Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–12.
ISSN-0037-8712 in press
doi:10.5269/bspm.66727

Fractional Navier-Stokes Equations With Delay Conditions

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ABSTRACT: Throught this paper,we study the cauchy problem for the conformable fractional Navier-Stokes Equations (FNSE) with finite delay external forces, contains some hereditary features, on a bounded domain. We prove the existence and uniqueness of local mild solutions for the initial datum by using semigroup theory, conformable fractional calculus and Banach contraction theorem. In the end, with more conditions on delay external forces, we establish the globality and continuation of the mild solutions.

Key Words: Conformable Fractional Derivative, Navier-Stokes equations, Fractional power of operator.

Contents

1. Introduction

In fact, there are many researches dealing with the local or global existence of mild solutions of the Cauchy problem for the Navier Stokes equations (NSE). In spite of the variation of researches about this, approaches to study this problems remains the same, the basic principal of this approaches is to transform the Navier-Stokes equations, with the finite delay external force f, charachterised by hereditary qualities related not only with the present state, but also on the past history of the system. The first mathematical study of NSE was carried by J. Leray [9]. In 2001, T. Caraballo and al. [2] was the first to consider several situations of ordinary NSE in which the external force contains some hereditary features and prove existence of solutions. After, the study of model with time fractional differential [3, 13, 14], become more advanced than the ordinary model [4, 7, 12], in the field of science and engineering researchers and applications.
In our work, we study the local existence and globality of mild solutions for time fractional NSE with finite delayed external forces in $\mathbb{R}^3$ with conformable fractional derivative. We find the existence result by using the Banach contraction mapping principle and fractional power of operators.
This paper will deal with the following sections. In section 2, we make a recapitulation of some basic facts on the conformable fractional calculus and Fractional Laplace transform, the other sections are specified for proving the mains results: Existence, uniqueness of mild solutions and example of delayed force function to illustrate our existence result in section 3, and globality of mild solution in section 4.

2010 *Mathematics Subject Classification:* 34K37, 35Q30.
Submitted January 18, 2023. Published April 26, 2023

2. Preliminaries

The system model presented as follows:
Let $\mathbf{B} \subset \mathbb{R}^3$ be a bounded domain with regular boundary $\partial \mathbf{B}$.

$$\begin{cases} D_t^\alpha v - \Delta v + (v \cdot \nabla) v = -\nabla p + f(t, v_t), t > 0, x \in \mathbf{B} \\ \nabla \cdot v = 0, t > 0, x \in \mathbf{B} \\ v|_{\partial \mathbf{B}} = 0, t > 0, \\ v(x,t) = \varphi(x,t), \quad -a \leq t \leq 0, \quad x \in \mathbf{B}, \end{cases} \tag{2.1}$$

such that:
•$v = v(.,t) = (v_i(.,t), v_j(.,t), v_k(.,t))$ is the velocity of the fluid,
•$p = p(x,t)$ is the associated pressure,
•$v_t(s) = v(t+s), -a \leq s \leq 0, f$ is an external force related to v_t,
•φ is the initial datum in the delayed interval $[-a, 0]$,
•$D_t^\alpha v$ is the conformable fractional derivative of order $\alpha \in (0,1)$ with respect to t.
First, we give some basic definitions and properties of the fractional calculus theory notations and preliminary results which will be used further in this paper.
we introduce the usual abstract space as follows:

$$\nu = \left\{ v \in (C_0^\infty(\mathbf{B}))^3 : \nabla \cdot v = 0 \right\}$$

•$H_\nu(\mathbf{B})$ = closure of ν in $\left(L^2(\mathbf{B})\right)^3$,with the norm $\| \cdot \|$,
Let $P : \left(L^2(\mathbf{B})\right)^3 \to H_\nu(\mathbf{B})$ be the Projection operator,
•$A = -P\Delta : D(A) \subset H_\nu(\mathbf{B}) \to H_\nu(\mathbf{B})$ with $D(A) = H_\nu(\mathbf{B}) \cap \left(H_0^1(\mathbf{B})\right)^3 \cap \left(H^2(\mathbf{B})\right)^3$ is a Stokes operator associated to the bilinear form define as $a(u,v) = \langle \nabla u, \nabla v \rangle$, where $u, v \in \vartheta$ and $\langle , \rangle$ is the user inner product on $H_\nu(\mathbf{B})$,
•$(-A)$ generates analytic semigroup of contractions $\{T(t)\}_{t \geq 0}$ on $H_\nu(\mathbf{B})$.
Using the projection operator and Stokes operatoron on (1), we can transform the system to the following evolution equation in a Banach space $H_\nu(\mathbf{B})$:

$$\begin{cases} D_t^\alpha v + Av = Fv + Pf(t, v_t), \quad t > 0, \\ v(t) = \varphi(t), \quad -a \leq t \leq 0, \end{cases} \tag{2.2}$$

where $Fv = -P(v \cdot \nabla)v$.
Now, we present the sectorial operators on $H_\nu(\mathbf{B})$, as follows:

Definition 2.1. *[11] $A : D(A) \subset X \longrightarrow X$ is said to be sectorial operator of type (M, ω, θ) if there exist $M > 0, \omega \in \mathbb{R}$ and $0 < \theta < \frac{\pi}{2}$ such that:*
1. *A be a closed and linear operator,*
2. *$\forall \lambda \notin \omega + S_\theta$, the resolvent $(\lambda I - A)^{-1}$ of A exists,*
3. *$\forall \lambda \notin \omega + S_\theta, \left|(\lambda I - A)^{-1}\right| \leq \frac{M}{|\lambda - \omega|}$.*

where

$$\omega + S_\theta := \{\omega + \lambda \mid \quad \lambda \in \mathbb{C} \text{ with } |\operatorname{Arg}(-\lambda)| < \theta\}.$$

Theorem 2.2. *[11] $(-A)$ densely sectorial operator generates a strongly analytic semigroup $(T(t))_{t \geq 0}$. Moreover, we have:*

$$T(t) = \frac{1}{2\pi i} \int_\Sigma e^{\lambda t} (\lambda I + A)^{-1} d\lambda, \tag{2.3}$$

with Σ being a suitable path $\lambda \notin \omega + S_\theta$.

