb{X}}\varphi_*\zeta\mathbb{U},\varphi_*f\mathbb{Y}).
\end{aligned}$$

Finally we show that λ is constant on $\Gamma(D_1)$. For $\mathbb{U}_1,\mathbb{U}_2\in\Gamma(D_1)$ and from Lemma 2.3. we obtain

$$\begin{aligned}
(\nabla\varphi_*)(\zeta\mathbb{U}_1,\zeta\mathbb{U}_2)&=\zeta\mathbb{U}_1(\ln\lambda)\varphi_*\zeta\mathbb{U}_2+\zeta\mathbb{U}_2(\ln\lambda)\varphi_*\zeta\mathbb{U}_1\\
&-g_1(\zeta\mathbb{U}_1,\zeta\mathbb{U}_2)\varphi_*(G(\ln\lambda)).
\end{aligned}$$

Replacing $\mathbb{U}_2$ by $\mathbb{U}_1$ in above equation, we get

$$\begin{aligned}
(\nabla\varphi_*)(\zeta\mathbb{U}_1,\zeta\mathbb{U}_1)&=2\zeta\mathbb{U}_1(\ln\lambda)\varphi_*\zeta\mathbb{U}_1\\
&-g_1(\zeta\mathbb{U}_1,\zeta\mathbb{U}_1)\varphi_*(G(\ln\lambda)).
\end{aligned}\tag{3.17}$$

Taking inner product with $\varphi_*\zeta\mathbb{U}_1$ in (3.17), we can write

$$\begin{aligned}
&2g_1(G(\ln\lambda),\zeta\mathbb{U}_1)g_2(\varphi_*\zeta\mathbb{U}_1,\varphi_*\zeta\mathbb{U}_1)\\
&-g_1(\zeta\mathbb{U}_1,\zeta\mathbb{U}_1)g_2(\varphi_*G(\ln\lambda),\varphi_\zeta\mathbb{U}_1)=0,
\end{aligned}$$

which shows that λ is constant on $\Gamma(D_1^{\bar{\theta}})$. Similarly, we can show that λ is constant on $\Gamma(D_2^{\bar{\theta}})$ and $\Gamma(\mu)$. This completes the proof of the Theorem. □

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