Definition 2.3. *[8] Let $\alpha \in]0,1]$. The conformable fractional derivative of order α of a function $x(.)$ for $t > 0$ is defined as follows:*

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = \lim_{\varepsilon \longrightarrow 0} \frac{x\left(t + \varepsilon t^{1-\alpha}\right) - x(t)}{\varepsilon}. \tag{2.4}$$

For $t=0$, we adopt the following definition:

$$\frac{\mathrm{d}^\alpha x(0)}{\mathrm{d}t^\alpha}=\lim_{t\to 0^+}\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha}.$$

The fractional integral $I^\alpha(.)$ associated with the conformable fractional derivative is defined by:

$$I^\alpha(x)(t)=\int_0^t s^{\alpha-1}x(s)\mathrm{d}s \tag{2.5}$$

Theorem 2.4. *[8] If $x(.)$ is a continuous function in the domain of $I^\alpha(.)$, then we have*

$$\frac{\mathrm{d}^\alpha\left(I^\alpha(x)(t)\right)}{\mathrm{d}t^\alpha}=x(t) \tag{2.6}$$

Definition 2.5. *[1] The Laplace transform of a function $x(.)$ is defined by:*

$$\mathcal{L}(x(t))(\lambda)=\int_0^{+\infty}e^{-\lambda t}x(t)\mathrm{d}t,\quad \lambda>0. \tag{2.7}$$

The adapted transform is given by the following definition.
The Fractional Laplace transform of order $\alpha\in]0,1]$ of a function $x(.)$ is defined by:

$$\mathcal{L}_\alpha(x(t))(\lambda)=\int_0^{+\infty}t^{\alpha-1}e^{-\lambda(t^\alpha/\alpha)}x(t)\mathrm{d}t,\quad \lambda>0 \tag{2.8}$$

Proposition 2.6. *[1] If $x(.)$ is a differentiable function, then we have the following results:*

$$I^\alpha\left(\frac{\mathrm{d}^\alpha x(.)}{\mathrm{d}t^\alpha}\right)(t)=x(t)-x(0) \tag{2.9}$$

$$\mathcal{L}_\alpha\left(\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha}\right)(\lambda)=\lambda\mathcal{L}_\alpha(x(t))(\lambda)-x(0) \tag{2.10}$$

For two functions $x(.)$ and $y(.)$, we have

$$\mathcal{L}_\alpha\left(x\left(\frac{t^\alpha}{\alpha}\right)\right)(\lambda)=\mathcal{L}(x(t))(\lambda) \tag{2.11}$$

$$\mathcal{L}_\alpha\left(\int_0^t s^{\alpha-1}x\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)y(s)\mathrm{d}s\right)(\lambda)=\mathcal{L}(x(t))(\lambda)\mathcal{L}_\alpha(y(t))(\lambda) \tag{2.12}$$

Now, Let $0\in\rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$,
then for $0<\beta\leq\overline{1}$, we can define a closed linear operator with the fractional power A^β on its domain $D\left(A^\beta\right)$.

Definition 2.7. *[11] Let A be a sectorial operator defined on a Banach space X, such that $\operatorname{Re}\sigma(A)>0$; for $\beta>0$, we note by $A^{-\beta}$ the operator defined by:*

$$A^{-\beta}=\frac{1}{\Gamma(\beta)}\int_0^{+\infty}t^{\beta-1}T(t)dt.$$

Definition 2.8. *[11]Let A be a sectorial operator defined on a Banach space X, such that $\mathcal{R}e\sigma(A)>0$. We define the family of cperators $\left(A^\beta\right)_{\beta\geq 0}$ as follows: $A^0=I_X$, and for $\beta>0$,*

$$A^\beta=\left(A^{-\beta}\right)^{-1},\quad D\left(A^\beta\right)=\operatorname{Im}\left(A^{-\beta}\right).$$

For analytic semigroup $\{T(t)\}_{t\geq 0}$, the following properties will be used.

Lemma 2.9. *[6,11,12] If $(-A)$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ and if $0 \in \rho(A)$, then:*
(a) $D\left(A^{\beta}\right)$ is a Banach space with the norm $\|x\|_D^{\beta} = \left\|A^{\beta}x\right\|$ for every $x \in D\left(A^{\beta}\right)$.
(b) There is a $M \geq 1$ such that:

$$M := \sup_{t\in[0,+\infty)} |T(t)| < \infty$$

(c) for any $\beta \in (0,1]$, there exists a positive constant M_β such that

$$\left\|A^{\beta}T(t)\right\| \leq \frac{M_\beta}{t^{\beta}}, \quad 0 < t \leq T$$

(d)For $x \in H_\nu(\Omega)$ and $t > 0$, $A^{\beta}T(t)x = T(t)A^{\beta}x$.
(e) For $0 < \alpha < \beta < 1, \quad D\left(A^{\beta}\right) \hookrightarrow D\left(A^{\alpha}\right)$.

Lemma 2.10. *[4] Let $u, v \in D\left(A^{\frac{1}{2}}\right)$, then following estimations hold:*

(1)There exists $l_1 > 0$ such that $\left\|A^{-\frac{1}{4}}Fu\right\| \leq l_1 \left\|A^{\frac{1}{2}}u\right\|^2$,
(2) $\left\|A^{-\frac{1}{4}}(Fu - Fv)\right\| \leq l_1 \left\|A^{\frac{1}{2}}(u-v)\right\| \left(\left\|A^{\frac{1}{2}}u\right\| + \left\|A^{\frac{1}{2}}v\right\|\right)$.

Lemma 2.11. *[14] Let $0 < \beta < 1$ and $T(t)$ is defined by (3). Then there exists $M_\beta > 0$ such that $\|A^{\beta}T(t) - A^{\beta}T(s)\| \leq M_\beta \left(t^{(1-\beta)} - s^{(1-\beta)}\right)$ for all $s, t > 0$ with $t > s$. implies that, $t \mapsto A^{\beta}T(t)$ is continuous for $t > 0$ with respect to uniform operator topology.*

Lemma 2.12. *[10] Let X be a Banach space and $A : D(A) \subset X \to X$ be a closed operator.*
For $-\infty \leq a < b \leq \infty$ and $f : I =]a; b[\to D(A)$ be such that the functions $t \to f(t), t \to Af(t)$ are integrable (Bochner sense) on I. Then

$$\int_a^b f(t)dt \in D(A), \quad A\int_a^b f(t)dt = \int_a^b Af(t)dt$$

3. Existence and uniqueness of mild solution

This section is speciefed to prove the local existence and uniqueness of mild solution to (2.2). First, by using the fractional Laplace transform in equation (2.2), we have:

$$\lambda\mathcal{L}_\alpha(v(t))(\lambda) + A\mathcal{L}_\alpha[v(t))](\lambda) = \varphi(0) + \mathcal{L}_\alpha(Fv(t))(\lambda) + \mathcal{L}_\alpha(Pf(t,x_t)(\lambda) \tag{3.1}$$

Then :

$$\mathcal{L}_\alpha(v(t) = (\lambda + A)^{-1}\left(\varphi(0)\right) + (\lambda + A)^{-1}\mathcal{L}_\alpha(Fv(t))(\lambda) + (\lambda + A)^{-1}\mathcal{L}_\alpha(Pf(t,x_t)(\lambda) \tag{3.2}$$

Applying the inverse fractional Laplace transform combined with (2.11) and (2.12), we obtain:

$$\begin{cases} v(t) = T\left(\dfrac{t^{\alpha}}{\alpha}\right)(\varphi(0)) + \displaystyle\int_0^t s^{\alpha-1}T\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)Fv(s)\mathrm{d}s \\ \qquad + \displaystyle\int_0^t s^{\alpha-1}T\left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right)Pf(s,x_s)\mathrm{d}s; & t \in [0,T] \\ v_0(t) = \varphi(t); & t \in [-a,0] \end{cases} \tag{3.3}$$

Now , we can introduce the following definition of mild solutions for the Cauchy problem (2.2);

Definition 3.1. *Let $0 < T < \infty$. We say that $v : [-a, T] \to D\left(A^{\frac{1}{2}}\right)$ is a local mild solution of the cauchy problem (2) if $v \in C\left([0, T]; D\left(A^{\frac{1}{2}}\right)\right)$ and v satisfies:*

$$\begin{cases} v(t) = T\left(\frac{t^\alpha}{\alpha}\right)(\varphi(0)) + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) Fv(s)\mathrm{d}s \\ \qquad + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) Pf(s, v_s)\mathrm{d}s; & t \in [0, T] \\ v_0(t) = \varphi(t); & t \in [-a, 0] \end{cases} \tag{3.4}$$

we denote by $\mathcal{C} := C\left([-a, 0]; D\left(A^{\frac{1}{2}}\right)\right)$ the Banach space of continuous functions from $[-a; 0]$ into $D\left(A^{\frac{1}{2}}\right)$ with the norm $\|.\|_*$, $V \subset \mathcal{C}$ be open, and $C([0, T]); D\left(A^{\frac{1}{2}}\right))$ is endowed with sup-norm topology. To obtain the existence of Mild Solution, we will introduce the foollowing assumptions:

(H_1) *: $Pf : [0, \infty) \times V \to H_\nu(\Omega)$ be such that : $\|Pf(t, \phi)\| \leq \mu(t)\|\phi\|_*$ for all $t \geq 0, \varphi \in V$ and $\mu \in L^p_{loc}[0, \infty)$, where $p > \frac{2}{\alpha}$,*

(H_2) *: $\|Pf(t, \phi) - Pf(t, \psi)\| \leq K_f\|\phi - \psi\|_*$ for all ϕ, $\psi \in V$ and for some $K_f > 0$.*

Theorem 3.2. *If $(H_1) - (H_2)$ are satisfied, Then for every $\varphi \in V$, there exists a unique mild solution $v : [-a, T] \to D\left(A^{\frac{1}{2}}\right)$ to the Cauchy problem with delay (2.2) , for $T = T_\varphi > 0$.*

Proof. Let $\varphi \in V$ and $R > 0$ be such that $\{\zeta \in \mathcal{C} : \|\zeta - \varphi\|_* \leq R\} \subset U$. Let $T > 0$. we define the following set:

$$Y_{\varphi;\frac{1}{2}} = \left\{u \in C([-a, T]); D\left(A^{\frac{1}{2}}\right) : u_0 = \varphi \text{ and } \|u_t - \varphi\|_* \leq R, \forall t \in [0, T]\right\}$$

$Y_{\varphi;\frac{1}{2}} \subset C\left([-a, T]); D(A^{\frac{1}{2}})\right)$ is non-empty and closed .
Now, we define an operator Γ on $Y_{\varphi;\frac{1}{2}}$ as follows,

$$\Gamma v(t) = \begin{cases} T\left(\frac{t^\alpha}{\alpha}\right)(\varphi(0)) + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) Fv(s)\mathrm{d}s \\ + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) Pf(s, v_s)\mathrm{d}s, & t \in [0, T], \\ \varphi(t), & t \in [-a, 0]. \end{cases}$$

<u>Step 1</u>: we prove that $\Gamma\left(Y_{\varphi;\frac{1}{2}}\right) \subset Y_{\varphi;\frac{1}{2}}$ Let $v \in Y_{\varphi;\frac{1}{2}}$.
So, we have $\|v(t)\|_{D\left(A^{\frac{1}{2}}\right)} = \|v_t(0)\|_{D\left(A^{\frac{1}{2}}\right)} \leq \|v_t\|_* \leq R + \|\varphi\|_*$ for all $t \in [0, T]$.
Now, $\forall t \in [0, T_1]$ and $\theta \in [-a, 0]$ such that $0 \leq t + \theta \leq T_1$, we have;

$$\begin{aligned}
\|(\mathit{\Gamma} v)_t(\theta)-\varphi(\theta)\|_{D\left(A^{\frac{1}{2}}\right)} &\leq \|T((t+\theta)^\alpha)\varphi(0)-\varphi(0)\|_{D\left(A^{\frac{1}{2}}\right)} \\
&+\left\|A^{\frac{1}{2}}\int_0^{t+\theta} s^{\alpha-1}A^{\frac{1}{4}}T(\frac{(t+\theta)^\alpha-s^\alpha}{\alpha})A^{-\frac{1}{4}}Fv(s)ds\right\| \\
&+\|\varphi(\theta)-\varphi(0)\|_{D\left(A^{\frac{1}{2}}\right)}+\left\|A^{\frac{1}{2}}\int_0^{t+\theta} s^{\alpha-1}T(\frac{(t+\theta)^\alpha-s^\alpha}{\alpha})Pf\left(s,v_s\right)ds\right\| \\
&\leq \|T((t+\theta)^\alpha)\varphi(0)-\varphi(0)\|_{D\left(A^{\frac{1}{2}}\right)} \\
&+M_{\frac{3}{4}}\alpha^{\frac{3}{4}}\int_0^{t+\theta} s^{\alpha-1}((t+\theta)^\alpha-s^\alpha)^{\frac{-3}{4}}\left\|A^{-\frac{1}{4}}Fv(s)\right\|ds \\
&+M_{\frac{1}{2}}\alpha^{\frac{1}{2}}\int_0^{t+\theta} s^{\alpha-1}((t+\theta)^\alpha-s^\alpha)^{\frac{-1}{2}}\mu(s)\left\|v_s\right\|_* ds+\|\varphi(\theta)-\varphi(0)\|_{D\left(A^{\frac{1}{2}}\right)} \\
&\leq \|T((t+\theta)^\alpha)\varphi(0)-\varphi(0)\|_{D\left(A^{\frac{1}{2}}\right)} \\
&+M_{\frac{3}{4}}\alpha^{\frac{3}{4}}l_1\int_0^{t+\theta} s^{\alpha-1}((t+\theta)^\alpha-s^\alpha)^{\frac{-1}{4}}\|v(s)\|^2_{D\left(A^{\frac{1}{2}}\right)}ds \\
&+\|\varphi(\theta)-\varphi(0)\|_{D\left(A^{\frac{1}{2}}\right)}+M_{\frac{1}{2}}\alpha^{\frac{1}{2}}\int_0^{t+\theta} s^{\alpha-1}((t+\theta)^\alpha-s^\alpha)^{\frac{-1}{2}}\mu(s)\left\|v_s\right\|_* ds \\
&\leq \|T((t+\theta)^\alpha)\varphi(0)-\varphi(0)\|_{D\left(A^{\frac{1}{2}}\right)} \\
&+M_{\frac{3}{4}}\alpha^{\frac{3}{4}}l_1\int_0^{t+\theta} s^{\alpha-1}((t+\theta)^\alpha-s^\alpha)^{\frac{-3}{4}}\left(R+\|\varphi\|_*\right)^2 ds \\
&+\|\varphi(\theta)-\varphi(0)\|_{D\left(A^{\frac{1}{2}}\right)} \\
&+M_{\frac{1}{2}}\alpha^{\frac{1}{2}}\int_0^{t+\theta} s^{\alpha-1}((t+\theta)^\alpha-s^\alpha)^{\frac{-1}{2}}\mu(s)\left(R+\|\varphi\|_*\right)ds
\end{aligned}$$

choose :
$t_1>0$ such that $\|\varphi(t+\theta)-\varphi(\theta)\|_{D(A^{\frac{1}{2}}}\leq\frac{R}{4}, \forall t\in[0,t_1]$ and $\theta\in[-a,0]$ such that $t+\theta\leq 0$,
$t_2>0$ such that $\left\|T\left(\frac{t^\alpha}{\alpha}\right)\varphi(0)-\varphi(0)\right\|_{D\left(A^{\frac{1}{2}}\right)}\leq\frac{R}{4}, \forall t\in[0,t_2]$,
$t_3>0$ such that $\int_0^t s^{\alpha-1}(t^\alpha-s^\alpha)^{\frac{-1}{2}}\mu(s)ds\leq\frac{R}{4M_{\frac{1}{2}}(R+\|\varphi\|_*)^2}\forall t\in[0,t_3]$. and ,
$t_4>0$ such that $\int_0^t s^{\alpha-1}(t^\alpha-s^\alpha)^{\frac{-1}{4}}ds\leq\frac{R}{4M_{\frac{3}{4}}l_1(R+\|\varphi\|_*)^2}\forall t\in[0,t_4]$.
Let $T_1=\min\{t_1,t_2,t_3,t_4\}$. For $t+\theta\in[-a,0], t\in[0,T_1]$, we have:
$\|(\mathit{\Gamma} v)_t(\theta)-\varphi(\theta)\|_{D\left(A^{\frac{1}{2}}\right)}=\|\varphi(t+\theta)-\varphi(\theta)\|_{D\left(A^{\frac{1}{2}}\right)}\leq\frac{R}{4}\leq R.$

Hence, $\|(\mathit{\Gamma} v)_t-\varphi\|_*\leq R$ for all $t\in[0,T_1]$.

Step 2 : we prove that $\mathit{\Gamma} v(t)$ is continuous on $(0,T_1]$ with respect to the topology induced by $D\left(A^{\frac{1}{2}}\right)$-norm.
First define $u(t):=\int_0^t s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)Fv(s)\mathrm{d}s$ and let $t_0\in]0,T_1]$

For $t > t_0$ and $\epsilon > 0$.

$$\begin{aligned}\left\|A^{\frac{1}{2}}\left(u(t)-u\left(t_0\right)\right)\right\| \leq & \left\|A^{\frac{1}{2}} \int_0^{t_0-\epsilon} s^{\alpha-1} A^{\frac{1}{4}}\left[T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)-T\left(\frac{t_0^\alpha-s^\alpha}{\alpha}\right)\right] A^{-\frac{1}{4}} F v(s) d s\right\| \\ & +\left\|A^{\frac{1}{2}} \int_{t_0-\epsilon}^{t_0} s^{\alpha-1} A^{\frac{1}{4}}\left[T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)-T\left(\frac{t_0^\alpha-s^\alpha}{\alpha}\right)\right] A^{-\frac{1}{4}} F v(s) d s\right\| \\ & +\left\|A^{\frac{1}{2}} \int_{t_0}^{t} s^{\alpha-1} A^{\frac{1}{4}} T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right) A^{-\frac{1}{4}} F v(s) d s\right\| \\ := & I_1+I_2+I_3 .\end{aligned}$$

For I_1, we get that:

$$\begin{aligned} I_1 & \leq l_1 \sup_{0 \leq s \leq t_0-\epsilon}\left\|A^{\frac{3}{4}}\left[T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)-T\left(\frac{t_0^\alpha-s^\alpha}{\alpha}\right)\right]\right\| \int_0^{t_0-\epsilon} s^{\alpha-1}\left\|A^{\frac{1}{2}} v(s)\right\|^2 d s \\ & \leq l_1 \sup_{0 \leq s \leq t_0-\epsilon}\left\|A^{\frac{3}{4}}\left[T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)-T\left(\frac{t_0^\alpha-s^\alpha}{\alpha}\right)\right]\right\|\left(R+\|\varphi\|_*\right)^2 \frac{\left(t_0-\epsilon\right)^\alpha}{\alpha} .\end{aligned}$$

So, by using Lemma 2.6, $t \mapsto A^{\frac{3}{4}} T(t)$ is continuous in the uniform operator topology on $[\epsilon, T_1]$ for every $\epsilon > 0$, there exists $\eta \in [0, t_0-\epsilon]$ such that,

$$\sup_{0 \leq s \leq t_0-\epsilon}\left\|A^{\frac{3}{4}}\left[T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)-T\left(\frac{t_0^\alpha-s^\alpha}{\alpha}\right)\right]\right\|=\sup_{0 \leq s \leq t_0-\epsilon}\left\|A^{\frac{3}{4}}\left[T\left(\frac{t^\alpha-\eta^\alpha}{\alpha}\right)-T\left(\frac{t_0^\alpha-\eta^\alpha}{\alpha}\right)\right]\right\|$$

hence $I_1 \to 0$ as $t \to t_0$. Now, consider I_2 and I_3 . Using Lemmas 1,2 we have,

$$\begin{aligned} I_2 & \leq \int_{t_0-\epsilon}^{t_0} s^{\alpha-1}\left\|A^{\frac{3}{4}}\left[T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)-T\left(\frac{t_0^\alpha-s^\alpha}{\alpha}\right)\right] A^{-\frac{1}{4}} F v(s)\right\| d s \\ & \leq M_{\frac{3}{4}} \alpha^{\frac{3}{4}} l_1 \int_{t_0-\epsilon}^{t_0} s^{\alpha-1}\left(t^\alpha-s^\alpha\right)^{\frac{-3}{4}}+s^{\alpha-1}\left(t_0^\alpha-s^\alpha\right)^{\frac{-3}{4}}\left\|A^{\frac{1}{2}} v(s)\right\|^2 d s \\ & \leq 2 M_{\frac{3}{4}} \alpha^{\frac{3}{4}} l_1\left(R+\|\varphi\|_*\right)^2 \int_{t_0-\epsilon}^{t_0} s^{\alpha-1}\left(t_0^\alpha-s^\alpha\right)^{\frac{-3}{4}} d s \rightarrow 0 \text { as } \epsilon \rightarrow 0 .\end{aligned}$$

Similaraly:

$$I_3 \leq M_{\frac{3}{4}} \alpha^{\frac{3}{4}} l_1 \int_{t_0}^t s^{\alpha-1}\left(t^\alpha-s^\alpha\right)^{\frac{-1}{4}}\left(R+\|\varphi\|_*\right)^2 d s \rightarrow 0 \text { as } t \rightarrow t_0 .$$

Therefore, $\left\|A^{\frac{1}{2}}\left(u(t)-u\left(t_0\right)\right)\right\| \to 0$ as $t \to t_0+$.
we can be proved that $\left\|A^{\frac{1}{2}}\left(u(t)-u\left(t_0\right)\right)\right\| \to 0$ as $t \to t_0-$.
Hence, $t \mapsto u(t)$ is continuous on $(0, T_1]$ with respect to the topology induced by $D\left(A^{\frac{1}{2}}\right)$-norm.
Now, define $u^{'}(t) := \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right) P f(s, v_s) \mathrm{d}s$ and let $t_0 \in (0, T_1]$ with $t > t_0$ and $\epsilon > 0$ small enough. Analogously, by using Lemma 2.6 and assumption H_1, H_2 we show that:

$$\left\|A^{\frac{1}{2}}\left(u^{'}(t)-u^{'}\left(t_0\right)\right)\right\| \rightarrow 0 \text { as } t \rightarrow t_0$$

Therfore, $t \mapsto u^{'}(t)$ is continuous on $(0, T_1]$ with respect to the topology induced by $D\left(A^{\frac{1}{2}}\right)$-norm.
Since $u(0) = \varphi(0) \in D\left(A^{\frac{1}{2}}\right)$, therefore by the continuity of $T(t)$, we can say that
$\left\|A^{\frac{1}{2}} T\left(\frac{t^\alpha}{\alpha}\right) \varphi(0)-A^{\frac{1}{2}} T\left(\frac{t_0^\alpha}{\alpha}\right) \varphi(0)\right\|=\left\|T\left(\frac{t^\alpha}{\alpha}\right) A^{\frac{1}{2}} \varphi(0)-T\left(\frac{t_0^\alpha}{\alpha}\right) A^{\frac{1}{2}} \varphi(0)\right\| \rightarrow 0$ as $t \to t_0$

Hence, we proved that $t \mapsto \Gamma v(t)$ is continuous on $[-r, T_1]$ with respect the topology induced by $D\left(A^{\frac{1}{2}}\right)$-norm, and $\Gamma\left(Y_{\frac{1}{2}}\right) \subset Y_{\frac{1}{2}}$

Step 3 : let $u, v \in Y_{\varphi,\frac{1}{2}}, t \in [0, T_1]$. Then using Lemmas (2.2),(2.3) and $(H_1), (H_2)$ we get,

$$\begin{aligned}
\left\|\Delta_{\frac{1}{2}}\right\| = \left\|A^{\frac{1}{2}}\{\Gamma u(t) - \Gamma v(t)\}\right\| &\leq \left\|A^{\frac{1}{2}} \int_0^t s^{\alpha-1} A^{\frac{1}{4}} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) A^{-\frac{1}{4}}(Fu(s) - Fv(s))ds\right\| \\
&+ \left\|\int_0^t s^{\alpha-1} A^{\frac{1}{2}} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\{Pf(s, u_s) - Pf(s, v_s)\}\, ds\right\| \\
&\leq M_{\frac{3}{4}} \alpha^{\frac{3}{4}} \int_0^t s^{\alpha-1}(t^\alpha - s^\alpha)^{\frac{-3}{4}} \left\|A^{-\frac{3}{4}}(Fu(s) - Fv(s))\right\| ds \\
&+ M_{\frac{1}{2}} \alpha^{\frac{1}{2}} \int_0^t s^{\alpha-1}(t^\alpha - s^\alpha)^{\frac{-1}{2}} \|(Pf(s, u_s) - Pf(s, v_s))\|\, ds \\
&\leq l_1 M_{\frac{3}{4}} \alpha^{\frac{3}{4}} \int_0^t s^{\alpha-1}(t^\alpha - s^\alpha)^{\frac{-3}{4}} l_1 \left\|A^{\frac{1}{2}}(u - v)\right\| \left(\left\|A^{\frac{1}{2}} u\right\| + \left\|A^{\frac{1}{2}} v\right\|\right) ds \\
&+ M_{\frac{1}{2}} \alpha^{\frac{1}{2}} K_f \int_0^t s^{\alpha-1}(t^\alpha - s^\alpha)^{\frac{-3}{4}} \|u_s - v_s\|_*\, ds \\
&\leq l_1 M_{\frac{3}{4}} \alpha^{\frac{3}{4}} \int_0^t s^{\alpha-1}(t^\alpha - s^\alpha)^{\frac{-3}{4}} \|u - v\|_{Y_{\varphi,\frac{1}{2}}} 2\left(R + \|\varphi\|_* ds\right. \\
&+ M_{\frac{1}{2}} \alpha^{\frac{1}{2}} K_f \int_0^t s^{\alpha-1}(t^\alpha - s^\alpha)^{\frac{-1}{2}} \sup_{0 \leq r \leq T_\varphi} \left\|A^{\frac{1}{2}}(\boldsymbol{u}(r) - v(r))\right\| ds
\end{aligned}$$

$\left\|\Delta_{\frac{1}{2}}\right\| \leq \left\{2\left(R + \|\varphi\|_*\right) M_{\frac{3}{4}} \alpha^{\frac{3}{4}} \int_0^t s^{\alpha-1}(t^\alpha - s^\alpha)^{\frac{-3}{4}} ds + M_{\frac{1}{2}} \alpha^{\frac{1}{2}} K_f \int_0^t s^{\alpha-1}(t^\alpha - s^\alpha)^{\frac{-1}{2}} ds\right\} \|u - v\|_{Y_{\varphi,\frac{1}{2}}}$.

Since: $M_{\frac{3}{4}} \alpha^{\frac{3}{4}} \int_0^t s^{\alpha-1}(t^\alpha - s^\alpha)^{\frac{-3}{4}} ds + M_{\frac{1}{2}} \alpha^{\frac{1}{2}} K_f \int_0^t s^{\alpha-1}(t^\alpha - s^\alpha)^{\frac{-1}{2}} ds \to 0$ as $t \to 0$, we can choose:
$T_\varphi \leq T_1$ such: $\left\|A^{\frac{1}{2}}\{\Gamma u(t) - \Gamma v(t)\}\right\| \leq L\|u - v\|_{Y_{\varphi,\frac{1}{2}}}$ for all $t \in [0, T_\varphi]$ and $0 < L < 1$.
This implies that $\|\Gamma u - \Gamma v\|_{Y_{\varphi,\frac{1}{2}}} \leq L\|u - v\|_{Y_{\varphi,\frac{1}{2}}}$ for $0 < L < 1$.
Therefore, $\Gamma : Y_{\varphi,\frac{1}{2}} \to Y_{\varphi,\frac{1}{2}}$ is a contraction map.

Hence, by Banach contraction theorem, Γ has a unique fixed point $v \in Y_{\varphi,\frac{1}{2}}$ which satisfies the integral equation (3.4). This proves the existence of uniqueness local mild solution of (2.2). □

3.1. Exemple of delay External force :

Let $\omega : [0, \infty) \times [-a, 0] \to \mathbb{R}$ a measurable function such that $|\omega(t, p)| \leq (-p)^{-\gamma}$ for all $t \geq 0, -a < p \leq 0$ and $\gamma < 1$.
For $\psi \in C\left([-a, 0]; D\left(A^{\frac{1}{2}}\right)\right)$, we define the project of a function f,

$$Pf(t, \psi) = \int_{-a}^0 \omega(t, p)\psi(s)ds, \quad t \geq 0.$$

Then $Pf(t, v_t) = \int_{-a}^0 \omega(t, p)v(t + p)ds.$ and $(t, \psi) \mapsto Pf(t, \psi)$ satisfy the assymptions $(H_1) - (H_2)$ of the Theorem 4 with $\mu(t) = K_f = \frac{a^{1-\gamma}}{1-\gamma}$ for all $t \geq 0$.

4. Globality of mild solution

Theorem 4.1. *Under the same assumptions as in Theorem* 3.1 *for* $V = X_{\frac{1}{2}}$ *and* $t > 0$, *for every* $\varphi \in X_{\frac{1}{2}}$, *the problem (2.2) has a unique mild solution on a maximal interval of existence* $[-a, t_{\max})$. *and either* $t_{\max} = \infty$ *or* $\limsup_{t \to t_{\max}} \|v(t)\|_{D(A^{\frac{1}{2}})} = \infty$.

Proof. according the result under the conditions , the mild solution of (2.2) exists in the interval $[-a, T]$. Now we prove that this solution can be extended to the interval $[-a, T+\theta]$ for some $\theta > 0$. Let v be the mild solution of (2.2) on $[-a, T]$. Define $u(t) = v(t+T)$ where $u(t)$ is a mild solution of:

$$\begin{cases} D_t^{\alpha} u + Au(t) = Fu(t) + Pf\left(t+T, u_t\right), \quad t > 0, \\ u_0 = v_T. \end{cases} \tag{4.1}$$

we have $v \in C\left([-a, T]; D\left(A^{\frac{1}{2}}\right)\right)$, then $u_0 = v_T \in X_{\frac{1}{2}}$. Therfore, according to the Theorem3.1, the mild solution of (4.1) exist on some interval $[-a, \tau]$, where $\tau > 0$. So , Let $R > 0$ fix and $C_\theta = C\left([-a, T+\theta]\right)$, Consider the following set:

$$\Lambda_{\varphi;\frac{1}{2}} = \left\{ \psi \in C_\theta : \psi(t) = v(t), \forall t \in [-a, T], \sup_{0 \le t \le T} \|\psi_t - \varphi\|_* \le R, \sup_{T \le t \le T+\theta} \|\psi(t) - \psi(T)\|_{D\left(A^{\frac{1}{2}}\right)} \le R \right\}$$

Similaraly as the prove in Theorem 3.1, we show that there exists a $\theta > 0$ such that the problem (2.2) has a unique mild solution in $\Lambda_{\varphi;\frac{1}{2}}$, wich prove the maximality of the interval of existence of mild solution the problem (2.2)

let $[-a, t_{\varphi max})$ be the maximal interval of existence of mild solution of (2.2). So:

-case 1: If $t_{\varphi max} = \infty$, then the mild solution is global.

-case 2: If $t_{\varphi max} < \infty$, we prove that $limsup_{t \to t_{\varphi max}} \|v(t)\|_{D\left(A^{\frac{1}{2}}\right)} = \infty$.

suppose that $limsup_{t \to t_{\varphi max}} \|v(t)\|_{D\left(A^{\frac{1}{2}}\right)} < \infty$. Then, $limsup_{t \to t_{\varphi max}} \|v_t\|_{X_{\frac{1}{2}}} < \infty$.

Therfore, there exists $B > 0$ such that $\|v_t\|_{X_{\frac{1}{2}}} \le B$,$\forall\ t \in [0, t_{\varphi max})$.

Then $\|Pf\left(t, v_t\right)\| \le B\mu(t)$, $\forall t \in [0, t_{\varphi, max})$.

Now, Let $0 < t < t_0 < t_{\max}$ and $\epsilon > 0$ be sufficiently small. Then we have

$$\begin{aligned}
\|v(t)-v(t_0\|_{D\left(A^{\frac{1}{2}}\right)} &= \left\|T(\frac{t^\alpha}{\alpha})\varphi(0) - T(\frac{t_0^\alpha}{\alpha})\varphi(0)\right\|_{D\left(A^{\frac{1}{2}}\right)} \\
&+ \left\|\int_t^{t_0} s^{\alpha-1}\left[T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)Pf\left(s,u_s\right)\right]ds\right\|_{D\left(A^{\frac{1}{2}}\right)} \\
&+ \left\|\int_0^{t-\epsilon} s^{\alpha-1}\left[T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_0^\alpha - s^\alpha}{\alpha}\right)\right]Pf\left(s,u_s\right)ds\right\|_{D\left(A^{\frac{1}{2}}\right)} \\
&+ \left\|\int_{t-\epsilon}^{t} s^{\alpha-1}\left[T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_0^\alpha - s^\alpha}{\alpha}\right)\right]Pf\left(s,u_s\right)ds\right\|_{D\left(A^{\frac{1}{2}}\right)} \\
&+ \left\|A^{\frac{1}{2}}\int_0^{t_0-\epsilon} s^{\alpha-1}A^{\frac{1}{4}}\left[T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_0^\alpha - s^\alpha}{\alpha}\right)\right]A^{-\frac{1}{4}}Fv(s)ds\right\| \\
&+ \left\|A^{\frac{1}{2}}\int_{t_0-\epsilon}^{t_0} s^{\alpha-1}A^{\frac{1}{4}}\left[T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_0^\alpha - s^\alpha}{\alpha}\right)\right]A^{-\frac{1}{4}}Fv(s)ds\right\| \\
&+ \left\|A^{\frac{1}{2}}\int_{t_0}^{t} s^{\alpha-1}A^{\frac{1}{4}}T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)A^{-\frac{1}{4}}Fv(s)ds\right\| \\
&\leq \left\|T(\frac{t^\alpha}{\alpha})\varphi(0) - T(\frac{t_0^\alpha}{\alpha})\varphi(0)\right\|_{D\left(A^{\frac{1}{2}}\right)} + BM_{\frac{1}{2}}\alpha^{\frac{1}{2}}\int_t^{t_0} s^{\alpha-1}(t_0-s)^{\frac{-1}{2}}\mu(s)ds \\
&+ B\sup_{0\leq s\leq t-\epsilon}\left\|A^{\frac{1}{2}}\left[T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_0^\alpha - s^\alpha}{\alpha}\right)\right]\right\|\int_0^{t-\epsilon} s^{\alpha-1}\mu(s)ds \\
&+ 2B\int_{t-\epsilon}^{t} s^{\alpha-1}(t_0-s)^{\frac{-1}{2}}\mu(s)ds \\
&+ B^2\frac{t_{\varphi max}^\alpha}{\alpha}\sup_{0\leq s\leq t-\epsilon}\left\|A^{\frac{3}{4}}\left[T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_0^\alpha - s^\alpha}{\alpha}\right)\right]\right\| \\
&+ 2B\int_{t-\epsilon}^{t} s^{\alpha-1}(t_0-s)^{\frac{-1}{4}}ds + B^2\int_t^{t_0} s^{\alpha-1}(t_0-s)^{\frac{-1}{4}}ds.
\end{aligned}$$

by applying Hölder's inequality with $\mu \in L^p\left[0, t_{\varphi max}\right)$ for $p > \frac{2}{\alpha}$ and the continuity of T(t) in the uniform-topology norm, we get that $t \mapsto v(t)$ is uniformly continuous on $(0, t_{\max})$ with respect to the topology induced by $D\left(A^{\frac{1}{2}}\right)$-norm.

Hence $\lim_{t\to t_{\max}} v(t) = v\left(t_{\varphi\max}\right)$ exists, so , one can extended the interval of existence which contradicts the maximality . the proof is complete. □

By the same procedure as the proof of theorem 4.1, we can proved the globality of mild solution by using more conditions, and get the following theorem.

Theorem 4.2. *Under the same assumptions as in Theorem3.1, if there exist $B > 0$ and a locally integrable functions $\xi(t)$ such that $\|Pf\left(t,\varphi\right)\| \leq B\xi(t)$ $\forall t > 0, \varphi \in X_{\frac{1}{2}}$ and $\xi \in L^p_{loc}[0,\infty)$, where $p > \frac{2}{\alpha}$, then Equation (2.2) has global mild solutions.*

5. Conclusion

In this paper, we proved the existence of $D(A^{\frac{1}{2}})$-valued local mild solution for a time fractional Navier-Stokes differential equations with the finite external forces involving conformable fractional derivative of order $0 < \alpha < 1$. The existence theorems is proved by using some Lipschitz conditions and Banach contraction theorem. As application, an example of delayed force function is presented to illustrate the applicability our main result. The globality has been established by using more assymptions on forces when the initial datum curve belong to the whole space $V = X_{\frac{1}{2}}$.

Acknowledgement

The authors would like to express their sincere apreciation to the referees for their very helpful suggestions and many kind comments.

Conflict of interest

The authors declare that they have no conflict of interest.

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